Appendix A: Velocity gradients within the source layer

The Rapid Distortion Theory was extended to turbulence distorted by a two-dimensional bluff body by Hunt (1973), and subsequently to turbulence distorted by a flat surface by HG. In the latter case, RDT assumptions imply that the velocity fluctuation \( \mathbf{v'}(x,y',z,t) \) at a distance \( y' \) from the surface simply differs from the free-stream fluctuation \( \mathbf{v}_\infty(x,y',z,t) \) by an irrotational fluctuation \( \nabla \Phi(x,y',z,t) \). Hunt (1973) showed that the two-dimensional Fourier transforms \( \hat{\Phi}(k_1,k_3,y',t) \) and \( \hat{v}_i(k_1,k_3,y',t) \) of the velocity potential \( \Phi \) and velocity fluctuation \( v' \) are related to the three-dimensional Fourier transform \( \hat{v}_\infty(k,t) \) of the free-stream velocity fluctuation through

\[
\hat{\Phi}(k_1,k_3,y',t) = \int_{-\infty}^{+\infty} \hat{v}_\infty(k,y') \hat{v}_{\infty}(k,t) dk_2, \quad (A1a)
\]
\[
\hat{v}'_i(k_1,k_3,y',t) = \int_{-\infty}^{+\infty} \hat{M}_{im}(k,y') \hat{v}_{\infty}(k,t) dk_2, \quad (A1b)
\]

where \( k_1, k_2 \) and \( k_3 \) are the components of the wavenumber \( k \) along directions \( x, y' \) and \( z \) respectively. From the relation \( v' = v_\infty + \nabla \Phi \) it is immediate to see that the "distortion" tensor \( M_{im} \) is given by

\[
M_{im}(k,y') = \delta_{im} e^{ik_2y'} + \left[ \begin{array}{c} \frac{i}{k_1} \beta_m(k,y') \\ \frac{\partial \beta_m(k,y')}{\partial y'} \\ \frac{i}{k_3} \beta_m(k,y') \end{array} \right] \quad (A1c)
\]

with \( i^2 = -1 \). If turbulence is homogeneous in planes \( (x,z) \) parallel to the surface, the Laplace equation (5) becomes in Fourier space

\[
\left[ \frac{\partial^2}{\partial y'^2} - k_H^2 \right] \hat{\Phi}(k_1,k_3,y',t) = 0,
\]

\( k_H = (k_1^2 + k_3^2)^{1/2} \) being the tangential wavenumber. The solution of this equation satisfying boundary conditions (8) and (12) is

\[
\left( e^{-k_H y'}/k_H \right) \int_{-\infty}^{+\infty} \hat{v}_{\infty}(k,t) dk_2, \quad \hat{v}_{\infty} \text{being the normal} \]
component of $\mathbf{v}_m(k,t)$. Comparing with (A1a) yields after HG

$$\beta_m(k,\gamma) = \frac{1}{k_H} e^{-k_H \gamma} \delta_{m2} \quad (A2)$$

From (A1)-(A2) one deduces that the variance of the components of the velocity gradient tensor

$$\left\langle \frac{\partial v_i}{\partial x_j} \right\rangle^2$$

(no summation on $i$ or $j$) is related to the three-dimensional velocity spectrum $\Phi_{ij}(k)$ of the free-stream turbulence through

$$\left\langle \frac{\partial v_i}{\partial x_j} \right\rangle^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d^3k}{\partial \gamma} \frac{d^3k_{im}}{\partial \gamma} \Phi_{mn}(k) d^3k \quad \text{for } j \neq 2$$

$$\int_{-\infty}^{\infty} \Phi_{ii}(k) + 2 k^2 - 2k_1 k_2 \cos(k_2 \gamma) e^{-k_H \gamma} \Phi_{12}(k) + \Phi_{22}(k) \right) d^3k \quad \text{for } j = 2 \quad (A3)$$

where $d^3k = dk_1 dk_2 dk_3$ and the star denotes the complex conjugate. Inserting (A1a-c) into (A3) yields

$$\left\langle \frac{\partial v_i}{\partial x_j} \right\rangle^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_j \left\{ \Phi_{ii}(k) - 2 k_1 \Phi_{12}(k) + k_2^2 \right\} \Phi_{22}(k) d^3k \quad \text{for } i \neq 2, j = 2$$

