Appendix to “On the stability of a compressible axisymmetric rotating flow in a pipe”

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Appendix A. Derivation of perturbation equations (30) and (31)

Elimination of pressure from (19) and (20) by cross differentiation in terms of \( x \) and \( y \), respectively, followed by subtraction gives a relationship between \( \psi_1 \), \( K_{1x} \) and \( \rho_1 \):

\[
\frac{\psi_{1xx}}{2y} + \psi_{1yy} + w_0 \left( \frac{\psi_{1xx}}{2y} + \psi_{1yy} \right)_x - w_{0yy} \psi_{1x} + \frac{\omega K_0 \rho_0}{2y^2} K_{1x} + \gamma M_0^2 \left( \frac{\omega^2 K_0^2}{4y^2} \rho_{1x} - 2w_0w_{0y} \rho_{1x} - w_0^2 \rho_{1xy} - \int_0^x \rho_{1yy} dx' - 2w_0 \rho_{1t} - 2w_0 \rho_{1yt} \right) = 0. \tag{A-1}
\]

Solving (A-1) for \( K_{1x} \) and substituting in linearized \( \theta \)–momentum equation (21) results in:

\[
K_{1t} = \frac{\omega K_0 \rho_0}{\rho_0} \psi_{1x} + \frac{2y_0^2 w_0}{\omega K_0 \rho_0} \left[ \left( \frac{\psi_{1xx}}{2y} + \psi_{1yy} \right)_t + w_0 \left( \frac{\psi_{1xx}}{2y} + \psi_{1yy} \right)_x - w_{0yy} \psi_{1x} \right.
\]
\[
+ \gamma M_0^2 \left( \frac{\omega^2 K_0^2}{4y^2} \rho_{1x} - 2w_0w_{0y} \rho_{1x} - w_0^2 \rho_{1xy} - \int_0^x \rho_{1yy} dx' - 2w_0 \rho_{1t} - 2w_0 \rho_{1yt} \right) \right] \tag{A-2}
\]

Elimination of \( K_1 \) from (A-1) and (A-2) by cross differentiation in terms of \( t \) and \( x \), respectively, followed by subtraction, and multiplying by \( \omega K_0 \rho_0 / (2y^2 w_0) \), gives

\[
2 \left( \frac{\psi_{1xx}}{2y} + \psi_{1yy} \right)_{xt} + \frac{1}{w_0} \left( \frac{\psi_{1xx}}{2y} + \psi_{1yy} \right)_{tt} + w_0 \left( \frac{\psi_{1xx}}{2y} + \psi_{1yy} \right)_{xx}
\]
\[
+ \left( \frac{\omega^2 K_0^2}{2y^2 w_0} - w_{0yy} \right) \psi_{1xx} - \frac{w_{0yy}}{w_0} \psi_{1xt}
\]
\[
= -\gamma M_0^2 \left[ \frac{\omega^2 K_0^2}{4y^2} \left( \rho_{1xx} + \frac{\rho_{1xt}}{w_0} \right) - 4w_{0y} \rho_{1xt} - 3 \rho_{1yyt} - 3w_0 \rho_{1xyt} 
\]
\[
- \frac{1}{w_0} \int_0^x \rho_{1yyt} dx' - \frac{2w_0}{w_0} \rho_{1tt} - 2w_0w_{0y} \rho_{1xx} - w_0^2 \rho_{1xy} \right]. \tag{A-3}
\]

Differentiation of (A-3) with respect to \( x \) gives (30).

