
A more detailed derivation than that in [NV, §§3.1-3.4] is given below. In order to make the derivation self-contained, some equations in [NV, §§2-3] are first listed:

\[ u_r + r^{-1}u + w_z = 0, \]  
\[ u_t + w(u_r - w_r) = -q_r + C(u_{rr} + r^{-1}u_r - r^{-2}u + u_{zz}), \]  
\[ w_t + u(w_r - u_z) = -q_z + C(w_{rr} + r^{-1}w_r + u_{zz}), \]  
\[ u = 0, \quad w = h'_\pm(t) \quad \text{at} \quad z = \pm\Lambda + h_\pm(t), \]  
\[ u = w_r = q_r = 0 \quad \text{at} \quad r = 0, \]  
\[ u = f_1 + f_2w \quad \text{at} \quad r = f, \]  
\[ (w_r + u_z)(1 - f_2^2) + 2(u_r - w_z)f_2 = 0 \]  
\[ q - \frac{w^2 + u^2}{2} + \frac{ff_1 - f_1 f_2^2}{f(1 + f_2^2)^{3/2}} = 2C(u_r - w_z)f_2 + w_z f_2^2 \quad \text{at} \quad r = f \]  
\[ f = 1 \quad \text{at} \quad z = \pm\Lambda + h_\pm(t). \]  
\[ \int_{-\Lambda + h_-}^{\Lambda + h_+} f(z,t)dz = 2\Lambda. \]  
\[ h_\pm(t) = \beta_{\pm\mu} \exp[i(\Omega + \omega_\pm\delta)t] + \text{c.c.}, \]  
\[ C \ll 1, \quad \mu \ll 1 \quad \text{and} \quad \delta \ll 1, \]  
\[ u = \varepsilon(\mathbf{A}U_0 e^{i\Omega t} + \text{c.c.}) + \varepsilon\sqrt{C}u_1 + \varepsilon^2u_2 + \varepsilon^3u_3 + \varepsilon^4u_4 + \mu u_0 + \text{HOT} \]  
\[ w = \cdots, \quad q - 1 = \cdots, \quad f - 1 = \cdots, \quad 0 < \varepsilon \ll 1. \]  
\[ \int_0^{2\pi/\Omega} \int_{-\Lambda}^{\Lambda} \int_0^1 (U_0 u_k + W_0 w_k) e^{-\ln r} rdrdt = 0 \quad \text{for all} \quad k \geq 1, \]  
\[ \varepsilon \delta dA/dt = -\left[(1 + i)\alpha_1 \sqrt{C} + \alpha_2 C\right] \varepsilon A + i\alpha_3 \varepsilon^3 A|A|^2 + i\mu \right( \alpha_4^+ \beta_4^+ e^{i\omega t} - \alpha_3^- \beta_- e^{i\omega t} \right) + \text{HOT}, \]  
\[ \delta \varepsilon dA/dr = H_1 \varepsilon \sqrt{C} + H_2 \varepsilon^2 + H_3 \varepsilon C + H_4 \varepsilon^3 \sqrt{C} + H_5 \varepsilon^3 + H_6 \mu + \text{HOT}, \]  
\[ \tau = \delta t. \]

1 The solution in the bulk

We shall only need to calculate the first three terms in the expansions (13). The (linear) inviscid eigenmode \((U_0, W_0, Q_0, F_0)\) is given by

\[ U_0 r + r^{-1}U_0 + W_0 z = i\Omega U_0 + Q_0 r = i\Omega W_0 + Q_0 z = 0, \]
\[ W_0 = 0 \text{ at } z = \pm \Lambda, \quad U_0 = W_{0r} = 0 \text{ at } r = 0, \] (18)

\[ U_0 - i\Omega F_0 = Q_0 + F_0 + F_0' = 0 \text{ at } r = 1, \] (19)

\[ F_0(\pm \Lambda) = \int_{-\Lambda}^{\Lambda} F_0(z) \, dz = 0. \] (20)

The terms of orders $\varepsilon \sqrt{C}$ and $\varepsilon^2$ in (13) are given by

\[ u_{kr} + r^{-1}u_k + w_{kr} = 0, \] (21)

\[ u_k + q_{kr} + (H_k U_0 e^{i\Omega t} + \text{c.c.}) = w_k + q_k + (H_k W_0 e^{i\Omega t} + \text{c.c.}) = 0, \] (22)

\[ w_k = G_k^+ \text{ at } z = \pm \Lambda, \quad u_k = w_{kr} = q_{kr} = 0 \text{ at } r = 0, \] (23)

\[ u_k = \phi_k, \quad q_k = \psi_k \text{ at } r = 1, \] (24)

\[ f_k = 0 \text{ at } z = \pm \Lambda, \quad \int_{-\Lambda}^{\Lambda} f_k \, dz + \gamma_k = 0 \] (25)

for $k = 1$ and 2, where

\[ \gamma_1 = 0, \quad \gamma_2 = \int_{-\Lambda}^{\Lambda} (A^2 P_0^2 e^{i\Omega t} + \text{c.c.} + 2|A| F_0|^2) \, dz / 2. \] (26)

Eqs. (21), (22) and (25) and the boundary conditions at $r = 0$ are obtained upon substitution of (13) and (15b) into (1)-(3), (5) and (10), when taking into account that $U_{0z} \equiv W_{0r}$. The remaining boundary conditions (and the functions $G_k^\pm, \phi_k$ and $\psi_k$) will be obtained below by applying matching conditions with the Stokes and the interface boundary layers.