Equations (A1)-(A3) reveal that terms involving $e^{-2k_H \gamma}$ in expressions (A4) correspond to the contribution $\left\langle (\partial^2 \Phi/\partial x_i \partial x_j)^2 \right\rangle$ whereas terms involving $e^{-k_H \gamma}$ correspond to velocity gradients produced by the interaction between the free-stream fluctuation and the perturbation $\nabla \Phi$.

In what follows we consider that the free-stream turbulence is homogeneous and isotropic. Since the essential contribution to the velocity gradients comes from the high-wavenumber part of the spectrum, we approximate $\Phi_{mn}(k)$ by a Kolmogorov spectrum with a sharp cut-off at the Kolmogorov wavenumber $k_K = \left( \frac{2}{3C_K} \frac{\varepsilon_\infty^{1/3}}{\nu} \right)^{3/4}$, so that

$$\Phi_{mn}(k) = C_K \varepsilon_\infty^{2/3} (k^2 \delta_{mn} - k_m k_n) k^{-17/3} \quad \text{for } k \leq k_K \quad (A5)$$

where $C_K$ is the Kolmogorov constant, $k = \|k\|$, and $\varepsilon_\infty$ is the dissipation rate of the free-stream
turbulence. Note that, since (A5) does not describe correctly the energy distribution at low wavenumber, it cannot be used to predict the distortion produced by the surface at distances such that \( \gamma L_\infty = O(1) \). Consequently, the results derived in this appendix are essentially valid in the limit \( \gamma L_\infty \to 0 \).

For \( \gamma L_\infty \) large compared to unity, terms involving \( e^{-kL}\gamma'\) and \( e^{-2kL}\gamma'\) are negligible in (A4). Under such conditions one recovers the well-known result (Monin & Yaglom 1975, p. 56)

\[
\left\langle \left( \frac{\partial v_i}{\partial x_j} \right)^2 \right\rangle_{\infty} = \frac{\varepsilon_\infty / \nu}{15} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}
\]  
(A6)

In the limit \( \Re_\infty \to \infty \), \( kKL_\infty \) is \( O(\Re_\infty^{-3/4}) \) (Monin & Yaglom 1975, p. 349) and \( \gamma L_\infty \) is \( O(\Re_\infty^{-1/2}) \) for \( \gamma \approx O(\delta \nu) \). Hence the source layer of the HG theory corresponds typically to \( kL_\gamma \) ranging from \( O(\Re_\infty^{-1/4}) \) to \( O(\Re_\infty^{-3/4}) \). To obtain the variance of velocity gradients in the part of this layer closest to the surface we thus consider the intermediate limit \( kL_\gamma \to \infty \), \( \gamma L_\infty \to 0 \). Under such conditions, Eqs. (A1b-c) show that eddies with tangential wavenumbers typically smaller than \( 1/\gamma \) are distorted by the surface whereas eddies with \( 1/\gamma \leq k_L \leq k_K \) are almost unaltered.

Let us first evaluate the terms of (A4) involving \( e^{-2kL}\gamma'\Phi_{22}(k) \), i.e. the contribution \( \left\langle \left( \nabla^2 \Phi / \partial x_i \partial x_j \right)^2 \right\rangle \). These terms may be written in the general form

\[
I_{ij}(kL_\gamma) = \frac{1}{3\pi} \frac{\varepsilon_\infty / \nu}{(2kL_\gamma)^{2/3}} 2\pi \int_0^{2\pi} \int_0^{2kL_\gamma} Z^{1/3} e^{-Z} dZ \right\} \int_0^{X_M} \frac{dX}{(1 + X^2)^{17/6}} dZ
\]