Differentiating (22) with respect to \( x \) gives

\[
\gamma M_0^2 (\rho_0 T_{1xt} + \rho_0 w_0 T_{1xx}) = \frac{\gamma - 1}{\gamma} \left[ \sqrt{2y_0 w_{0y}} u_{1x} + \gamma M_0^2 (P_{1xt} + w_0 P_{1xx}) \right] - \rho_0 \sqrt{2y_0 w_{0y}} u_{1x} \tag{A-4}
\]
From (16) we have \( \rho_0 T_1 = P_1 - \rho_1 T_0 \). Substituting this in (A-4) gives
\[
M_0^2 (P_{1xt} + w_0 P_{1xx}) - \gamma M_0^2 T_0 (\rho_{1xt} + w_0 \rho_{1xx}) = \left( \frac{\gamma - 1}{\gamma} \frac{P_{0y}}{\rho_0} - T_{0y} \right) \sqrt{2y \rho_0 u_{1x}}. \tag{A-5}
\]
Using (20) to express \( P_{1xt} \) and \( P_{1xx} \) in (A-5) and multiplying by \(-\frac{1}{w_0}\) gives
\[
\gamma M_0^2 \left[ \left( \frac{T_0}{w_0} - 3M_0^2 w_0 \right) \rho_{1xt} + (T_0 - M_0^2 w_0^2) \rho_{1xx} - 3M_0^2 \rho_{1tt} - \frac{M_0^2}{w_0} \int_0^x \rho_{1tt} dx' \right]
= \left( M_0^2 \frac{w_{0y}}{w_0} - \frac{T_{0y}}{w_0} + \frac{\gamma - 1}{\gamma} \frac{P_{0y}}{\rho_0 w_0} \right) \psi_{1xx} - M_0^2 w_0 \psi_{1xyy}
- \frac{M_0^2}{w_0} \psi_{1ytt} + M_0^2 \frac{w_{0y}}{w_0} \psi_{1xt} - 2M_0^2 \psi_{1yxt}. \tag{A-6}
\]
Differentiating (A-6) with respect to \( x \) gives (31).

**Appendix B. Boundary conditions for (43) and (44)**

The substitution of (42) into (32)-(41) gives boundary conditions for \( \tilde{\phi} \) and \( \tilde{\rho} \)
\[
\tilde{\phi}(x, 0) = 0, \quad \tilde{\phi}(x, 1/2) = \tilde{\phi}(0, 1/2) \tag{B-1}
\]
for \( 0 \leq x \leq x_0 \) and
\[
\frac{\tilde{\phi}_{xx}(0, y)}{2y} + \tilde{\phi}_{xy}(0, y) + \left( \frac{\omega^2 K_0 K_{0y}}{4y^2 w_0} - \frac{w_{0yy}}{w_0} \right) \tilde{\phi}_{x}(0, y) + \frac{\phi_{yy}(0, y)}{w_0} = 0, \tag{B-2}
\]
\[
\left( \frac{T_0}{w_0} \tilde{\phi}_{y}(0, y) \right)_y = \gamma M_0^2 \left( \frac{\sigma \tilde{\phi}_x(0, y)}{2y} + \omega^2 K_0^2 \tilde{\phi}_{y}(0, y) \right)
\]
with \( \tilde{\phi}(0, 0) = \tilde{\phi}_y(0, 0) = 0 \),
\[
\tilde{\phi}_{xxx}(0, y) + \tilde{\phi}_{xy}(0, y) + \left( \frac{\omega^2 K_0 K_{0y}}{4y^2 w_0} - \frac{w_{0yy}}{w_0} \right) \tilde{\phi}_{x}(0, y) + \frac{\phi_{yy}(0, y)}{w_0} = 0, \tag{B-4}
\]
\[
\left( \frac{\omega^2 K_0^2}{4y^2 w_0} - 2w_{0yy} \right) \tilde{\rho}_x(0, y) - w_0 \tilde{\rho}_xy(0, y) - 2\sigma \frac{w_{0yy}}{w_0} \tilde{\rho}_y(0, y) - 2\sigma \tilde{\rho}_y(0, y)
\]
\[
\frac{\tilde{\rho}_{xxx}(0, y)}{2y} + \frac{\tilde{\rho}_{xy}(0, y)}{w_0} + \frac{\tilde{\rho}_{yy}(0, y)}{2y} + \frac{\phi_{yy}(0, y)}{w_0} = 0, \tag{B-5}
\]
\[
\gamma \left[ \sigma \left( \frac{T_0}{w_0} - 3M_0^2 w_0 \right) \tilde{\rho}_x(0, y) + (T_0 - M_0^2 w_0^2) \tilde{\rho}_{xx}(0, y) - 3\sigma^2 M_0^2 \tilde{\rho}(0, y) \right]
= \sigma \frac{w_{0yy}}{w_0} \tilde{\rho}_x(0, y) - \sigma^2 \frac{1}{w_0} \tilde{\phi}_y(0, y) - 2\sigma \tilde{\phi}_{xy}(0, y), \tag{B-7}
\]
\[
\tilde{\phi}_x(x_0, y) = 0, \quad \sigma \tilde{\rho}(x_0, y) + w_0 \tilde{\rho}_x(x_0, y) = 0 \tag{B-8}
\]
for $0 \leq y \leq 1/2$.

