To derive (2.13) it is convenient to first Fourier expand \( u \) and \( p(s) \) in \( \langle u u p(s) \rangle \) and \( \langle uu p(s) \rangle / \partial x_3 \). This derivation is limited to a weakly inhomogeneous turbulence for which \( \langle uu \rangle \) is proportional to \( \exp(2a \cdot x) \) where \( a \) is constant and directed along \( x_3 \); (i.e., \( a \cdot x = ax_3 \)). A simple model for this \( u \) is

\[
u = u_0 \exp(ax_3)
\]

(C.1)

where \( u_0 \) is statistically homogeneous; i.e., \( \partial \langle uu_0 \rangle / \partial x_3 = 0 \). The Fourier expansion of \( u \) is given by

\[
u = (2\pi)^{-3} \int dk u_k^0 \exp(i k \cdot x + ax_3), \quad (\text{real } k)
\]

(C.2)

where \( k \) is real, and \( u_k^0 \) is the Fourier transform of \( u_0 = u \exp(-ax_3) \).

We also need the Fourier expansion of \( p(s) \), which, since it is given by

\[
\nu^2 p(s) = -\nabla \cdot (\nu \cdot \nabla u),
\]

(C.3)

is proportional to \( \exp(2a \cdot x) \) when (C.2) is substituted for \( u \).

Therefore, we may define a homogeneous pressure field \( p^0 \) by

\[
p(s) = p^0 \exp(2a \cdot x)
\]

(C.4)

and Fourier expand \( p(s) \) as

\[
p(s) = \left( \int \frac{d^3k}{(2\pi)^3} k^0 \exp(i k \cdot x + 2a \cdot x) \right)
\]

where \( k \) is real and \( p_k^0 \) is the Fourier transform of \( p^0 = p(s) \exp(-2a \cdot x) \).

The value of \( p_k^0 \) is determined from (C.3) by multiplying both sides with \( \exp(-2a \cdot x) \) and then taking the Fourier transform of both sides utilizing (C.2) and (C.4): The result is
\[ p_k^0 = - \int \frac{d^3k_a}{(2\pi)^3} \frac{\left(ik+2a\right) \cdot \left(u^0_{k_a} u^0_{k_b}\right) \cdot (ik_a + a)}{(ik+2a) \cdot (ik+2a)} \]  

(C.5)

where the prime on \(u^0_{k_a} u^0_{k_b}\)' denotes that we exclude \(k_a + k_b = 0\). (We caution that the symbol \(a\) is used for both the scale length of inhomogeneity and as an index in \(k_a\), and ask the readers indulgence for this poor notation.) This equation together with (C.4) gives the Fourier expansion of \(p^a(s)\).

The Fourier expansion of \(<u_u u_v^p(s)\>\) can now be written by substitution of (C.2), (C.4), and (C.5) to yield

\[ <u_u u_v^p(s)> = - \int \frac{d^3k_a d^3k_b d^3k_c d^3k_d}{(2\pi)^{12}} \frac{<w_{k_c} u_{k_d} (u_{k_a} u_{k_b})> : (ik_a + a)(ik + 2a)^2}{(ik+2a) \cdot (ik+2a)} \]

\[ \times \exp[-i(k + k_c + k_d) \cdot x] \]  

(C.6)

\[ u_{k_a} = u^0_{k_a} \exp(a \cdot x) \] as defined by (C.1),

where \(w_{k_c}\) is the vertical component of \(u_{k_c}\) and we have used the convenient vector notation \(<w_{u} (u)^{\prime}> : (ik_a + a)(ik + 2a) = <w_{u}(ik + 2a) \cdot (u)^{\prime} :: (ik_a + a)\>\)

The correlation in (C.6) is evaluated by cumulant expansion

\[ <w_{k_c} u_{k_d} (u_{k_a} u_{k_b})> = <w_{k_c} u_{k_a} > <u_{k_d} u_{k_b}> \]

\[ <w_{k_c} u_{k_d} > <u_{k_d} u_{k_a}> + Q^*, \]  

(C.7)

where \(Q^*\) is the forth-order cumulant. Our basic approximation is to neglect \(Q^*\) as well as all other forth-order cumulants. Substitution of (C.7) in (C.6) and use of the quasi-homogeneous condition (B.5)—justified since \(u^0_k\) is statistically homogeneous—we have

\[ <u_u u_v^p(s)> = -2 \int \frac{d^3k_a d^3k_b}{(2\pi)^6} \frac{x(ik_a + a) : S(k_b) S(k_a) \cdot (ik + 2a)^2}{-k^2 + 4i\alpha \cdot k + 4a^2} \]  

(C.8)
\[ S(\mathbf{k}) = V^{-1} \langle \mathbf{u}_k^0 (\mathbf{u}_k^0)^* \rangle \exp(2a \cdot \mathbf{x}) \]

where the asterisk denotes the complex conjugate and we have used the vector notation \[ \langle \mathbf{w}_k \mathbf{u}_k^* \rangle = \mathbf{x}_i \langle \mathbf{w}_k \mathbf{u}_k^* \rangle. \]