where \( X_M = \left( \frac{2kL_\gamma^2}{Z} \right)^{1/2} - 1 \) and \( \chi_{ij}(\theta) \) denotes trigonometric functions depending on the values of \( i \) and \( j \). Since \( kL_\gamma \) is large, the essential contributions to \( I_{ij} \) come from values of \( Z \) of order unity or less. Hence \( I_{ij} \) may be approximated as

\[
I_{ij}(kL_\gamma) = \frac{1}{3\pi} \frac{\varepsilon_\infty / \nu}{(2kL_\gamma)^{2/3}} \int_0^{2\pi} \int_0^{\infty} \chi_{ij}(\theta) d\theta Z^{1/3} e^{-Z} dZ \int_0^{\infty} \frac{dX}{(1 + X^2)^{17/6}}
\]

which yields (Abramovitz & Stegun 1964, p. 258-260)
\[ I_{ij}(k_{K\gamma}) = \frac{2}{495} 2^{1/3} \left( \frac{\Gamma(1/3)}{\Gamma(2/3)} \right)^3 Q_{ij} \left( \frac{\varepsilon_\infty}{v} \right) \frac{1}{(k_{K\gamma})^{4/3}} \]  

where \( \Gamma \) denotes the Gamma function and \( Q_{ij} = 3/4 \) for \( i=j \neq 2 \), \( Q_{ij} = 1/4 \) for \( i \neq 2 \), \( j \neq 2 \) and \( i \neq j \), \( Q_{ij} = 1 \) for \( i = 2 \) and \( j \neq 2 \) or \( i \neq 2 \) and \( j = 2 \), \( Q_{ij} = 2 \) for \( i = j = 2 \). From (A7) we deduce that within the source layer, the variance of the velocity gradients associated with the velocity perturbation \( \partial \Phi / \partial x_i \) is

\[
\left\langle \left( \partial^2 \Phi / \partial x_i \partial x_j \right)^2 \right\rangle (\gamma) = 0.0181 \left[ \begin{array}{ccc}
3 & 4 & 1 \\
4 & 8 & 4 \\
1 & 4 & 3 \\
\end{array} \right] \\
\]

In (A4), integrals involving \( e^{-k \partial \gamma \sin(k_{2\gamma})} \Phi_{2\gamma}(k) \) may be written in the form

\[
J_{ij}(k_{K\gamma}) = -\frac{2}{3\pi} \left( \frac{\varepsilon_\infty}{v} \right) \frac{2\pi}{\Gamma(2/3)} \frac{k_{K\gamma}}{U^{1/3} e^{-U}} \int_0^\infty X_M X \sin(XU) dX \\
\]

with now \( X_M = \left\{ \left( \frac{k_{K\gamma}}{U} \right)^2 - 1 \right\}^{1/2} \). Using the same approximation and integrating by parts with respect to \( X \) results in

\[
J_{ij}(k_{K\gamma}) = -\frac{3}{595} 2^{1/3} \frac{\Gamma(1/3)}{\Gamma(2/3)} \left( \frac{\varepsilon_\infty}{v} \right) \frac{2\pi}{\Gamma(2/3)} \frac{k_{K\gamma}}{U^{1/3} e^{-U}} K_{4/3}(U) dU \\
\]

where \( K_{4/3}(U) \) denotes the modified Bessel function of order \( \mu \) (Abramowitz & Stegun 1965, p. 374). Integration with respect to \( U \) can be performed analytically (Gradshteyn & Ryzhik 1980, p. 712), yielding

\[
J_{ij}(k_{K\gamma}) \approx -\frac{2592}{95045} \frac{\varepsilon_\infty}{v} \frac{Q_{ij}}{(k_{K\gamma})^{4/3}} \\
\]  

(A9a)

Evaluation of the numerical prefactors indicates that

\[
I_{ij}(k_{K\gamma}) + J_{ij}(k_{K\gamma}) \approx -0.0006 Q_{ij} \frac{\varepsilon_\infty}{v} \frac{1}{(k_{K\gamma})^{4/3}} \\
\]  