**Appendix C. Analysis of imaginary parts of (43) and (44)**

Substituting (60) into (43) and (44), collecting terms of the orders $\epsilon_I$, $\sigma_I$, $\epsilon_I\sigma_R$, $\epsilon_I\Delta\Omega$ and neglecting terms of the orders $O(\sigma_R^2$, $\sigma_I^2$, $\sigma_R\sigma_I$, $\sigma_I\epsilon_R$, $\sigma_I\Delta\Omega)$ and higher gives

$$
\epsilon_I \left\{ \frac{\phi_{Ixx}}{2y} + \phi_{Iyy} + \left( \frac{\Omega_1 K_0 w_0}{2y^2 w_0^2} - \frac{w_{0y}}{w_0} \right) \phi_I + \gamma M_0^2 \left[ \left( \frac{\Omega_1 K_0^2}{4y^2 w_0^2} - 2w_{0y} \right) \rho_I - w_0 \rho_{Iy} \right] \right\} \quad \text{xxx}
$$

$$
+ \sigma_I \left\{ \frac{2}{w_0} \left( \frac{\psi_{Ixx}}{2y} + \psi_{Iyy} \right) - \frac{w_{0y}}{w_0} \psi_{Ic} + \gamma M_0^2 \left[ \left( \frac{\Omega_1 K_0^2}{4y^2 w_0^2} - 4 \frac{w_{0y}}{w_0} \right) \rho_{Ic} - 3 \rho_{Icy} \right] \right\} \quad \text{xx}
$$

$$
+ \epsilon_I \sigma_R \left\{ \frac{2}{w_0} \left( \frac{\phi_{Ixx}}{2y} + \phi_{Iyy} \right) - \frac{w_{0y}}{w_0} \phi_I + \gamma M_0^2 \left[ \left( \frac{\Omega_1 K_0^2}{4y^2 w_0^2} - 4 \frac{w_{0y}}{w_0} \right) \rho_I - 3 \rho_{Iy} \right] \right\} \quad \text{xx}
$$

$$
+ \epsilon_I \Delta\Omega \left\{ \frac{K_0 K_{0y}}{2y^2 w_0} \phi_I + \gamma M_0^2 \frac{K_0^2}{4y^2 w_0} \rho_I \right\} \quad = 0, \quad \text{(C-1)}
$$

$$
\epsilon_I \left\{ \frac{1}{\gamma M_0^2 (T_0 - M_0^2 w_0^2)} \left[ \left( M_0^2 w_{0y} - \frac{T_{0y}}{w_0} + \frac{(\gamma - 1) M_0^2 \Omega_1 K_0^2}{4y^2 w_0} \right) \phi_I - M_0^2 w_0 \phi_{Iy} \right] \right\} \quad \text{xxx}
$$

$$
+ \frac{\sigma_I}{T_0 - M_0^2 w_0^2} \left\{ \frac{T_0 - 3 M_0^2 w_0^2}{w_0} \rho_{Ic} - \frac{w_{0y}}{\gamma w_0} \psi_{Ic} + \frac{2}{\gamma} \psi_{Icy} \right\} \quad \text{xx}
$$

$$
+ \epsilon_I \sigma_R \left\{ \frac{T_0 - 3 M_0^2 w_0^2}{w_0} \rho_I - \frac{w_{0y}}{\gamma w_0} \phi_I + \frac{2}{\gamma} \phi_{Iy} \right\} \quad \text{xx}
$$

$$
+ \epsilon_I \Delta\Omega \left\{ \frac{\gamma - 1}{\gamma (T_0 - M_0^2 w_0^2)} \frac{K_0^2}{4y^2 w_0} \phi_I \right\} \quad = 0. \quad \text{(C-2)}
$$