2 The solution in the Stokes boundary layers

For the sake of brevity we give details only for the boundary layer near $z = \Lambda$, where we use the stretched coordinate

\[ \xi = [z - \Lambda - h_+ (t)] / \sqrt{C}, \]

with $h_+$ as given in (11). We seek the expansions

\[ u = \varepsilon A \tilde{U}_0 e^{i\Omega t} + \text{c.c.} + \varepsilon \sqrt{C} \tilde{u}_1 + \varepsilon^2 \tilde{u}_2 + \text{HOT}, \]

\[ w = h'_+ + \sqrt{C} \left( \varepsilon A \tilde{W}_0 e^{i\Omega t} + \text{c.c.} + \text{HOT} \right) \]

\[ q - 1 = \varepsilon A \tilde{Q}_0 e^{i\Omega t} + \text{c.c.} + \varepsilon \sqrt{C} \tilde{q}_1 + \varepsilon^2 \tilde{q}_2 + \text{HOT} + O(\mu). \] (27)

Substitution of (27) and (15b) into (1)-(4) yields

\[ \tilde{Q}_{0\xi} + i\Omega \tilde{U}_0 - \tilde{U}_{0\xi \xi} + \tilde{Q}_{0r} = \tilde{U}_{0r} + r^{-1} \tilde{U}_0 + \tilde{W}_{0\xi} \]

\[ = \tilde{u}_{1\xi} + \tilde{u}_{1\xi \xi} + \tilde{q}_{1r} + \left( H_1 \tilde{U}_0 e^{i\Omega t} + \text{c.c.} \right) = \tilde{q}_{1\xi} = 0, \] (28)

\[ \tilde{q}_{0\xi} - \left( A \tilde{U}_0 e^{i\Omega t} + \text{c.c.} \right) \left( A \tilde{U}_{0\xi} e^{i\Omega t} + \text{c.c.} \right) = 0, \] (29)

\[ \tilde{u}_{2\xi} + \tilde{u}_{2\xi \xi} + \tilde{q}_{2r} + \left( H_2 \tilde{U}_0 e^{i\Omega t} + \text{c.c.} \right) + \left( A \tilde{W}_0 e^{i\Omega t} + \text{c.c.} \right) \left( A \tilde{U}_{0\xi} e^{i\Omega t} + \text{c.c.} \right) = 0, \] (30)

\[ \tilde{U}_0 = \tilde{W}_0 = \tilde{u}_1 = \tilde{u}_2 = 0 \quad \text{at} \quad \xi = 0, \] (31)

and integration of (28)-(31) leads to

\[ \tilde{U}_0 = K_0^+ (r) (1 - \Gamma), \quad \tilde{W}_0 = - (dK_0^+ / dr + r^{-1} K_0^+) \left[ \xi + (1 - i)(1 - \Gamma) / \sqrt{2\Omega} \right], \] (32)

\[ \tilde{u}_1 = \left[ K_0^+ (1 - \Gamma) - (1 - i) H_1 K_0^+ \xi / 2\sqrt{2\Omega} \right] A e^{i\Omega t} + \text{c.c.}, \] (33)
\[ \tilde{u}_2 = \tilde{U}_{22}(r, \xi, \lambda) e^{2i\Omega r} + \left[ K_2^+(1 - \Gamma) - (1 - i)H_2 K_0^+ \xi \Gamma / 2\sqrt{2\Omega} \right] e^{i\Omega \xi} \]
\[ + |A|^2 \tilde{K}_0^+(dK_0^+/dr + r^{-1} K_0^+) \left[ i \left[ |\Gamma|^2 - 1 \right] + 2(1 - 2i)(\Gamma - 1) + (1 + i)\sqrt{2\Omega} \xi \Gamma \right] / 2\Omega \]
\[ + |A|^2 \tilde{K}_0^+(dK_0^+/dr) \left[ |\Gamma|^2 - 1 + 2i(\Gamma - 1) \right] / 2\Omega + \text{c.c.} \]

(34)

where overbars and c.c. stand for the complex conjugate, \( K_s^+ = K_s^+(r) \) is an arbitrary function for \( s = 0, 1 \) and 2, that is to be calculated, the function \( \Gamma = \Gamma(\xi) \) is given by

\[ \Gamma(\xi) = \exp \left[ (1 + i)\sqrt{\Gamma / 2\xi} \right], \]

(35)

and the function \( \tilde{U}_{22} \) is not calculated (because it will be not needed in the sequel).

Now, the functions \( K_0^+, K_1^+ \) and \( K_2^+ \), and the functions \( G_1^+ \) and \( G_2^+ \) appearing in the boundary conditions (23) are obtained by applying matching conditions between the solutions in the bulk (13) and in this boundary layer (27). After applying a similar procedure to the boundary layer near \( z = -\Lambda \) we obtain

\[ U_0(r, \pm \Lambda) = K_0^+(r), \quad u_1(r, \pm \Lambda) = AK_1^+(r) e^{i\Omega \xi} + \text{c.c.}, \]

(36)

\[ G_1^+ = \pm \left[ (1 - i)AV_0 \xi e^{i\Omega \xi} + \text{c.c.} \right] / \sqrt{2\Omega}, \quad G_2^+ = 0, \]

(37)

\[ u_2 = K_2^+(r) e^{i\Omega \xi} + K_2^-(r) e^{2i\Omega r} + \text{c.c.} \]
\[ - |A|^2 \left[ 3(1 - i)\tilde{U}_0 u_{0r} + \text{c.c.} + 4r^{-1}|U_0|^2 \right] / 2\Omega, \quad \text{at } z = \pm \Lambda, \]

(38)

where the first continuity equation (17) has been taken into account and \( u_2 \) has been also obtained for convenience. Again the functions \( K^\pm \) will not appear in the sequel and are not considered here.

3 The solution in the interface boundary layer

Here we use the stretched coordinate

\[ \eta = [r - f(z, t, r)]/\sqrt{C}, \]

and seek the expansions

\[ u = f_1 + \delta f_1 + f_2 w + \sqrt{C} \left( \varepsilon AW_0 e^{i\Omega \xi} + \text{c.c.} + \varepsilon \sqrt{C} u_1^{*} + \text{HOT} \right), \]
\[ w = \varepsilon AW_0 e^{i\Omega \xi} + \text{c.c.} + \varepsilon \sqrt{C} w_1^{*} + \varepsilon^{2} w_2^{*} + \varepsilon^{2} \sqrt{C} w_3^{*} + \text{HOT}, \]
\[ q - 1 = (u^2 + w^2)/2 + \left( \varepsilon AW_0 e^{i\Omega \xi} + \text{c.c.} \right) + \varepsilon \sqrt{C} p_1^{*} + \varepsilon^{2} p_2^{*} + \varepsilon^{2} \sqrt{C} p_3^{*} + \text{HOT}. \]