(A9b)

Since \( k_{K\gamma} \) is large in the source layer, (A9b) shows that the presence of the surface has a negligible influence on \( \left\langle \left( \partial v_i / \partial x_j \right)^2 \right\rangle \) for \( i \neq 2 \) and \( j \neq 2 \), as well as on \( \left\langle \left( \partial v_i / \partial y \right)^2 \right\rangle \). In particular it is worth noting that for \( \gamma = O(\delta_Y) \), \( \left\langle \left( \partial v_i / \partial x_j \right)^2 \right\rangle \) is almost identical to its free-stream value while, according to the result of HGM and to (A16) below, the value of \( \left\langle v^2 \right\rangle \) itself is reduced by a
For \( i=1 \) or \( 3 \) and \( j=2 \) an integral involving \( e^{k_K \gamma} \cos (k_2 \gamma) \psi_{i2}(k) \) must be evaluated in the corresponding expressions of (A4). This integral may be written in the form

\[
G(k_K \gamma) = \frac{2}{3} \frac{\varepsilon_{\infty}/v}{(k_K \gamma)^{4/3}} \int_0^{1/3} U^{1/3} e^{-U} \left\{ \frac{X_0}{(1+X^2)^{17/6}} \right\} dU
\]

After integrating with respect to \( X \) one obtains

\[
G(k_K \gamma) = \frac{1}{5} \frac{2^{1/3} \Gamma(1/3) \varepsilon_{\infty}/v}{\Gamma(2/3)} \int_0^{5/3} U^{1/3} e^{-U} K_{4/3}(U) dU - \frac{3}{11} \int_0^{8/3} U^{1/3} e^{-U} K_{7/3}(U) dU
\]

which yields

\[
G(k_K \gamma) = \frac{1566}{95095} \frac{\Gamma(1/3) \varepsilon_{\infty}/v}{(k_K \gamma)^{4/3}}
\]

(A10)

Finally, for \( i=2 \) and \( j=1 \) or \( 3 \), the integral involving \( e^{k_K \gamma} \cos (k_2 \gamma) \psi_{22}(k) \) is

\[
H(k_K \gamma) = -\frac{2}{3} \frac{\varepsilon_{\infty}/v}{(k_K \gamma)^{4/3}} \int_0^{1/3} U^{1/3} e^{-U} \left\{ \frac{X_0 \cos(XU)}{(1+X^2)^{17/6}} \right\} dU
\]

which yields

\[
H(k_K \gamma) = \frac{1944}{19019} \frac{\Gamma(1/3) \varepsilon_{\infty}/v}{(k_K \gamma)^{4/3}}
\]

(A11)

Combining (A7) (with \( i=1 \) or \( 3 \) and \( j=2 \)) and (A10) shows that, as the surface is approached, \( \langle (\partial u / \partial y)^2 \rangle \) and \( \langle (\partial w / \partial y)^2 \rangle \) increase approximately by

\[
I_{12}(k_K \gamma) + G(k_K \gamma) = \frac{2}{55} \left( \frac{1}{9} \left( \frac{\Gamma(1/3)^2}{\Gamma(2/3)} \right) + \frac{783}{1729} \right) \Gamma(1/3) \frac{\varepsilon_{\infty}/v}{(k_K \gamma)^{4/3}} = 0.1164 \frac{\varepsilon_{\infty}/v}{(k_K \gamma)^{4/3}}
\]

(A12)

Similarly, combining (A7) (with \( i=2 \) and \( j=1 \) or \( 3 \)) and (A11) shows that \( \langle (\partial v / \partial x)^2 \rangle \) and \( \langle (\partial v / \partial z)^2 \rangle \) are reduced approximately by

\[
I_{2j}(k_K \gamma) + H(k_K \gamma) = \frac{2}{45} \left( \frac{1}{45} \left( \frac{\Gamma(1/3)^2}{\Gamma(2/3)} \right) - \frac{972}{1729} \right) \Gamma(1/3) \frac{\varepsilon_{\infty}/v}{(k_K \gamma)^{4/3}} = -0.2015 \frac{\varepsilon_{\infty}/v}{(k_K \gamma)^{4/3}}
\]