From (B-1)-(B-8), the boundary conditions for these equations are:

$$
\phi_I(x, 0) = 0, \quad \phi_I(x, 1/2) = \phi_I(0, 1/2) \quad \text{(C-3)}
$$

for $0 \leq x \leq x_0$ and

$$
\phi_{Ixx}(y, 0) = 0, \quad \text{(C-4)}
$$

$$
\gamma M_0^2 w_0 \phi_I(0, y) = \phi_{Iy}(0, y), \quad \text{(C-5)}
$$

$$
\epsilon_I \left\{ \frac{T_0}{w_0} \phi_{Iy}(0, y) \right\} - \gamma M_0^2 \frac{\Omega_1 K_0^2}{4y^2 w_0} \phi_{Iy}(0, y) \right\} - \sigma_I \left\{ \gamma M_0^2 \frac{\psi_{Icy}(0, y)}{2y} \right\}
$$

$$
- \epsilon_I \sigma_R \left\{ \gamma M_0^2 \frac{\phi_{Ix}(0, y)}{2y} \right\} - \epsilon_I \Delta\Omega \left\{ \gamma M_0^2 \frac{K_0^2}{4y^2 w_0} \phi_{Iy}(0, y) \right\} = 0 \quad \text{with} \quad \phi_I(0, 0) = \phi_{Iy}(0, 0) = 0, \quad \text{(C-6)}
$$

$$
\epsilon_I \left\{ \frac{\phi_{Ixx}(0, y)}{2y} + \phi_{Ixy}(0, y) + \left( \frac{\Omega_1 K_0 w_0}{2y^2 w_0^2} - \frac{w_{0y}}{w_0} \right) \phi_{Ix}(0, y) \right\}
$$

$$
+ \gamma M_0^2 \left[ \left( \frac{\Omega_1 K_0^2}{4y^2 w_0^2} - 2w_{0y} \right) \rho_{Ix}(0, y) - w_0 \rho_{Ixy}(0, y) \right] \right\} \quad + \epsilon_I \sigma_R \left\{ \frac{\phi_{Iyy}(0, y)}{w_0} - 2 \gamma M_0^2 \frac{w_{0y}}{w_0} \rho_I(0, y) + \rho_{Iy}(0, y) \right\}
$$

$$
= 0.
$$
\[ + \epsilon_I \Delta \Omega \left\{ \frac{K_0 K_{0y}}{2y^2 w_0^2} \phi_{Ix}(0, y) + \gamma M_0^2 \frac{K_0^2}{4y^2 w_0} \rho_{Ix}(0, y) \right\} = 0, \quad \text{(C-7)} \]

\[ \epsilon_I \left\{ \frac{\phi_{Ixxx}(0, y)}{2y} + \gamma M_0^2 \left[ \left( \frac{\Omega_1 K_0^2}{4y^2 w_0} - 2w_0 \right) \rho_{Ixx}(0, y) - w_0 \rho_{Ixy}(0, y) \right] \right\} \]

\[ + \sigma_I \left\{ \frac{\psi_{1xxx}(0, y)}{y w_0} + \frac{2\psi_{1xyy}(0, y)}{w_0} - \frac{w_{yy} \psi_{1x}}{w_0^2} \right\} \]

\[ + \gamma M_0^2 \left[ \left( \frac{\Omega_1 K_0^2}{4y^2 w_0} - 4 \frac{w_0}{w_0} \right) \rho_{Icx}(0, y) - 3 \rho_{Icy}(0, y) \right] \]

\[ + \epsilon_I \sigma_R \left\{ \frac{\phi_{Ixy}(0, y)}{y w_0} + \frac{2\phi_{Iyy}(0, y)}{w_0} - \frac{w_{yy} \phi_{Ix}}{w_0^2} \right\} \]