(39)

Substitution of (39) and (15b) into (1)-(3) and (6)-(8) yields the following equations

\[ P_{m}^{*} = i\Omega W_{m}^{*} - W_{m}^{*} + P_{m}^{*} = U_{m}^{*} + W_{m}^{*} + i\Omega F_{0} = 0, \]

(40)

\[ p_{m}^{*} = \Omega^{2} \left( (A F_{0}) e^{i\Omega \xi} + \text{c.c.} \right), \]

(41)

\[ w_{11}^{*} - w_{11}^{*} + \left[ (H_1 W_0^{*} - A W_0^{*}) e^{i\Omega \xi} + \text{c.c.} \right] = 0, \]

(42)

\[ u_{11}^{*} + w_{12}^{*} + f_{12}^{*} \left[ (H_1 F_0 - i\Omega F_0 + A U_0^{*}) e^{i\Omega \xi} + \text{c.c.} \right] = p_{11}^{*} = 0, \]

(43)

\[ w_{21}^{*} - w_{22}^{*} + P_{22}^{*} + \left[ (H_2 W_0^{*} - A W_0^{*}) e^{i\Omega \xi} + \text{c.c.} \right] + (A U_0^{*} e^{i\Omega \xi} + \text{c.c.}) (A F_{0} e^{i\Omega \xi} + \text{c.c.}) - \Omega^{2} \left( (A F_{0}) e^{i\Omega \xi} + \text{c.c.} \right) (A F_{0} e^{i\Omega \xi} + \text{c.c.}) = 0, \]

(44)

\[ p_{m}^{*} + (2i\Omega H_2 F_0 e^{i\Omega \xi} + \text{c.c.}) + f_{21}^{*} + 2(2W_0^{*} e^{i\Omega \xi} + \text{c.c.})(i\Omega A F_{0} e^{i\Omega \xi} + \text{c.c.}) + (A F_{0} e^{i\Omega \xi} + \text{c.c.}) [A (i\Omega W_0^{*} - W_0^{*}) e^{i\Omega \xi} + \text{c.c.}] = 0, \]

(45)
and the following boundary conditions at $\eta = 0$

\begin{align*}
    p_0^* + F_0'' + F_0 &= W_{0n}^* = U_0^* = 0 \\
p_1^* + f_1 + f_{1zz} &= w_{1n}^* + (i\Omega AF_0^e e^{i\eta t} + \text{c.c.}) = u_1^* = 0 \\
p_2^* + p_3 + f_{2zz} &= (AF_0^e e^{i\eta t} + \text{c.c.})^2 + (AF_0^e e^{i\eta t} + \text{c.c.})^2/2 = w_{2n}^* = 0, \\
p_3^* + f_3 + f_{3zz} &= 2f_1(AF_0^e e^{i\eta t} + \text{c.c.}) - f_{1zz}(AF_0^e e^{i\eta t} + \text{c.c.}), \\
w_{3n}^* + f_{3zz} + (H_2 F_0^0 e^{i\eta t} + \text{c.c.}) + (AF_0^e e^{i\eta t} + \text{c.c.})(AF_0^e e^{i\eta t} + \text{c.c.}) \\
    + (AF_0^e e^{i\eta t} + \text{c.c.}) [A(U_{0n}^* - W_{0n}^*) e^{i\eta t} + \text{c.c.}] &= 0.
\end{align*}

Integration of (40)-(51) yields

\begin{align*}
P_0^* &= -i(F_0 + F_0'')/\Omega, \quad W_0^* = -i(F_0^0 + F_0')/\Omega, \quad U_0^* = i(F_0^0 + F_0') - \Omega^2 F_0/\eta/\Omega, \\
p_1^* &= -f_1 - f_{1zz} + \Omega^2 (AF_0^e e^{i\eta t} + \text{c.c.}) \eta, \\
u_1^* &= -\sqrt{2}X [(1 + i)AF_0^0 \Gamma(\eta)e^{i\eta t} + \text{c.c.}] + (i\Omega AF_0^e e^{i\eta t} + \text{c.c.}) \eta + \text{POL} \\
u_1^* &= 2[AF_0^0 \Gamma(\eta)e^{i\eta t} + \text{c.c.}] + \text{POL} \\
p_2^* &= -f_2 - f_{2zz} + (AF_0^e e^{i\eta t} + \text{c.c.})^2 - (AF_0^e e^{i\eta t} + \text{c.c.})^2/2, \\
w_{3n}^* &= \eta \text{POL}, \quad p_3^* = \text{POL} \\
w_{3n}^* &= i|A|^2 [3(2F_0'' + 2F_0' - \Omega^2 F_0)F_0' + (\tilde{F}_0'' + \tilde{F}_0')F_0'']/\Omega \\
    - 2i|A|^2 [(2F_0'' + 2F_0' - \Omega^2 F_0)F_0' + (\tilde{F}_0'' + \tilde{F}_0')F_0''] \Gamma(\eta)/\Omega \\
    - (1 - i)\sqrt{2}/\Omega |A|^2 (\tilde{F}_0'' + \tilde{F}_0') - \Omega^2 \tilde{F}_0 F_0' \eta \Gamma(\eta) + \text{c.c.} + \eta \text{POL} + \text{OT},
\end{align*}

where the function $\Gamma$ is as defined in (35), POL stands for a polynomial in the $\eta$ variable (whose coefficients may depend on the remaining variables) and OT stands for oscillatory terms in the short time variable, of the type $\text{OT} = W_{3n}(\eta, z, \tau)e^{i\eta t} + W_{3n}(\eta, z, \tau)e^{i\eta t} + \text{c.c.}$. In fact, when taking into account the actual expressions for POL and some results below, it may be seen (after a careful, involved analysis that is omitted for the sake of brevity) that POL identically vanishes in the expressions giving $w_{3n}^*$ and $\tilde{w}_{3n}$; but this assertion is not essential to obtain the results in this Section.