For small inhomogeneity \( (a \ll k_0) \), the integrand of (C.8) is expanded in powers of \( a \), retaining terms to first order:

\[
\langle \mathbf{u}_s \mathbf{u}_p (s) \rangle = -2 \int \frac{dk_a dk_b}{(2\pi)^2} \left[ a \cdot \mathbf{S}(k_b) \cdot \mathbf{x}_s \mathbf{S}(k_a) \cdot \mathbf{x}_b k_b + a \cdot \mathbf{S}(k_a) \mathbf{x}_s \mathbf{S}(k_b) \cdot \mathbf{x}_b k_b \right] \\
+ 2(a - \frac{2a \cdot k_a}{k_a^2}) \mathbf{x}_b \mathbf{S}(k_a) \mathbf{x}_s \mathbf{S}(k_b) \cdot \mathbf{x}_b k_b^2, \tag{C.9}
\]

where we have used the incompressibility condition \( (ik^*_a + a) \mathbf{u}_k^* = 0 \) [i.e., \( (ik^*_a + a) \mathbf{u}_k = 0 \) which follows from the Fourier transform of \( \nabla \cdot \exp(-a \cdot \mathbf{x}) = 0 \)], and the zeroth order term vanishes because it is an odd power of \( k \) and of \( k \). This vanishing is as should be since \( \langle \mathbf{u}_s \mathbf{u}_p (s) \rangle \) is zero for a homogeneous turbulence (i.e., for \( a = 0 \)). The integrations in (C.9) are straightforwardly evaluated in the small anisotropy limit when \( \mathbf{S}(k_i) \) is given by

\[
\mathbf{S}(k_a) = 2\pi^2 (1 - \frac{k_a}{k^2}) \frac{E(k_a)}{k_a^3}, \quad \text{isotropic} \mathbf{S}, \tag{C.10}
\]

where \( E(k_a) \) is the scalar energy spectrum normalized by \( \int dk_a E(k_a) = (3/2)\nu_0^2 \).

Substitution of (C.10) in (C.9) allows us to readily calculate the \( \mathbf{x}_i \mathbf{x}_j \), \( \mathbf{x}_i \mathbf{x}_j \), and \( \mathbf{x}_i \mathbf{x}_j \), components of the pressure correlation. These are found to

\[
\langle \mathbf{u}_s \mathbf{u}_p (s) / \mathbf{x}_i \rangle = \frac{1}{15} \left( \frac{3}{2} \nu_0^2 \right)^2 a
\]

\[
\langle \mathbf{u}_s \mathbf{u}_p (s) / \mathbf{x}_1 \rangle = \frac{2}{225} \left( \frac{3}{2} \nu_0^2 \right)^2 a \tag{C.11}
\]

\[
\langle \mathbf{u}_s \mathbf{u}_p (s) / \mathbf{x}_3 \rangle = 0
\]

These components are seen to be very small.
Before comparison of (C.11) with $A^O$ we must have the components of $\langle uuap(s)/a_3s \rangle$ to confirm (2.13). An expression for $\langle uuap(s)/a_3s \rangle$ is obtained in the same manner as was done for (C.8) by substitution of the Fourier expansions (C.2) and (C.5): The result is

$$\langle uuap(s)/a_3s \rangle = -2 \frac{dk_a dk_b}{(2\pi)^4} \frac{(ik_a + a) \cdot S(k_b)S(k_a) \cdot (ik_0 + a)}{\Lambda - k^2 + 4ia \cdot k + 4a^2 \Lambda}$$  \hspace{1cm} \text{(C.12)}$$

Evaluation of this integral in the limits of small $a$ and small anisotropy yields the $\hat{x}, \hat{y}, \hat{z}$ components of (C.12) as

$$\langle u_xu_yap(s)/a_3s \rangle = \frac{1}{15} \left( \frac{3}{2} v_o^2 \right)^2 a$$ \hspace{1cm} \text{(C.13)}$$

$$\langle u_xu_zap(s)/a_3s \rangle = -\frac{8}{225} \left( \frac{3}{2} v_o^2 \right)^2 a$$

$$\langle u_yu_zap(s)/a_3s \rangle = 0.$$

For comparison, the components of $A^O$ are also evaluated in the limit of small anisotropy to obtain, with $\frac{2}{\Lambda}$,

$$A_{0x} = \frac{24}{9} \left( \frac{3}{2} v_o^2 \right)^2 a$$

$$A_{0y} = \frac{8}{9} \left( \frac{3}{2} v_o^2 \right)^2 a$$  \hspace{1cm} \text{(C.14)}$$

$$A_{0z} = 0$$

Upon comparison, it is seen that the diagonal components of $\Pi(s)$

$$(1 + T_p) \langle u_xu_y + u_yu_z \rangle$$

are much smaller in magnitude than the corresponding diagonal components of $A^O$, in conformity with (2.13). (The off-diagonal elements are all zero). Additionally, $\Pi(s)$ is not proportional to $A^O$ since $\Pi_{11}(s)/\Pi_{33}(s) = A_{11}/A_{33}$.
PART 1