\[ + \gamma M_0^2 \left[ \left( \frac{\Omega_1 K_0^2}{4y^2 w_0} - 4 \frac{w_0}{w_0} \right) \rho_{Ix}(0, y) - 3 \rho_{Ixy}(0, y) \right] \]

\[ + \epsilon_I \Delta \Omega \left\{ \gamma M_0^2 \frac{K_0^2}{4y^2 w_0} \rho_{Ixx}(0, y) \right\} = 0, \quad \text{(C-8)} \]

\[ \epsilon_I \rho_{Ixx}(0, y) + \frac{\sigma_I}{T_0 - M_0^2 w_0^2} \left\{ T_0 - 3M_0^2 w_0^2 \rho_{Icx}(0, y) - w_0 \psi_{Icx}(0, y) + \frac{2}{\gamma} \psi_{Icy}(0, y) \right\} \]

\[ + \frac{\epsilon_I \sigma_R}{T_0 - M_0^2 w_0^2} \left\{ T_0 - 3M_0^2 w_0^2 \rho_{Ixx}(0, y) - w_0 \psi_{Ixy}(0, y) + \frac{2}{\gamma} \psi_{Ixy}(0, y) \right\} = 0, \quad \text{(C-9)} \]

\[ \phi_{Ix}(x, y) = 0, \quad \sigma_I \rho_{Ic}(x, y) + w_0 \epsilon_I \rho_{Ie}(x, y) + \epsilon_I \sigma_R \rho_{Ie}(x, y) = 0 \quad \text{(C-10)} \]

for \( 0 \leq y \leq 1/2 \).

Two integrations with respect to \( x \) of (C-1) and (C-2) and the use of boundary conditions (C-4) and (C-7)-(C-10) result in

\[ \epsilon_I \left\{ \phi_{Ix} - \frac{1}{\gamma M_0^2 (T_0 - M_0^2 w_0^2)} \left[ \left( M_0^2 w_0 - \frac{T_{0y}}{w_0} \right) \frac{\Omega_1 K_0^2}{4y^2 w_0} \right] \phi_{I} - M_0^2 \frac{K_0^2}{4y^2 w_0} \phi_{Iy} \right\} \]

\[ + \frac{\sigma_I}{T_0 - M_0^2 w_0^2} \left\{ T_0 - 3M_0^2 w_0^2 \phi_{Ic} - w_0 \psi_{Ic} + \frac{2}{\gamma} \psi_{Icy} \right\} \]

\[ + \frac{\epsilon_I \sigma_R}{T_0 - M_0^2 w_0^2} \left\{ T_0 - 3M_0^2 w_0^2 \rho_{I} - w_0 \psi_{I} + \frac{2}{\gamma} \psi_{Iy} \right\} \]

\[ + \epsilon_I \Delta \Omega \left\{ \frac{\gamma - 1}{\gamma (T_0 - M_0^2 w_0^2)} \frac{K_0^2}{4y^2 w_0} \phi_{I} \right\} = \sigma_I f_2(y) + \epsilon_I \sigma_R f_3(y), \quad \text{(C-11)} \]
Here
\[ f_1(y) = \frac{\phi_{y y}(0, y)}{w_0} - \frac{w_{y y}}{w_0^2} \phi_I(0, y) + \gamma M_0^2 \left( \frac{\Omega_1 K_0^2}{4 y^2 w_0^2} - \frac{w_{y y}}{w_0} \right) \rho_I(0, y) - \rho_{1 y}(0, y), \]
\[ f_2(y) = -\frac{1}{T_0 - M_0^2 w_0^2} \left( 2 M_0^2 w_0 S(y) + \frac{w_{y y}}{\gamma w_0} \Phi(y) - \frac{2}{\gamma} \Phi_y(y) \right), \]
\[ f_3(y) = -\frac{1}{T_0 - M_0^2 w_0^2} \left( 2 M_0^2 w_0 \phi_I(x_0, y) + \frac{w_{y y}}{\gamma w_0} \phi_I(x_0, y) - \frac{2}{\gamma} \phi_{1 y}(x_0, y) \right). \]

Solution of (C-12) for \( \epsilon_I \rho_{l z} \), substitution of the result in (C-11), multiplication by \( (T_0 - M_0^2 w_0^2)/T_0 \) and an additional integration with respect to \( x \) gives (61).