Now, in order to apply matching conditions between the solution in the bulk (13) and the solution in this boundary layer (39), we take into account that the solution in the bulk satisfies

\begin{align*}
    q(f, z; t, \tau) &= q(1, z; t, \tau) + (f - 1)q_e(1, z; t, \tau) + O(e^3) \\
u(f, z; t, \tau) &= \ldots, \quad w_e(f, z; t, \tau) = \ldots,
\end{align*}

where

\begin{align*}
    \phi_1 &= u_1 = f_{11} + (H_1 F_0 e^{i\eta t} + \text{c.c.}), \quad \psi_1 = q_1 = f_1 - f_{1zz}, \\
    \phi_2 &= u_2 = f_{2z} + (H_2 F_0 e^{i\eta t} + \text{c.c.}) - (AF_0^e e^{i\eta t} + \text{c.c.})(AU_{0e} e^{i\eta t} + \text{c.c.})
\end{align*}

with $e = 1, 2$.
\[ \psi_2 \equiv q_2 = -f_2 - f_{2zz} - (AF_0e^{i\Omega t} + c.c.) (AQ_{0x}e^{i\Omega t} + c.c.) + (AF_0e^{i\Omega t} + c.c.)^2 
- (AF_0e^{i\Omega t} + c.c.)^2 f_{2zz} + (AQ_{0x}e^{i\Omega t} + c.c.)^2 f_{2zz} + (i\Omega AQ_{0x}e^{i\Omega t} + c.c.)^2 f_{2zz}/2, \]
\[ w_{2r} = |A|^2 \left[ 3(2\tilde{F}_0 + 2\tilde{F}_0 - \Omega^2 \tilde{F}_0) F_0 + (\tilde{F}_0 + \tilde{F}_0) F_0 \right]/\Omega - |A|^2 \tilde{W}_{0rr} F_0 + c.c. + \text{OT}, \]

where \( w_{2r} \) has been also obtained for convenience and \( \text{OT} \) stands for oscillatory terms in the short time scale, of the type \( k_1(z) e^{i\Omega t} + k_2(z) e^{-i\Omega t} + c.c. \), where the functions \( k_1 \) and \( k_2 \) need not being calculated.

### 4 Solvability conditions

Here we shall compute the coefficients in the amplitude equation (15b) by eliminating secular terms in the short time scale, \( t \sim 1 \). To this end, we first obtain an integral solvability condition as follows. First, introduce into (1)-(8) the time scales \( t \sim 1 \) \( \tau = \delta t \) by replacing in (2)-(4) and (6) the time derivative by \( \partial/\partial r + \partial/\partial \tau \). Then multiply (2) by \( ru_0e^{-i\Omega t} \), (3) by \( rw_0e^{-i\Omega t} \), the second and third equations in (17) by \( -ru_0e^{-i\Omega t} \) and \( -rw_0e^{-i\Omega t} \) respectively, add, integrate in \( 0 < r < f_0 \), \( -\Lambda - h_0 < z < \Lambda + h_0 \), integrate by parts and take into account the boundary conditions (4) and (6)-(9) to obtain

\[ \frac{\partial}{\partial t} (e^{-i\Omega t} I_1) + e^{-i\Omega t} I_2 = e^{-i\Omega t} (I_3 + I_4 + I_5^+ - I_5^-), \]  

where

\[ I_1 = \int_{\Lambda + h_0}^{\Lambda + h_+} \int_0^{t} \left( u_0 + w_0 \right) r dr dz - \int_{-\Lambda + h_-}^{\Lambda + h_+} Q_0(f_0, f) f(f - 1) dz, \]
\[ I_2 = \int_{\Lambda + h_0}^{\Lambda + h_+} \int_0^{t} \delta(u_0 + w_0) r dr dz - \int_{-\Lambda + h_-}^{\Lambda + h_+} \delta f f Q_0(f_0, f) dz, \]
\[ I_3 = \int_{\Lambda + h_0}^{\Lambda + h_+} \int_0^{t} (u_0 - w_0)(u_0 - w_r) r dr dz, \]
\[ I_4 = \int_{\Lambda + h_0}^{\Lambda + h_+} \int_0^{t} \left[ (u_0 + w_0)f - f(f - 1) \right] Q_0(f_0, f - 1) dz, \]
\[ I_5^+ = \int_{\Lambda + h_0}^{\Lambda + h_+} \int_0^{t} \left[ \frac{\partial h^+}{\partial t} W_0 + Q_0 \right] \frac{\partial h^+}{\partial t} - W_0 q_0 \left( g - 1 \right) + C \left( W_0 u_0 + W_0 w_0 \right) \right] r dr dz. \]

Now, secular terms are eliminated by integrating (55) in the short time scale in the interval \( ]0, t[ \), dividing by \( t \), letting \( t \to \infty \) and requiring \( I_1 \) to be bounded as \( t \to \infty \). Then we obtain the following solvability condition

\[ \lim_{t \to \infty} t^{-1} \int_0^t e^{-i\Omega t} (I_2 - I_3 - I_4 - I_5^+ + I_5^-) dt = 0. \]

In order to apply (57) we first take into account eqs. (6), (11) and (17)-(19), and the structure of the solution in the bulk and in the Stokes and the interface boundary layers to obtain
\[ l_2 = \delta \int_{-\Lambda}^{\Lambda} \int_{-\Lambda}^{\Lambda} (U_0 u_r + W_0 w_r) r dr dz \\
- \delta \int_{-\Lambda}^{\Lambda} \left[ f_{-1} f_{-1} (f_{-1} + f_{-1} - (f_{-1} - (U_0 u_r + W_0 w_r)) \right]_{r=1} dz + 2 \varepsilon C \left\{ H_1 e^{i \mathcal{C}} \int_{-\Lambda}^{\Lambda} \int_{-\Lambda}^{\Lambda} U_0 (r, \Lambda) \left[ \tilde{U}_0 (r, \xi) - U_0 (r, \Lambda) \right] r dr d\xi + c.c. \right\} + \text{HOT} \]  
(58)