(A13)
Collecting all the above results to evaluate expressions (A4) reveals that in the common limit $k \gamma \to \infty \gamma L_{\infty} \to 0$, the variance of the velocity gradients evolves as

$$
\langle \left( \partial v_i / \partial x_j \right)^2 \rangle (\gamma) \approx \frac{\varepsilon_{\infty}}{15} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \frac{\varepsilon_{\infty}}{\nu} + \frac{\varepsilon_{\infty}}{(k \gamma)^{4/3}} \begin{bmatrix} -0 & 0.116 & -0 \\ -0.201 & 0 & -0.201 \\ -0 & 0.116 & -0 \end{bmatrix}
$$

(A14)

Summation of results (A14) over $i$ and $j$ shows that the local pseudo-dissipation per unit mass is

$$
\varepsilon(\gamma) = \varepsilon_{\infty} (1 - 0.173 (k \gamma)^{-4/3})
$$

(A15)

Hence RDT predicts that the pseudo-dissipation is slightly reduced in the source layer. In contrast, since in this approach the difference between the local value and the undisturbed value of the velocity fluctuation reduces to an irrotational field, the variance of vorticity fluctuations is not altered. Interestingly, the kinematic relation $S^2 = \varepsilon(\gamma) / \nu - \partial^2 \langle v^2 \rangle / \partial \gamma^2$ linking the local value of the pseudo-dissipation $\varepsilon(\gamma)$ to the enstrophy $2S^2$ can be used to recover the near-surface evolution of $\langle v^2 \rangle$. Combining (A15) with the fact that $S^2 = \varepsilon_{\infty} / \nu$ everywhere yields after integrating twice and requiring that $\langle v^2 \rangle \to 0$ for $\gamma \to 0$

$$
\langle v^2 \rangle (\gamma) = 1.784 \varepsilon_{\infty} 2^{2/3} \gamma^{2/3} + O(\varepsilon_{\infty} 2^{1/3} L_{\infty}^{-1/3} \gamma^{1/3})
$$

(A16)

where we have set $C_K = 0.25 \frac{55}{9} \approx 1.528$ (Townsend 1976 p. 98). The leading-order term in (A16) agrees with the result found by Hunt (1984). Note that this term is independent of the low-wavenumber part of the energy spectrum, since only the high-wavenumber shape was specified in (A5). Moreover, expanding $M_{22}(k, \gamma) = M_{22}^*(k, \gamma)$ in the limit $k \gamma \to 0$ shows that the contribution of low wavenumbers to $\langle v^2 \rangle (\gamma)$ is $O(\gamma^2)$, i.e. the linear term in (A16) is actually zero.

To determine $\partial^2 \langle v_i^2 \rangle / \partial \gamma^2$ (i≠2) in the limit $\gamma L_{\infty} \to 0$, let us begin by evaluating

$$
\langle v_i \partial^2 v_i / \partial \gamma^2 \rangle = \int_{-\infty}^{+\infty} M_{im}^*(k, \gamma) \frac{\partial^2 M_{in}(k, \gamma)}{\partial \gamma^2} \Phi_{mn}(k) d^3 k
$$

(A17)