To help evaluate $\Pi^{(U)}_a$ we express $p^{(U)}$ as a Fourier integral, valid for periodic boundary conditions or for an asymptotically large system:

$$p^{(U)} = 2i \int \frac{dk}{(2\pi)^3} \frac{k_i u^*(k)}{k^2} \frac{\partial U_0}{\partial x_3} \exp(ik \cdot \vec{R}), \quad (D.1)$$

where $u^*_i(k)$ is the Fourier transform of $u_i(x,t)$, and we have limited ourselves to a unidirectional mean flow $\bar{U} = [U_0(x),0,0]$ which may vary only with $x_3$. The spatial inhomogeneity factor $\exp(a \cdot \chi)$ is ignored in (D.1) since it would only contribute to a higher than first order dependence of $\Pi^{(U)}_a$ on $a$; i.e., the triple-moment in $\Pi^{(U)}_a$ is, itself, first order in $a$. Substitution of (D.1) into (2.11), the definition of $\Pi^{(U)}_a$, we have

$$\Pi^{(U)}_a = -2i \frac{\partial U_0}{\partial x_3} \int \frac{dk}{(2\pi)^3} \left[ (1 + T_r) \langle u_i u^*_i(k) \rangle \frac{k_i k_j}{k^2} + \langle u_i u^*_j(k) \rangle \frac{k_i k_j}{k^2} \right] \exp(ik \cdot \chi), \quad (D.2)$$

where the quantities $u_i$ and $\bar{u}$ without a super plus are not Fourier transformed.

Our goal is to determine the 13 off-diagonal element and the order of magnitude of the diagonal elements of $\Pi^{(U)}_a$ in the limit of small forcing, small $|\partial U_0/\partial x_3|$ and $|\partial \bar{u}/\partial x_3|$, and small anisotropy. Each element is considered separately: $\Pi^{(U)}_{13}$, $\Pi^{(U)}_{11}$ and $\Pi^{(U)}_{33}$, in that order.

The 13 component is expressed by (D.2) as

$$\Pi^{(U)}_{13} = -\frac{\partial U_0}{\partial x_3} \int \frac{dk}{(2\pi)^3} \left[ 2\langle \bar{u}_i u^*_j(k) \rangle \frac{k_i^2}{k^2} + 4\langle u_i u_j u^*_k(k) \rangle \frac{k_i k_j k_k}{k^4} \right]. \quad (D.3)$$

To evaluate this integral we need the triple-moments $\langle u_i u^*_j(k) \rangle$ and $\langle u_i u_j u^*_k(k) \rangle$; later in this appendix we will also need $\langle \bar{u}_i u^*_j(k) \rangle$. These moments can be obtained with the aid (2.10). To lowest order in $\partial U_0/\partial x_3$, and/or $\bar{g}/\theta_0$ -- and neglect of $\theta^{(')}$ -- (2.10) gives

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\begin{align}
3\langle u_j u_i^* \rangle &= -\tau_0 A_{1j}^0, \quad \text{(D.4)} \\
3\langle u_i^* \rangle &= -\tau_0 A_{33}^0 \\
3\langle u_i^* u_j \rangle &= -\tau_0 (A_{13}^0 + \Pi_{13}^{(U)} + \langle u_i^* \rangle \frac{\partial u_i}{\partial x_3}). \quad \text{(D.5)}
\end{align}

The first order terms \( \langle u_i^* \rangle \partial U_0 / \partial x_3 \) and \( \Pi_{13}^{(U)} \) are included in (D.5) since \( A_{13}^0 \) is, itself, first order; i.e., \( A_{13}^0 = \langle u_i u_j \rangle \neq \partial U_0 / \partial x_3 \). The \( g / \Theta_0 \) terms have been neglected in (D.5) as second order in the forcing since these terms have the form \( \langle u_i u_j \rangle g / \Theta_0 \) and \( \langle u_i u_j \rangle \) is of order in \( \partial U_0 / \partial x_3 \) by virtue of the fact it contains \( u_i \). The \( \Pi_{13}^{(g)} \) and \( \Pi_{13}^{(e)} \) are also second order and therefore neglected.

What is required for (D.3) are the Fourier components \( \langle u_j u_j u_i^* (k) \rangle \) where \( j = 1, 3 \). These components can be derived in a formal procedure by taking the Fourier transform of (2.3) and substituting it for \( u_j^*(k) \) in \( \langle u_j u_j u_i^* (k) \rangle \) -- a lengthy procedure. Instead, we use a simpler, but more heuristic, method based on (D.4) and (D.5) which gives the same result. In this method, Fourier expansions of \( A_{33}^0 \) and \( A_{13}^0 \) in the right side of (D.4) are written, after substitution of our inhomogeneity relation \( \partial \langle u_j u_j \rangle / \partial x_3 = 2a \langle u_j u_j \rangle \) into (2.8), as follows:

\begin{equation}
A_{j3}^0 = 3a \int \frac{dk}{(2\pi)^3} \langle u_j^2 \rangle \langle u_j u_j^* (k) \rangle \exp(ik \cdot x) \quad \text{(D.6)}
\end{equation}

(where \( \langle u_j^2 \rangle \) and \( u_j \) are not Fourier expanded). In addition, in the left side of (D.4) is Fourier expanded as \( \langle u_j u_j u_j^* \rangle = (2\pi)^{-3} \int dk \langle u_j u_j u_j^* (k) \rangle \exp(ik \cdot x) \). The Fourier components of each side of (D.4) are then equated to yield

\begin{align}
3\langle u_j u_j^* (k) \rangle &= -\tau_0 A_{1j}^0 (k) \quad \text{(D.7)} \\
3\langle u_j u_j^* (k) \rangle &= -\tau_0 A_{33}^0 (k),
\end{align}

\text{47}
where
\[ A_{i,i}^+(k) = 2a<u_i><u_i u_i^+(k)> + 4a<u_i u_i><u_i u_i^+(k)>, \]  
\[ A_1^+(k) = 6a<u_i><u_i u_i^+(k)>. \]  
(D.8)

It may be heuristic to equate the Fourier components of both sides of (D.4), but we have also derived the same (D.7) in more rigorous, and lengthy fashion.