\[ l_3 = 2 \varepsilon^2 \sqrt{C} \int_{-\Lambda}^{\Lambda} \int_{-\Lambda}^{\Lambda} \left\{ A \left( \xi W_0 (r, \Lambda) \tilde{U}_0 - U_0 (r, \Lambda) \tilde{W}_0 \right) e^{i \mathcal{N}} + c.c. \right\} \left( A \tilde{U}_0 \tilde{W}_0 e^{i \mathcal{N}} + c.c. \right) r dr d\xi \\
- 2 \varepsilon C \int_{-\Lambda}^{\Lambda} \int_{-\Lambda}^{\Lambda} U_0 (r, \Lambda) \left[ A \tilde{U}_0 (r, \xi) e^{i \mathcal{N}} + c.c. \right] r dr d\xi \\
- C \varepsilon A e^{i \mathcal{N}} \int_{-\Lambda}^{\Lambda} \int_{-\Lambda}^{\Lambda} \left\{ (W_0^2 + W_0^2 + W_0^2 + W_0^2) r + U_0^2 \right\} r dr dz + \text{HOT}, \]  
(59)

\[ l_4 = \int_{-\Lambda}^{\Lambda} \left\{ \varepsilon (f W_0 (1, z) + (f-1) W_0 (1, z)) (A W_0^2 (1, z) e^{i \mathcal{N}} + c.c.) + i \Omega (1 - f/2) F_0 f_0 \\
+ (f-1) \left[f_0 (U_0 (1, z) + U_0 (1, z)) - f Q_0 (1, z) + F_0 + F_0^* \right] - (f-1)^2 \left(F_0^* + F_0^* \right) \right\} f_0 dz \\
- i \Omega \int_{-\Lambda}^{\Lambda} \left\{ (f-1)^2 (1 - \Omega^2 f) + (f-1) f_2 + 3 f_2 (1 + f_2) + \frac{\varepsilon^2}{2} \left( W_0^2 (0, z) e^{i \mathcal{N}} + c.c. \right)^2 \right\} F_0 dz \\
+ \int_{-\Lambda}^{\Lambda} U_0 (f-1) - f_1 W_0_{r=1} \left[ f_1 - f_2 f_2 + f_2 f_2 - \frac{1}{2} \left[ f_2^2 + \varepsilon^2 \left( A W_0^2 (0, z) e^{i \mathcal{N}} + c.c. \right)^2 \right] \right] \right\} d\xi \\
+ \frac{\Omega^2}{2} \int_{-\Lambda}^{\Lambda} (f-1)^2 U_0 (1, z) dz + \int_{-\Lambda}^{\Lambda} U_0 (f-1)^2 (2 - W_0 f_1) (f-1) \right\} d\xi \\
- C \int_{-\Lambda}^{\Lambda} f_1 W_0 (1, z) dz + C \int_{-\Lambda}^{\Lambda} f_1 + \varepsilon (A W_0^2 (0, z) e^{i \mathcal{N}} + c.c.) \right\} U_0 (1, z) dz + \text{HOT}, \]  
(60)

\[ I_5^+ = (i \Omega \mu \beta \pm e^{i (\mathcal{N} + \omega x r)} + c.c.) \int_{-\Lambda}^{\Lambda} r Q_0 (r, \Lambda) dr \\
\pm \sqrt{C} \varepsilon \int_{-\Lambda}^{\Lambda} U_0 (r, \Lambda) \left[ A \tilde{U}_0 (0, r) e^{i \mathcal{N}} + c.c. + \sqrt{C} \tilde{U}_0 (r, 0) + \varepsilon \tilde{U}_0 (r, 0) \right] r dr + \text{HOT}, \]  
(61)

where \( u \) and \( w \) (the velocity components in the bulk) and \( f \) are as given in (13), \( \tilde{U}_0, \tilde{W}_0, \tilde{u}_1, \tilde{u}_2 \) and \( W_0^* \) are as given in (32)-(34) and the second equation below (51) and \( \text{HOT} \) stands for

\[ \text{HOT} = o(\mu + \varepsilon C + \varepsilon^2 \sqrt{C} + \varepsilon^3) \]  
(62)

Also, in order to obtain (58) and (59) we have used the solution in the Stokes boundary layer near \( z = -\Lambda \) (not given in §2), and the fact that the functions \( U_0^2 \) and \( U_0 W_0z \) are even in the \( z \) variable; in order to obtain (60) we have taken into account the equation

\[ i \Omega \int_{-\Lambda}^{\Lambda} fQ_0 (1, z) dz = -i \Omega \int_{-\Lambda}^{\Lambda} \left[ f(f-1) + 2 f_2 + (2 f-1) f_2 \right] F_0 dz + o(\varepsilon \mu) \]

that is obtained upon substitution of (9), (11), (19) and (20), and integration by parts.

Now, \( H_1, H_2 \) and \( H_5 \) are readily calculated upon substitution of the resulting equations into (57) and setting to zero the coefficients of \( \varepsilon \sqrt{C}, \varepsilon^2 \) and \( \mu \), to obtain

\[ H_1 = -(1 + i) \alpha_1 A, \quad H_2 = 0, \quad H_5 = i \left( \alpha_1^+ \beta_+ e^{i \omega x r} - \alpha_1^- \beta_- e^{i \omega x r} \right), \]  
(63)
where

\[ \alpha_1 = \int_{0}^{1} Q_0(r, \Lambda)^2 r dr \left[ \frac{\sqrt{2} \Omega}{2} \int_{-\Lambda}^{\Lambda} F_0(z) Q_0(1, z) dz \right]^{-1}, \] (64)

\[ \alpha_{\mp} = -\Omega \int_{0}^{1} Q_0(r, \Lambda) r dr \left[ \frac{1}{2} \int_{-\Lambda}^{\Lambda} F_0(z) Q_0(1, z) dz \right]^{-1}, \] (65)

and we have taken into account (17) and the following equations

\[ \int_{-\Lambda}^{\Lambda} \int_{0}^{1} (U_0^2 + W_0^2) r dr = - \int_{-\Lambda}^{\Lambda} F_0(z) Q_0(1, z) dz, \] (66)

\[ \int_{0}^{1} U_0(r, \Lambda) U_0(r, 0) r dr = (1 + i) \sqrt{\Omega/2} \int_{0}^{1} U_0(r, \Lambda)^2 r dr \]

Eq. (66) is obtained when multiplying the second and third equations (17) by \( r U_0 \) and \( r W_0 \) respectively, adding, integrating, integrating by parts and taking into account the first equation (17) and (18)-(19); the second equation is readily obtained when taking into account (32) and (36).