**Appendix D. Analysis of real parts of (43) and (44)**

Substituting expansions (60) into (43) and (44) and neglecting terms of the orders \( O(\sigma_R^2, \sigma_I^2, \sigma_R \epsilon_R, \sigma_I \epsilon_I, \sigma_R \Delta \Omega) \) and higher gives
\[
\epsilon_R \left\{ \frac{\phi_{R x x}}{2y} + \phi_{R y y} + \left( \frac{\Omega_1 K_0 K_{0 y}}{2 y^2 w_0^2} - \frac{w_{y y}}{w_0} \right) \Phi(y) + \gamma M_0^2 \left[ \left( \frac{\Omega_1 K_0^2}{4 y^2 w_0^2} - \frac{w_{y y}}{w_0} \right) \rho_R - \rho_{0 R} \right] \right\} = 0, \quad (D-1)
\]
\[
\epsilon_R \left\{ \rho_R - \frac{1}{\gamma M_0^2 (T_0 - M_0^2 w_0^2)} \left( M_0^2 w_{0 y} - \frac{T_{0 y}}{w_0} + (\gamma - 1) \frac{M_0^2 \Omega_1 K_0^2}{4 y^2 w_0^2} \Phi(y) - \rho_R \right) \right\} = 0, \quad (D-2)
\]
\[ \phi_R(x, 0) = 0, \quad \phi_R(x, 1/2) = \phi_R(0, 1/2) \quad (D-3) \]

for \( 0 \leq x \leq x_0 \) and
\[ \phi_{R x x}(0, y) = 0, \quad \gamma M_0^2 w_0 \rho_R(0, y) = \phi_{R y y}(0, y), \]
\[ \epsilon_R \left\{ \left( \frac{T_0}{w_0} \phi_{R y y}(0, y) \right) - \gamma M_0^2 \frac{\Omega_1 K_0^2}{4 y^2 w_0^2} \phi_{R y y}(0, y) \right\} = 0 \]
with \( \phi_R(0, 0) = \phi_{R y y}(0, 0) = 0, \quad (D-6) \]
\[ \epsilon_R \left\{ \frac{\phi_{R x x x}(0, y)}{2y} + \phi_{R y y y}(0, y) + \left( \frac{\Omega_1 K_0 K_{0 y}}{2 y^2 w_0^2} - \frac{w_{y y}}{w_0} \right) \Phi(y) + \gamma M_0^2 \left[ \left( \frac{\Omega_1 K_0^2}{4 y^2 w_0^2} - \frac{w_{y y}}{w_0} \right) \rho_{R x x}(0, y) - \rho_{0 R x y}(0, y) \right] \right\} = 0. \]
\[ \epsilon_R \left( \frac{\phi_{Rxx}(0, y)}{2y} + \frac{(1 - K_0K_{0y})}{2y^2w_0^2} \phi_{Rx} + \gamma M_0^2 \left( \frac{\Omega_1K_0^2}{4y^2w_0^2} - 2w_{0y} \right) \phi_R + \gamma M_0^2 \left( \frac{\Omega_1K_0^2}{4y^2w_0^2} - 2w_{0y} \right) \phi_R - w_{0y} \psi_{1cx}(0, y) + \frac{2}{\gamma} \psi_{1cy}(0, y) \right) = 0, \]  
\[ (D-7) \]

\[ \sigma_R \left( \frac{\psi_{1xx}(0, y)}{y w_0} + \frac{2 \psi_{1xy}(0, y)}{w_0} - \frac{w_{0y}}{w_0^2} \psi_{1cx}(0, y) \right) + \gamma M_0^2 \left( \frac{\Omega_1K_0^2}{4y^2w_0^2} - 4w_{0y} \right) \psi_{1cx}(0, y) - 3 \rho_{1cy}(0, y) \right) = 0, \]  
\[ (D-8) \]

\[ \epsilon_R \rho_{Rxx}(0, y) + \sigma_R \left( \frac{T_0 - 3M_0^2w_0^2}{w_0} \right) \rho_{Rxx}(0, y) - \frac{w_{0y}}{w_0} \psi_{1cx}(0, y) + \frac{2}{\gamma} \psi_{1cy}(0, y) \right) = 0, \]  
\[ (D-9) \]

\[ \phi_{Rx}(x, y) = 0, \quad \sigma_{R1c}(x, y) + \epsilon_R w_0 \rho_{Rx}(x, y) = 0 \]  
\[ (D-10) \]

for \( 0 \leq y \leq 1/2 \).