Similarly, Fourier expansion of both sides of (D.5), with use of (D.3) for \( \Pi_{12}^{(U)} \), we have
\[ 3<u_3 u_i u_i^+(k)> = -\tau_0 [A_{i,i}^+(k) + \frac{3}{2} \tau_0 \frac{aU_0}{\partial x_i} A_{i,i}^+(k) \left( \frac{2k_i^2}{k^2} - 1 \right)] + 4<u_3 u_i u_i^+(k) \left( \frac{k_i k_j}{k^2} \right) \]  
(Eqn's. (D.7) and (D.8) give \( <u_i u_i^+(k)> \) and \( <u_3 u_i u_i^+(k)> \) in terms of second-moments. To obtain \( <u_i u_i u_i^+(k)> \) in terms of second-moments, (D.7) is substituted in (D.9) to yield
\[ 3<u_3 u_i u_i^+(k)> = -\tau_0 [A_{i,i}^+(k) + \frac{1}{3} \frac{aU_0}{\partial x_i} A_{i,i}^+(k) \left( \frac{2k_i^2}{k^2} - 1 \right)] \times \left( 1 - \frac{4}{3} \tau_0 \frac{aU_0}{\partial x_i} \frac{k_i k_j}{k^2} \right)^{-1}. \]  
(D.10)

Evaluation of \( \Pi_{10}^{(U)} \) can now be made by substitution of (D.7) and (D.10) in (D.3) which, to lowest order in \( \frac{aU_0}{\partial x_i} \), yields
\[ \Pi_{10}^{(U)} = \tau_0 \frac{aU_0}{\partial x_i} \int \frac{dk}{(2\pi)^3} \left( \frac{2k_i^2}{3k^2} + \frac{4}{9} \tau_0 \frac{aU_0}{\partial x_i} \left( \frac{2k_i^2}{k^2} - 1 \right) \left( \frac{k_i k_j}{k^2} \right) \right) A_{i,i}^+(k) \]  
\[ + \tau_0 \frac{aU_0}{\partial x_i} \left( \frac{2}{3} \frac{k_i k_j}{k^2} \right) A_{i,i}^+(k) \].  
(D.11)

The quantity \( A_{i,i}^+(k) \) is expressed in terms of spectra as
\[ A_{i,i}^+(k) = 6a<u_i><S_{i,i}(k)>. \]  
(D.12)
This expression obtained from (D.8), (B.5) and use of
\[ u_j^* u_i^* (k) = (2\pi)^{-3} \int d\vec{k}_a \langle u_j^*(k_a) u_i^*(k) \rangle \exp(-i\vec{k} \cdot \vec{x}) = S_{3j}(k) \exp(-i\vec{k} \cdot \vec{x}). \]

For \( S_{3j}(k) \) we use the isotropic expression (C.10). Finally, by substitution of
(D.12) and (C.10), the integrations in (D.11) are readily performed to yield
\[ \Pi_{13}^{(U)} = \tau_o \frac{\partial U}{\partial x_j} (6a \langle u_j^2 \rangle \frac{2}{15} \langle u_j^2 \rangle), \quad (D.13) \]
where we used \( \int d\vec{k} E(k) = (3/2) \langle u_j^2 \rangle \) near isotropy. This relation is expressed
in terms of \( A_0^j \), by substitution of (2.8):
\[ \Pi_{13}^{(U)} = \frac{2}{15} \tau_o \frac{\partial U}{\partial x_j} A_0^j, \quad (D.14) \]
the desired result. This expression gives \( \Pi_{13}^{(U)} \) in terms of second-moments to
first order in \( \partial U / \partial x_j \), and \( \partial U / \partial x_i \), for small anisotropy. To judge the
importance of this term we note that its magnitude is less than that of \( A_0^j \),
the principal term of the 13 component of (2.15), but only by a factor of about
1/3 to 1/2; i.e., for the magnitude of \( A_0^j = 6a \langle u_j^2 \rangle \langle u_iu_j \rangle \) we use \( \langle u_iu_j \rangle = \langle u_iu_j \rangle \)
\(-\tau_o/3 (\partial U / \partial x_j) \langle u_j^2 \rangle \) (e.g., Tennekes and Lumley, 1972) to get \( A_0^j = -(1/3) \)
\( \tau_o (\partial U / \partial x_j) A_0^j \).

In comparison, \( \Pi_{13}^{(Y)} \) is not negligible.