In order to calculate \( \alpha_1 \) and \( \alpha_{\mp} \) we need the nontrivial eigenfunctions of (17)-(20) that were first calculated in a semianalytical form by Sanz [1985]; we simply collect his results here. The eigenfrequencies \( \Omega \) are exactly the (real) solutions of one of the following equations

\[ \Lambda \tan \Lambda = \sum_{n \text{ odd}} a_n r_n \quad \text{or} \quad \Lambda \cot \Lambda = \sum_{n \text{ even}} a_n r_n = 0 \] (67)

where

\[ a_0 = 1, \quad a_n = \frac{2\Omega^2/(\Omega^2 q_n - s_n)} {\text{if } n \geq 1,} \] (68)

\[ q_n = I_0(l_n), \quad r_n = q_n/(l_n^2 - 1), \quad s_n = l_n(l_n^2 - 1) I_1(l_n), \quad l_n = n \pi/2 \Lambda \quad \text{if } n \geq 0, \] (69)

and \( I_0 \) and \( I_1 \) are the first two modified Bessel functions. If the first equation (67) holds (odd modes), \( Q_0 \) and \( F_0 \) are defined (up to a constant factor) by

\[ Q_0 = \sum_{n \text{ odd}} a_n I_0(l_n r) \cos[l_n(z + \Lambda)], \quad F_0 = \Lambda \sin z / \cos \Lambda + \sum_{n \text{ odd}} a_n r_n \cos[l_n(z + \Lambda)], \] (70)

while if the second equation holds (even modes), then

\[ Q_0 = \sum_{n \text{ even}} a_n I_0(l_n r) \cos[l_n(z + \Lambda)], \quad F_0 = \Lambda \cos z / \sin \Lambda + \sum_{n \text{ even}} a_n r_n \cos[l_n(z + \Lambda)]. \] (71)

\( U_0 \) and \( W_0 \) are readily calculated by means of the second and third equations (16).

Now, \( \alpha_1 \) and \( \alpha_{\mp} \) are readily calculated. Notice that these constants are real and that \( \alpha_{\mp} = \alpha_{\mp}^* \) and \( \alpha_{\mp} = -\alpha_{\mp}^* \) for even and odd modes respectively, i.e., for the \( m \)-th mode we have

\[ \alpha_{\mp} = (-1)^m \alpha_{\mp}^*. \] (72)

The coefficients \( H_3 \) and \( H_5 \) of the amplitude equation (15b) will depend on the terms of orders \( \epsilon \sqrt{C} \) and \( \epsilon^2 \) in the expansions (13), that are considered now. When taking into account (26), (37), (52)-(54) and (63), the solutions of (14b), (17)-(25) for \( k = 1 \) and 2 are seen to be given by

\[ u_1 = A U_1 e^{i\nu x} + \text{c.c.}, \quad w_1 = A W_1 e^{i\nu x} + \text{c.c.}, \quad q_1 = A Q_1 e^{i\nu x} + \text{c.c.}, \quad f_1 = A F_1 e^{i\nu x} + \text{c.c.}, \] (73)
\[
\begin{align*}
   u_2 &= A^2 U_{22}e^{2\Omega t} + \text{c.c.} + u_{20}, \quad u_2 = A^2 W_{22}e^{2\Omega t} + \text{c.c.} + w_{20},
   \quad (74) \\
   q_2 &= A^2 Q_{22}e^{2\Omega t} + \text{c.c.} + |A|^2 Q_{20}, \quad f_2 = A^2 F_{22}e^{2\Omega t} + \text{c.c.} + |A|^2 F_{20},
   \quad (75)
\end{align*}
\]
where the nonoscillatory (in the short time scale) components of the velocity field, \(u_{20}\) and \(w_{20}\), will be considered in \(\S 4\), while \((U_1, W_1, Q_1, F_1), (U_{22}, W_{22}, Q_{22}, F_{22})\) and \((Q_{20}, F_{20})\) are given by

\[
\begin{align*}
   U_1 + r^{-1}U_1 + W_{1z} &= U_{22} + r^{-1}U_{22} + W_{22} = 0, 
   \quad (76) \\
   i\Omega U_1 + Q_{1z} - (1+i)\alpha_1 U_0 &= 2i\Omega U_{22} + Q_{22z} = Q_{20z} = 0, 
   \quad (77) \\
   i\Omega W_1 + Q_{1z} - (1+i)\alpha_1 W_0 &= 2i\Omega W_{22} + Q_{22z} = Q_{20z} = 0, 
   \quad (78) \\
   W_1 &= \pm(1-i)W_{0z}/\sqrt{2\Omega}, \quad W_{22} = 0 \quad \text{at} \quad z = \pm \Lambda, 
   \quad (79) \\
   U_1 - i\Omega F_1 + (1+i)\alpha_1 F_0 &= U_{22} - 2i\Omega F_{22} + F_0 U_{0z} - F_0^2 W_0 = 0 \quad \text{at} \quad r = 1, 
   \quad (80) \\
   Q_1 + F_1 + F_1^* + F_{22} + F_{22}^* + F_0 Q_{0z} - F_0^2 + (F_0^2 - W_0^2 + \Omega^2 F_0^2)/2 &= 0 \quad \text{at} \quad r = 1, 
   \quad (81) \\
   Q_{20} + F_{20} + F_{20}^* + (F_0 Q_{0z} + \text{c.c.}) - 2|F_0|^2 - |F_0|^2 - |W_0|^2 - |\Omega|^2 F_0^2 &= 0 \quad \text{at} \quad r = 1, 
   \quad (82) \\
   F_1(\pm\Lambda) &= F_{22}(\pm\Lambda) = \int_{-\Lambda}^{\Lambda} F_1 dz = \int_{-\Lambda}^{\Lambda} F_{22}^2/2 dz = \int_{-\Lambda}^{\Lambda} (F_{20} + |F_0|^2) dz = 0, 
   \quad (83) \\
   \int_{-\Lambda}^{\Lambda} \int_{0}^{1} (U_0 U_1 + W_0 W_{1z}) r dr dz &= 0. 
   \quad (84)
\end{align*}
\]