Two integrations of (D-1) and (D-2) with respect to \( x \) and the use of boundary conditions (D-4), (D-7)-(D-10) result in

\[ \epsilon_R \left( \frac{\phi_{Rxx}}{2y} + \phi_{Ryy} + \frac{(1 - K_0K_{0y})}{2y^2w_0^2} \phi_{Rx} + \gamma M_0^2 \left( \frac{\Omega_1K_0^2}{4y^2w_0^2} - 2w_{0y} \right) \phi_R - w_{0y} \psi_{1cx}(0, y) + \frac{2}{\gamma} \psi_{1cy}(0, y) \right) = 0, \]  
\[ (D-11) \]

\[ \epsilon_R \left( \rho_R - \frac{1}{\gamma M_0^2(T_0 - M_0^2w_0^2)} \left( \frac{M_0^2w_{0y}}{w_0} - \frac{T_{0y}}{w_0} + (\gamma - 1) M_0^2 \frac{\Omega_1K_0^2}{4y^2w_0^2} \phi_R - M_0^2w_0 \phi_{Ry} \right) \right) = 0, \]  
\[ (D-12) \]

Here \( f_2(y) \) is defined in (C-13). Also, the conditions in (D-5) and (D-6) can be solved and show that

\[ \epsilon_R \phi_{R}(0, y) = \sigma_R \gamma M_0^2 \frac{\pi}{4x_0} \int_0^y \exp(\alpha(y')) \left[ \int_0^{y'} g(y'') \exp(-\alpha(y'')) dy'' \right] dy', \]

\[ \gamma M_0^2 w_0 \rho_R(0, y) = \sigma_R \gamma M_0^2 \frac{\pi}{4x_0} \exp(\alpha(y)) \left[ \int_0^{y'} g(y'') \exp(-\alpha(y'')) dy'' \right] \]  
\[ (D-13) \]

where

\[ \alpha(y) = - \int_0^y p(y') dy', \quad p(y) = \frac{w_0}{T_0} \left( \frac{T_{0y}}{w_0} \right)_y - \gamma M_0^2 \Omega_1 \frac{K_0^2}{4y^2T_0}, \quad g(y) = \frac{\Phi(y) w_0(y)}{y} / T_0(y). \]

Note that \( \phi_R(0, y) = 0 \) when \( M_0 = 0 \). Also, note that in the general case \( \phi_R(0, 1/2) \) is now determined and may not be zero. For example, in the case of a solid-body rotation
profile where \( K_0 = 2y \) and \( w_0 = T_0 = 1 \) we find \( p(y) = -\gamma M_0^2 \Omega_1 \), \( \alpha(y) = \gamma M_0^2 \Omega_1 y \), and then

\[
\phi_R(0, y) = \gamma M_0^2 \frac{\pi}{4x_0} \int_0^y \exp(\gamma M_0^2 \Omega_1 y) \left[ \int_0^{y'} \frac{\Phi(y)}{y} \exp(-\gamma M_0^2 \Omega_1 y'') dy'' \right] dy'.
\] (D-14)

Examples of calculating \( \phi_R(0, y) \) according to (D-14) are shown in Fig. D-1 for various Mach numbers. It is clear that \( \phi_R(0, 1/2) \) is not zero.

Solving (D-12) for \( \epsilon_{RPR_x} \) and substituting in (D-11), multiplying by \((T_0 - M_0^2 w_0^2)/T_0\), and integrating again with respect to \( x \) gives (63).
Figure 1: D-1: Solutions of $\phi_R(0,y)$ for a solid-body rotation flow at various Mach numbers.