As for the element \( \Pi_{13}^{(Y)} \), its order of magnitude is estimated by
substitution of (D.7) and (D.10) into the 11 component of (D.2)
\[ \Pi_{13}^{(U)} = \frac{2\tau_o}{3} \frac{\partial U}{\partial x_j} \int \frac{dk}{(2\pi)^3} \left[ 2[A_{13}^i(k)] + \frac{\tau_o}{3} \frac{\partial U}{\partial x_j} (A_{13}^i(k)) \frac{k_i^2}{k^2} \right. \]
\[ + A_{13}^i(k) \frac{k_i k_j}{k^2} \]. \quad (D.15)
For asymptotically small isotropy the $A^+_1$, integral term is asymptotically small since, with (D.12) and (C.10), $A^+_1 \sim S_{11}(k)$ and $\int dk S_{11}(k) k^2/k^2$ is zero when the isotropic $S_{11}$ is used. Similarly, for the $A^+_1$, integral term. The remaining $A^+_1$, term does not vanish at isotropy, but is second order in $\tau_0 \partial U_0 / \partial x_1$.

Therefore

$$\Pi^{(U)}_{11} = 0[(\tau_0 \partial U_0 / \partial x_1)^2], \quad (D.16)$$

in the limit of small anisotropy and small $(\tau_0 \partial U_0 / \partial x_1)^2$.

The last element we consider is $\Pi^{(U)}_{33}$. It is given by substitution of (D.7) into (D.2)

$$\Pi^{(U)}_{33} = 2\tau_0 \frac{\partial U_0}{\partial x_3} \int \frac{dk}{(2\pi)^2} A^+_3(k) \frac{k_1 k_3}{k^2}. \quad (D.17)$$

This integral, too, is small near isotropy since $A^+_3 \sim S_{33}(k)$. However, we can say that $\int dk S_{33}(k) k_1 k_3/k^2$ is of order $\partial U_0 / \partial x_3$, since a small shear magnitude will cause a deviation from isotropy (proportional to the shear). Therefore

$$\Pi^{(U)}_{33} = 0[(\tau_0 \partial U_0 / \partial x_3)^2], \quad (D.18)$$

in our limit of weak shear and small anisotropy.

To end this part of the appendix, we note that (D.14), (D.16) and (D.18) can be combined in the tensorial form

$$\Pi^{(U)} = \frac{2}{15} \tau_0 A^o_{03}(\widehat{UU} + \widehat{UU}^T) + \delta_o, \quad (D.19)$$

where $\delta_o$ denotes terms of order $(\partial U_0 / \partial x_3)^2$ and $(\partial U_0 / \partial x_3)(\partial \Phi / \partial x_3)$ neglected in (D.4) and other places, $\widehat{UU}^T$ denotes the transpose of $\widehat{UU}$, we have used $\Pi^{(U)} = \Pi^{(U)}_{11}$. The form used in (2.14') is obtained by substitution of (D.4) into (D.19):
\[ u_s^{(U)} = - \frac{2}{5} \langle u_s \rangle (vU + vU^T) + \delta_o, \]  
\text{(D.20)}

as we set out to prove.

**PART 2**

To derive (2.23) we substitute (D.1) into \( \langle u_0^{(U)} \delta \rangle \) and obtain

\[ (1+T_o)u_0^{(U)} \delta = -2 \frac{\partial u_0}{\partial x_3} \int \frac{dk}{(2\pi)^3} \langle u_0^* u_3 \rangle \frac{k_3}{k^2} \exp(ik \cdot x). \]  
\text{(D.21)}

To lowest order in mean gradients, \( \langle u_0^* u_3 \rangle \) is given by (2.26) as

\[ \langle u_0^* u_3 \rangle = -(\tau_o/3)A_{13}^{(U)}. \]  
\text{(D.22)}

Substitution of (2.19) for \( A_{13}^{(U)} \) and use of \( \partial u_0^*/\partial x_3 = 2a \langle uu \rangle \), we afterwards Fourier expand both sides of (D.22) and heuristically equate Fourier components to obtain

\[ \langle u_0^* u_3 \rangle = -(2/3) a \tau_o [2\langle uu \rangle \langle x \rangle u_0^* u_3 \rangle + \langle uu \rangle \langle u_0^* u_3 \rangle]. \]  
\text{(D.23)}

This could be established by a more rigorous but much more complex derivation.

Substitution of (D.23) and

\[ \langle u_0^* u_3 \rangle = \delta_{13} \langle ku_0^* \rangle \exp(-ik \cdot x) \]  
\text{(D.24)}

into the 13 component of (D.21) we have

\[ \langle \theta(u, \frac{\partial p}{\partial x_3} + u, \frac{\partial p}{\partial x_1}) \rangle = \frac{4 \tau_o}{3} \frac{\partial u_0}{\partial x_3} \int \frac{dk}{(2\pi)^3} \left[ 2\langle uu \rangle \langle S_{33} \rangle(k) \frac{k_3}{k^2} + \langle uu \rangle \langle S_{33} \rangle(k) \frac{k_1^2}{k^2} \right]. \]  
\text{(D.25)}

In the asymptotic limit of small anisotropy, the integral is readily evaluated to be

\[ \langle \theta(u, \frac{\partial p}{\partial x_3} + u, \frac{\partial p}{\partial x_1}) \rangle = \frac{4}{5} \frac{\tau_o}{3} \frac{\partial u_0}{\partial x_3} \langle uu \rangle \langle u_3 \rangle \langle u_3 \rangle \]  
\text{(D.26)}
\[ = - \frac{2}{5} \frac{\partial ^2 u_o}{\partial x_3} \langle u_3^2 \theta \rangle, \]

the desired expression for the 13 element.