Notice that \(u_2\) has no oscillatory terms with frequency \(\Omega\) in the short time scale; this is a consequence of condition (14b) for \(k = 2\). Then \(K_r^+ = 0\) in (34) (see (38)); since, in addition, \(H_2 = 0\) (see (63)), we have

\[
\int_{0}^{1} U_0(r, \Lambda) \bar{u}_{2z}(r, 0) r dr = \text{independent of} \ t. 
\]

Also, the problem posed by (76)-(81) and (83)-(84) giving \((U_1, W_1, Q_1, F_1)\) possesses a solution if and only if the constant \(\alpha_1\) is as given in (64), as is readily seen (upon multiplication of (77), (78) and the second and third equations in (17) by \(r U_0, r W_0, -r U_1\) and \(-r W_1\) respectively, addition, integration in \(-\Lambda < z < \Lambda, 0 < r < 1\), integration by parts and substitution of (17)-(20), (76), (78), (81) and (83)).

If (64) holds then (76)-(81) and (83)-(84) uniquely define \(U_1, W_1, Q_1\) and \(F_1\), that may be obtained in a semianalytical form. In particular, \(Q_1\) and \(F_1\) are given by

\[
\begin{align*}
   Q_1 &= -(1-i)[b Q_0 + \partial Q_0/\partial \Omega] + \sum_{n \text{ odd}} b_n I_0(n \pi r) \cos[n(z + \Lambda)]/\sqrt{2\Omega}, 
   \quad (86a) \\
   F_1 &= -(1-i)[b F_0 + \partial F_0/\partial \Omega + 2\Lambda(\partial \Omega/\partial \Omega - \alpha_1 \sqrt{2\Omega}) \sin z/(\Omega \cos \Lambda)] \\
   &\quad + \sum_{n \text{ odd}} b_n r_n \cos[n(z + \Lambda)]/\sqrt{2\Omega}, \quad (87a)
\end{align*}
\]
if \(\Omega\) is a solution of the first equation in (67), or

\[
\begin{align*}
   Q_1 &= -(1-i)[b Q_0 + \partial Q_0/\partial \Omega] + \sum_{n \text{ even}} b_n I_0(n \pi r) \cos[n(z + \Lambda)]/\sqrt{2\Omega}, 
   \quad (86b) \\
   F_1 &= -(1-i)[b F_0 + \partial F_0/\partial \Omega + 2\Lambda(\partial \Omega/\partial \Omega - \alpha_1 \sqrt{2\Omega}) \cos z/(\Omega \sin \Lambda)] \\
   &\quad + \sum_{n \text{ even}} b_n r_n \cos[n(z + \Lambda)]/\sqrt{2\Omega}, 
   \quad (87b)
\end{align*}
\]
if \(\Omega\) is a solution of the second equation in (67); \(U_1\) and \(W_1\) are given by

\[
\begin{align*}
   U_1 &= [iQ_{1r} + (1-i)\alpha_1 U_0]/\Omega, \quad W_1 = [iQ_{1z} + (1-i)\alpha_1 W_0]/\Omega. 
   \quad (88)
\end{align*}
\]
Here the constant $b_n$ is given by

$$b_n = 2\Omega (\partial \Omega / \partial \Lambda - \alpha_1 \sqrt{2\Omega}) q_n a_n / (\Omega^2 q_n - s_n)$$

for $n = 0, 1, \ldots$, and the constants $a_n, l_n, g_n, r_n$ and $s_n$ are as given in (68)-(69). The results below do not depend on the constant $b$, that is uniquely determined by (84).

Similarly, if neither $\Omega_1 = 2\Omega$ nor $\Omega_1 = 0$ are solutions of the second equation (67), then (76)-(83) uniquely define $(U_{22}, W_{22}, Q_{22}, F_{22})$ and $(Q_{20}, F_{20})$, that may be written in a semianalytical form as

$$Q_{22} = 2\Omega^2 c_0 + 4\Omega^2 \sum_{k=1}^{\infty} c_{2k} l_0 (2l_k r) \cos[2l_k (x + \Lambda)] \tag{89}$$

$$F_{22} = i [F_0(z) W_0(1, z) - F_0(z) U_{0r}(1, z)] / 2\Omega + 2 \sum_{k=1}^{\infty} c_{2k} l_k l_1 (2l_k) \cos[2l_k (x + \Lambda)], \tag{90}$$

$$U_{22} = i Q_{22r} / 2\Omega, \quad W_{22} = i Q_{22s} / 2\Omega, \tag{91}$$

$$Q_{20} = D_3 - g_0 / 2, \tag{92}$$

$$F_{20} = D_4 \cos z - D_3 - \sum_{k=1}^{\infty} g_{2k} (1 - 4l_k^2)^{-1} \cos[2l_k (z + \Lambda)], \tag{93}$$

where

$$D_1 = \int_{-\Lambda}^{\Lambda} F_0(z) \, dz / 4\Lambda + \sum_{k=1}^{\infty} \left[ s_{2k} e_{2k} + 2(1 - 4l_k^2) d_{2k} \Omega^2 q_{2k} \right] / (4\Omega^2 q_{2k} - s_{2k}) (1 - 4l_k^2),$$

$$D_2 = \Lambda \cot \Lambda - 1 + 8\Omega^2 \sum_{k=0}^{\infty} r_{2k} / (4\Omega^2 q_{2k} - s_{2k}),$$

$$D_3 = \left[ \tan \Lambda \sum_{k=1}^{\infty} (1 - 4l_k^2)^{-1} g_{2k} + \int_{-\Lambda}^{\Lambda} |F_0(z)|^2 \, dz / 2 \right] / (\Lambda - \tan \Lambda),$$

$$D_4 = \left[ \Lambda \sum_{k=1}^{\infty} (1 - 4l_k^2)^{-1} g_{2k} + \int_{-\Lambda}^{\Lambda} |F_0(z)|^2 \, dz / 2 \right] / (\Lambda \cos \Lambda - \sin \Lambda),$$