Using (A1c) and (A2) we obtain
\[ \left\langle \frac{\partial^2 v_i}{\partial y^2} \right\rangle = -\frac{2}{15} \frac{\varepsilon_{\infty}}{v} + i_{11} + i_{13} + j_{11} + j_{13} - \int_{-\infty}^{\infty} \left( k_H^2 + k_2^2 \right) \frac{k_i}{k_H} e^{-k_H Y} \sin(k_2 Y) \Phi_{i2}(k) d^3 k \]

with \( i_{ij} + j_{ij} \) given by (A9b). The last integral may be written in the form

\[ \frac{1}{3} \frac{\varepsilon_{\infty}/v}{(k_{\kappa} \gamma)^{4/3}} \int_0^{11/6} \frac{X \sin(XU)}{U(U+1)^{1/6}} dX \frac{\pi^{1/2}}{\Gamma(5/6)} \frac{\varepsilon_{\infty}/v}{(k_{\kappa} \gamma)^{4/3}} \int_0^{11/3} e^{-U} K_{1/3}(U) dU \]

Evaluation of this integral yields

\[ \left\langle \frac{\partial^2 v_i}{\partial y^2} \right\rangle \approx -\frac{2}{15} \frac{\varepsilon_{\infty}}{v} + \frac{18}{455} \frac{\varepsilon_{\infty}/v}{(k_{\kappa} \gamma)^{4/3}} \]

(A18)

Now, adding results (A14) (for \( i \neq 2 \) and \( j = 2 \)) and (A18), we find

\[ \frac{\partial^2 \left\langle v_i^2 \right\rangle}{\partial y^2} = 2 \left( \left\langle v_i \frac{\partial^2 v_i}{\partial y^2} \right\rangle + \left( \frac{\partial v_i}{\partial y} \gamma \right)^2 \right) \approx 0.372 \frac{\varepsilon_{\infty}/v}{(k_{\kappa} \gamma)^{4/3}} \text{for} \ i \neq 2 \]

(A19)

This result can also be used to obtain the leading-order terms governing the variations of \( \left\langle v_i^2 \right\rangle \)
\( (i \neq 2) \) in the limit \( \gamma L_\infty \to 0 \). Integrating twice and requiring that the surface value of \( \left\langle v_i^2 \right\rangle \)
matches with the kinetic energy found at \( \gamma = 0 \) in (20) implies

\[ \left\langle u^2 \right\rangle = \left\langle w^2 \right\rangle = \frac{3}{2} u^2 - 3.876 \varepsilon_{\infty} 2^{1/3} \gamma^{2/3} + O(\varepsilon_{\infty} 2^{1/3} L_\infty^{-1/3} \gamma) \]

(A20)

Expanding the products \( M_{ij}(k, \gamma) M_{ij}^*(k, \gamma) \) involved in the expression of \( \left\langle v_i^2 \right\rangle (\gamma) \) in the limit \( k_0 \gamma \to 0 \) reveals that low wavenumbers provide a nonzero linear contribution to the variance of the tangential fluctuations. The exact magnitude of this contribution depends on the shape of the energy spectrum in the low-wavenumber range; its sign is always positive, implying the existence of a minimum of the tangential r.m.s. velocities within the source layer.

Equations (A14), (A16) and (A20) can be used to deduce the evolution of the Taylor microscales within the source layer. Since \( \left\langle u^2 \right\rangle \) and \( \left\langle w^2 \right\rangle \) increase by a factor of 3/2 when \( \gamma \)
ranges from \( O(L_\infty) \) to \( O(L_\infty R_{\infty}^{-1/2}) \), the longitudinal and transverse microscales of the tangential fluctuations increase by a factor of \( (3/2)^{1/2} = 1.22 \). In contrast, since \( \left\langle v^2 \right\rangle \) decreases by a factor of \( R_{\infty}^{1/3} \) between \( \gamma = O(L_\infty) \) and \( \gamma = O(L_\infty R_{\infty}^{-1/2}) \), the transverse microscale associated
with the vertical velocity, $\lambda_{2T}$, decreases by a factor of $O(Re_{\infty}^{1/6})$. More precisely, using (A16) we find that for $\gamma/L_{\infty}=2Re_{\infty}^{-1/2}$, $\lambda_{2T}$ decreases by $(1.78)^{-1/2}Re_{\infty}^{1/6}$ compared to its free-stream value. These results compare well with the numerical evolutions of the Taylor microscales obtained by WLGR.