The 11 element of (D.21) is evaluated by substitution of (D.23) and (D.24):

\[ 2\langle u_1 \frac{\partial U}{\partial x_1} \theta \rangle = \frac{4\pi \tau}{3} \langle u_0 \frac{\partial U}{\partial x_3} \rangle \int \frac{dk}{(2\pi)^3} \left[ \langle u_1 \theta \rangle S_{13}(k) + \langle u_1 \theta \rangle S_{33}(k) \right] \frac{k_1^2}{k^2}. \quad (D.27) \]

The second term in the integrand is second order since \( \langle u_1 \theta \rangle \) is of the order \((\partial u_0/\partial x_3)(\partial \theta_0/\partial x_3)\) in the asymptotic limits of small anisotropy and small mean gradients. The first term in the integrand integrates to zero near isotropy since \( S_{13}k_1^2 \) is an odd function of \( k_1 \) and \( k_3 \).

Similarly, the 33 component of \( \langle u \tilde{V}^p(U) \theta \rangle \) is also found to be asymptotically small in the limit of small anisotropy and small mean gradients. Hence, in those limits, the diagonal elements of \( \langle u \tilde{V}^p(U) \theta \rangle \) are negligible and, as we have seen, the off-diagonal 13 element is given by (D.26). These diagonal and off-diagonal elements can be combined in the tensorial form

\[ (1 + T_p)\langle u \tilde{V}^p(U) \theta \rangle = -\frac{2}{5} \langle u_0 \theta \rangle (\tilde{V}V + VV^T), \quad (D.28) \]

for our case of \( \tilde{V}V = \hat{x}_1 \hat{x}_1 \partial u_0/\partial x_3 \). This equation is (2.23) as we wished to prove.

To derive (2.23'), (D.1) is substituted into the Fourier expansion of

\[ \langle \tilde{V}p(U) \theta^2 \rangle = -2 \frac{\partial u_0}{\partial x_3} \int \frac{dk}{(2\pi)^3} \langle \theta^2 u_3^r(k) \rangle \frac{kk_1}{k^2} \exp(ik \cdot x). \quad (D.29) \]

The integral is evaluated by use of (2.27) in the limit of small mean gradients:

\[ 3\langle u_3 \theta^2 \rangle = -\tau_o A_3(\theta^2). \quad (D.30) \]
As for (D.7), the Fourier expansions of both sides of (D.30) combined with use of \( \partial \theta / \partial x = 2a \langle u, \theta \rangle \) yields
\[
\langle \theta^2 u^\dagger(k) \rangle = -\frac{2}{3} \alpha \tau \left[ \langle \theta^2 \rangle \langle u, u^\dagger \rangle(k) + 2 \langle u, \theta \rangle \langle \theta u^\dagger \rangle(k) \right].
\] (D.31)

Substitution of (D.31) and (D.24) in (D.29) we have
\[
\langle \nu_p(U) \theta^2 \rangle = \frac{4a \tau \alpha}{3} \int \frac{dk}{(2\pi)^3} \left[ \langle \theta^2 \rangle S_{33}(k) + 2 \langle u, \theta \rangle R_{33}(k) \right] \frac{k_k}{k^2},
\] (D.32)

where \( R_{33}(k) = \langle \theta^+(k) u^+(k) \rangle V^{-1} \) and \( \theta^+(k) \) is the Fourier transform of \( \theta(x) \).

The integration can be done in the small anisotropy limit and use of \( R_{33}(k) = S_{33}(k) \langle u \rangle / \langle u^2 \rangle \) in that limit. The result is
\[
\frac{\partial E(U)}{\partial x_j} = \frac{4a \tau \alpha}{3} \int \frac{dk}{(2\pi)^3} \left[ \langle u \rangle (2 \langle \theta^2 \rangle + \frac{2}{5} \langle u^2 \rangle + \frac{4}{5} \langle u, \theta \rangle) \delta_{ij} \right],
\] (D.32)

where \( \delta_{ij} \) is the Kronecker delta.

After substitution of (2.20), this equation can be written in the vector form
\[
\langle \nu_p(U) \theta^2 \rangle = -\frac{4}{5} \langle \theta^2 u \rangle \cdot \tilde{w},
\]

which is (2.23').
PART 1

The purpose of this appendix is to derive (2.16), (2.17) and (2.18). To derive (2.16) we substitute into \( \langle u u \theta \rangle \) an expression for \( \theta \) obtained from the fluctuating part of the thermodynamic equation. That equation is

\[
\frac{\partial \theta}{\partial t} - \sigma \nabla^2 \theta = - (u \cdot \nabla \theta)' - u \cdot \nabla \theta_0 - U \cdot \nabla \theta,
\]

where \((u \cdot \nabla \theta)' = u \cdot \nabla \theta - \langle u \cdot \nabla \theta \rangle\), \(\sigma\) is the thermal conductivity, and the conductivity term has been placed on the left side for later convenience.