and for $n \geq 0$, the constants $l_n, r_n, g_n$ and $s_n$ are as given in (69), while $c_n, d_n, e_n$ and $g_n$ are as given by

$$c_n = [2D_1 / D_2 + (1 - l_0^2) d_0 / 2 + c_n] / (4\Omega^2 q_n - s_n),$$

$$d_n = -i \int_{-\Lambda}^{\Lambda} \left[ F_0(z) W_0(1, z) - F_0(z) U_{0r}(1, z) \right] \cos[l_n (z + \Lambda)] \, dz / (\Lambda \Omega),$$

$$e_n = \int_{-\Lambda}^{\Lambda} [(2 - 3\Omega^2) F_0(z)^2 + W_0(1, z)^2 - F_0(z)^2] \cos[l_n (z + \Lambda)] \, dz / 2\Lambda,$$

$$g_n = \int_{-\Lambda}^{\Lambda} (\Omega^2 - 2) |F_0(z)|^2 - |W_0(1, z)|^2 + |F_0(z)|^2 \cos[l_n (z + \Lambda)] \, dz / \Lambda.$$

Now $H_3, H_4$ and $H_5$ are readily obtained upon substitution of (13) (with the terms of orders $\varepsilon \sqrt{C}$ and $\varepsilon^2$ as given by (73)-(75)) and (15b) into (58)-(61), substitution of the resulting equations into (57) and setting to zero the coefficients of $\varepsilon C$, $\varepsilon^2 \sqrt{C}$ and $\varepsilon^3$, to obtain (after some algebraic manipulations)

$$H_3 = -\alpha_2 A, \quad H_4 = 0, \quad H_5 = i\alpha_3 A|A|^2, \tag{94}$$

where the real constants $\alpha_2$ and $\alpha_3$ are given by
\begin{align}
(\alpha_2 - 2 - 2\alpha_2^2 / \Omega) \int_{-\Lambda}^{\Lambda} F_0(z) Q_0(1, z) \, dz &= -4F'_0(\Lambda) F''_0(\Lambda) \\
&+ (1 + i) \int_{0}^{1} Q_{2r}(r, \Lambda) Q_{1r}(r, \Lambda) r \, dr / \sqrt{2\Omega^3} \\
&+ \int_{-\Lambda}^{\Lambda} [2Q_0(1, z)^2 - \Omega^2 F_0(z)^2 - (1 + i) \alpha_1 F_1(z) Q_0(1, z)] \, dz \\
2(\alpha_3 / \Omega) \int_{-\Lambda}^{\Lambda} F_0(z) Q_0(1, z) \, dz &= \int_{-\Lambda}^{\Lambda} [(2 - 3\Omega^2) F_0 F_{22} - F'_0 F''_{22} + (2 - \Omega^2) F_0 F_{20} - F'_0 F''_{20} \\
&- 2W_0(1, z) W_{22}(1, z) - (F_{22} + F_{20}) Q_0(1, z)] \, dz + \int_{-\Lambda}^{\Lambda} (F_{22} - F_{20}) W_0(1, z)^2 \, dz \\
&+ \int_{-\Lambda}^{\Lambda} (10F_0 + 17F''_0) F'_0 + 4(F_0 + F'_0) F''_0 F_0 - (6 + \Omega^2) F_0^3 - F_0 W_0(1, z)^2 \, dz / 2.
\end{align}

In order to obtain (93)-(96) we have taken into account that \( tU_0, iW_0, Q_0 \) and \( F_0 \) are real (see (68)-(70) or (71)); also we have used (63), (66), (85) and the following equations

\begin{align}
\int_{0}^{1} \int_{-\infty}^{0} U_0(r, \Lambda) [\bar{U}_0(r, \xi) - U_0(r, \Lambda)] \, rdrd\xi &= -(1 - i) \int_{0}^{1} r U_0(r, \Lambda)^2 \, dr / \sqrt{\Omega}, \\
U_{0z}(r, \Lambda) &= W_{0r}(r, \Lambda) = 0 \\
\int_{-\Lambda}^{\Lambda} \int_{0}^{1} (U_0 U_1 + W_0 W_1) rdrdz &= \int_{-\Lambda}^{\Lambda} [-\alpha_1 (1 - i) F_0 Q_0(1, z) / \Omega - F_1 Q_0(1, z)] \, dz \\
\int_{-\Lambda}^{\Lambda} \int_{0}^{1} [(U'_0 + U''_0 + W'_0 + W''_0) r + U_0 / r] \, rdrdz &= 4F'_0(\Lambda) F''_0(\Lambda) + \int_{-\Lambda}^{\Lambda} [\Omega^2 F_0^2 + 2F'_0 Q_0(1, z)] \, dz, \\
\int_{0}^{1} U_0(r, \Lambda) \bar{u}_{1z}(r, 0) \, rdr &= \left[(1 + i) \sqrt{\Omega / 2} \int_{0}^{1} U_0(r, \Lambda) U_1(r, \Lambda) \, rdright] \\
&+ \alpha_1 \int_{0}^{1} U_0(r, \Lambda)^2 \, rdr / \sqrt{2\Omega^3} \right] \, r \, dr + c.c.
\end{align}

Eqs. (97) and (98) readily follow from (18), (19), (32) and (35). Eq. (99) is obtained by multiplying the first equation in (77) and (78) by \( rU_0 \) and \( rW_0 \) respectively, adding, integrating in \(-\Lambda < z < \Lambda, 0 < r < 1\) and integrating by parts. Similarly, (100) is obtained by multiplying the equations

\begin{equation}
U_{0rr} + U_{0zz} + r^{-1}U_{0r} - r^{-2}U_0 = W_{0rr} + W_{0zz} + r^{-1}W_0 = 0
\end{equation}

(that are readily obtained from (17)) by \( rU_0 \) and \( rW_0 \) respectively, adding, integrating in \(-\Lambda < z < \Lambda, 0 < r < 1\) and integrating by parts. Equation (101) is readily obtained when taking into account (33) and (36).