Eq'n. (E.1) is formally integrated in the same way as was the Navier-Stokes equation in Sec. 2 with the result

\[
\theta(t) = G_0(t) \theta(0) - \int_0^t dt_1 G_0(t-t_1) [(u \cdot \nabla \theta)' + u \cdot \nabla \theta_0 + U \cdot \nabla \theta]_{t_1}
\]

(E.2)

\[
G_0(t) = \exp[-(t-t_1)\nu \nabla^2]
\]

where the subscript \(t_1\) in the integrand is to remind us that the terms in square brackets are all to be evaluated at time \(t_1\); e.g., \(u = u(t_1)\).

Substitution of (E.2) into \(\langle u u \theta \rangle\) we have

\[
\langle u u \theta \rangle = (I.V.)_1 - \int_0^t dt_1 \langle u(t)u(t)G_0 [(u \cdot \nabla \theta)' - u \cdot \nabla \theta_0 + U \cdot \nabla \theta]t_1
\]

(E.3)

where \((I.V.)\) denotes the initial value term \(\langle u(t)u(t)G_0(t)\theta(0) \rangle\). A needed additional expression for \(\langle u u \theta \rangle\) is obtained by substitution of (2.3) for \(u\) in \(\langle u u \theta \rangle\) to obtain

\[
\langle u u \theta \rangle = (I.V.)_2 - \int_0^t dt_1 \langle u(t) \theta(t)G_0 [(u \cdot \nabla u)' + \frac{\nabla p}{\rho_0} + u \cdot \nabla U \cdot u + \frac{g \theta}{\theta_0}]t_1\rangle
\]

(E.4)
where \((I.V.)_2 = <u(t)\theta(t)G_v(t)u(0)>\). We next add (E.4) and the transpose of (E.4) to (E.3). The result is

\[
3 <uu\theta> = I.V. - \int_0^t dt_1 \left[ B(t-t_1) + (1 + T_p) <u(t)\theta(t)G_v(t-t_1)\frac{\nabla p(t_1)}{\rho_o}> \right] \\
+ <u(t)\theta(t)G_v(t-t_1)\theta(t_1) \frac{\nabla}{\rho_o} + <u(t)\theta(t)G_v(t-t_1)u(t)> \cdot \nabla > \\
+ <u(t)u(t)G_v(t-t_1)u(t_1) > \cdot \nabla \\
\frac{B(t-t_1)}{\rho_o} = \left( 1 + T_p <u(t)\theta(t)G_v(t-t_1)[u(t_1) \cdot \nabla u(t_1)]' > \right) \\
+ <u(t)u(t)G_v(t-t_1)[u(t_1) \cdot \nabla \theta(t_1)]' >
\]

(E.5)

where \(I.V. = (I.V.)_1 + (1 + T_p)(I.V.)_2\) is henceforth ignored as small for \(t > \tau_o\) and, as for (2.5), the terms containing \(U \cdot \nabla\) are collectively vanishing since their sum in (E.5) is of the form \(U \cdot \nabla <uuG_\theta>\) when \(\nabla) << k_o \nu_o\).

The quantity \(\int dt_1 B(t-t_1)\) is evaluated in the same manner as the evaluation of \(\int dt_1 A(t-t_1)\) in App. B; i.e., cumulant expansion of the fourth-moments in \(B(t-t_1)\) in terms of second-moments, followed by time integration of second-moments as in (B.9) but with the velocity spectrum \(S_{k_b}(t-t_1)\) of the cross-spectrum \(<u^*(t)\theta(t)\nu(t_1)\nabla>\). A simplifying approximation made is \(\sigma = \nu\). The result of this evaluation of \(B\) is

\[
\int_0^t dt_1 B(t-t_1) = \tau_o A_\theta + \int_0^t dt_1 Q\theta(t-t_1) \\
\]

(E.6)

where \(A_\theta\) is defined by (2.19) and \(Q\theta(t-t_1)\) is defined in the paragraph following (2.21).

The remaining time integrations in (E.5) can all be expressed as

\[
\int_0^t dt_1 <u(t)\theta(t)G_v(t-t_1)\frac{\nabla p(t_1)}{\rho_o}> = \tau_1 <u(t)\theta(t)\frac{\nabla p(t)}{\rho_o}> 
\]

(E.7)
\[
\int_0^t dt_1 \langle \dot{y}(t) \dot{y}(t) G_\alpha(t-t_1) \dot{y}(t_1) \rangle = \tau_2 \langle \dot{y}(t) \dot{y}(t) \rangle \\
\int_0^t dt_1 \langle \dot{y}(t) y(t) G_\alpha(t-t_1) y(t_1) \rangle = \tau_3 \langle \dot{y}(t) y(t) \rangle 
\]

where \( \tau_1, \tau_2, \tau_3 \) are to be determined. This determination is similar to the evaluation of \( \int dt_1 A \) and \( \int dt_1 \langle \dot{y} G_\alpha y \rangle \) in App. B; i.e., we express the integrands in terms of (two-time) fourth-moments, expand the fourth-moments in terms of products of second-moments \( S(k; t, t_1) \) and fourth-cumulants, neglect the fourth-cumulants, and, finally, evaluate the time integrals by use of (E.8).

The result is given by
\[
\tau_1 = \begin{cases} 
  t, & t << \tau_k \\
  \tau_0, & t >> \tau_k 
\end{cases} \quad (i = 1, 2, 3)
\]

Finally, substitution of (E.6) to (E.10) in (E.5) yields (2.16) as we set out to prove.

As for (2.17) or (2.18) their derivations are similar to that of (2.16), i.e., for (2.17) we substitute (E.2) for \( \theta \) into \( \langle \dot{y} \theta \dot{y} \rangle \) followed by substitution of (2.3) for \( \dot{y} \) and proceed with same steps as (E.3) to (E.10) but with \( y(t) \) replaced by \( \theta(t) \) in the appropriate places. Similarly for (2.18), except that we begin with substitution of (E.2) into \( \langle \theta^3 \rangle \).

**PART 2**

To derive (2.22), we substitute (C.5) for \( p(S) \) into the Fourier expansion of \( \langle \dot{y} \dot{y} p(S) \theta \rangle \) to obtain
\[
\langle \dot{y} \dot{y} p(S) \theta \rangle = -\int \frac{dkdkdkdkd}{(2\pi)^2} \langle \dot{y}_k \dot{y}_k (u_{-k_c} u_{-k_d}) \rangle \cdot (ik + \alpha)(ik + 2\alpha)^2 \\
\frac{(1k + 2\alpha) \cdot (1k + 2\alpha)}{(1k + 2\alpha)(1k + 2\alpha)} \\
x \exp [-i(k_c + k_d) \cdot \chi], 
\]

(E.11)
where $k = k_a + k_b$. Upon comparison, it can be seen that the right side of (E.11) is exactly the same as the right side of (C.6) when $w_{k_c}$ is replaced by $\Theta_{k_c}$. For this reason, (E.11) is converted to (C.9) when $u_\xi$ in the left side of (C.9) is replaced by $\Theta$ and $\chi_s \cdot S(k_b)$ in the right side is replaced by $\chi_s \cdot R(k_b)$, where $R$ is defined by $R(k_b) = V^{-1} < \hat{\xi}, \Theta, u^* >$ (by its definition $\hat{R}$ is obtained by replacing $u_{k_b}$ in $S(k_b)$ by $\hat{\xi}, \Theta_{k_b}$). With these replacements, the elements of (E.11) are the same as (C.11) with $u_s$ on the left side of (C.11) replaced by $\Theta$ and $(v'_o^2)$ on the right side replaced by $<\Theta u_\xi>$ -- assuming near isotropy for $\hat{R}$ as well as $S$ to derive (C.11). As a result, the 33, 11 and 13 components of (E.11) are implied by (C.11) to be as follows:

$$<u_{\xi} \Theta p(s)/\partial x_s> = \frac{1}{15} \left( \frac{3}{2} v'_o^2 \right) \left( \frac{3}{2} <u_{\xi} \Theta> \right) a$$

$$<\Theta u_{\xi} p(s)/\partial x_s> = \frac{2}{225} \left( \frac{3}{2} v'_o^2 \right) \left( \frac{3}{2} <u_{\xi} \Theta> \right) a$$

(E.12)

$$<\Theta u_{\xi} p(s)/\partial x_s> = 0.$$  

These relations can be combined in the tensor form of (2.22)

$$(1 + t_c) <u_{\xi} p(s) \Theta> = f_{ij}^{(\Theta)}, \quad (\text{near isotropy})$$

(E.13)

where $f_{ij}$ are numerical constants determined by (E.12).

The derivation of (2.22') is similar to the derivation of (E.13). Thus, we substitute (C.5) for $p(s)$ into the Fourier expansion of $<\Theta p(s) \Theta>$ and note that it would be the same as the Fourier expansion of the $\chi_s$ component of $<u_{\xi} p(s) u_\xi>$ given by (C.6) if $w_{k_a}$ and $\hat{\xi}, u_{k_b}$ were replaced by $\Theta_{k_a}$ and $\Theta_{k_b}$, respectively.
Therefore, \( \langle Y_p(s) \theta^2 \rangle \) is given by the \( x_3 \) component of (C.9) with \( u_x u_3 \) on the left side replaced by \( \theta^2 \) and \( x_3 \cdot S \cdot x_3 \), on right side replaced by \( x_3 \cdot S R \cdot x_3 \), where \( R \) is defined after (E.11).

It follows that the elements of \( \langle Y_p(s) \theta^2 \rangle \) are given by (C.11) with \( u_x \) appropriately replaced by \( \theta \). These elements are

\[
\begin{align*}
\langle \theta^2 y_p(s) \theta x_3 \rangle & = \frac{1}{15} \left( \frac{3}{2} \langle u_x \theta \rangle^2 \right) \\
\langle \theta^2 y_p(s) \theta x_1 \rangle & = 0
\end{align*}
\]  

(E.14)

in the limit of small anisotropy. This expression can be written in the form given by (2.22'), since, for isotropy, \( \langle u_x \rangle \langle \theta^2 \rangle - \langle u_x \theta \rangle^2 \) and \( A_3(\theta^2) - 6 \langle u_x \theta \rangle^2 \), using \( 3 \langle u_x \theta \rangle \langle \theta x_3 \rangle = 2 \langle u_x \theta \rangle \). The essential point of (2.22') and (E.14) is that \( \langle \theta^2 y_p(s) \rangle \) is negligibly small.