Appendix A Proof of Lemma 2

We will make use of certain facts established in (´Esik and Rondogiannis 2014).

Suppose that \( L \) is a basic model. For each \( x \in L \) and \( \alpha < \kappa \), we define \( x|_{\alpha} = \bigcup_{\alpha} \{ x \} \). It was shown in (´Esik and Rondogiannis 2014) that \( x = x|_{\alpha} \) and \( x|_{\alpha} \leq x|_{\beta} \) for all \( \alpha < \beta < \kappa \). Moreover, \( x = \bigvee_{\alpha < \kappa} x|_{\alpha} \). Also, for all \( x, y \in L \) and \( \alpha < \kappa \), it holds \( x = y \) iff \( x|_{\alpha} = y|_{\alpha} \), and \( x \sqsubseteq y \) iff \( x|_{\alpha} \sqsubseteq y|_{\alpha} \). And if \( x \sqsubseteq y \), then \( x|_{\alpha} \leq y|_{\alpha} \).

It is also not difficult to prove that for all \( x \in L \) and \( \alpha, \beta < \kappa \), \( x|_{\alpha} \sqsubseteq x|_{\beta} \). Moreover, whenever \( X \subseteq (z)_{\alpha} \) and \( \beta \leq \alpha < \kappa \), it holds \( (\bigcup_{\alpha} X)|_{\beta} = \bigcup_{\beta} X \). And if \( \alpha < \beta \), then \( (\bigcup_{\alpha} X)|_{\beta} = \bigcup_{\alpha} X \). Finally, we will make use of the following two results from (´Esik and Rondogiannis 2014):

**Proposition 1**

Let \( A, B \) be basic models and let \( \alpha < \kappa \). If \( f_{j} : A \rightarrow B \) is an \( \alpha \)-monotonic function for each \( j \in J \), then so is \( f = \bigvee_{j \in J} f_{j} \) defined by \( f(x) = \bigvee_{j \in J} f_{j}(x) \).

**Lemma 2**

Let \( Z \) be an arbitrary set and \( L \) be a basic model. Then, \( Z \rightarrow L \) is a basic model with the pointwise definition of the order of relations \( \leq \) and \( \sqsubseteq \) for all \( \alpha < \kappa \).

Suppose that \( A, B \) are basic models. By Lemma 2 the set \( A \rightarrow B \) is also a model, where the relations \( \leq \) and \( \sqsubseteq \), \( \alpha < \kappa \), are defined in a pointwise way (see (´Esik and Rondogiannis 2014, Subsection 5.3) for details). It follows that for any set \( F \) of functions \( A \rightarrow B \), \( \bigvee F \) can be computed pointwisely. Also, when \( F \subseteq (f)_{\alpha} \) for some \( f : A \rightarrow B \), \( \bigcup_{\alpha} F \) for \( \alpha < \kappa \) can be computed pointwisely.

We want to show that whenever \( f : A \rightarrow B \), \( \beta < \kappa \) and \( F \subseteq (f)_{\beta} \) is a set of functions such that \( F \subseteq [A \rightarrow B] \), then \( \bigcup_{\beta} F \in [A \rightarrow B] \). We will make use of a lemma.

**Lemma 3**

Let \( L \) be a basic model. For all \( x, y \in L \) and \( \alpha, \beta < \kappa \) with \( \alpha \neq \beta \), \( x|_{\beta} \sqsubseteq_{\alpha} y|_{\beta} \) iff either \( \beta < \alpha \) and \( x|_{\beta} = y|_{\beta} \) (or equivalently, \( x =_{\beta} y \)), or \( \beta > \alpha \) and \( x|_{\alpha} \sqsubseteq_{\alpha} y|_{\alpha} \).

**Proof**

Let \( x|_{\beta} \sqsubseteq_{\alpha} y|_{\beta} \). If \( \beta < \alpha \) then \( x|_{\beta} = (x|_{\beta})|_{\beta} = (y|_{\beta})|_{\beta} = y|_{\beta} \). If \( \beta > \alpha \) then \( x|_{\alpha} = (x|_{\beta})|_{\alpha} \sqsubseteq_{\alpha} (y|_{\beta})|_{\alpha} = y|_{\alpha} \).

Suppose now that \( \beta < \alpha \) and \( x|_{\beta} = y|_{\beta} \). Then \( (x|_{\beta})|_{\alpha} = x|_{\beta} = y|_{\beta} = (y|_{\beta})|_{\alpha} \) and thus \( x|_{\beta} =_{\alpha} y|_{\beta} \). Finally, let \( \beta > \alpha \) and \( x|_{\alpha} \sqsubseteq_{\alpha} y|_{\alpha} \). Then \( (x|_{\beta})|_{\alpha} = x|_{\alpha} \sqsubseteq_{\alpha} y|_{\alpha} = (y|_{\beta})|_{\alpha} \) and thus \( x|_{\beta} \sqsubseteq_{\alpha} y|_{\beta} \).

**Remark 1**

Under the above assumptions, if \( \beta < \alpha \), then \( x|_{\beta} \sqsubseteq_{\alpha} y|_{\beta} \) iff \( x|_{\beta} =_{\alpha} y|_{\beta} \) iff \( x|_{\beta} = y|_{\beta} \).

**Corollary 1**

For all \( X, Y \subseteq L \) and \( \alpha \neq \beta \), \( \bigcup_{\beta} X \sqsubseteq_{\alpha} \bigcup_{\beta} Y \) iff \( \beta < \alpha \) and \( \bigcup_{\beta} X = \bigcup_{\beta} Y \), or \( \beta > \alpha \) and \( \bigcup_{\alpha} X \sqsubseteq_{\alpha} \bigcup_{\alpha} Y \).
Let \( \alpha \subseteq \kappa \) be a set of functions in \( A \) and \( \beta < \alpha \) is a set of functions in \( B \). Then \( x = \bigcup_\beta X \) and \( y = \bigcup_\beta Y \). Let \( \beta < \alpha \). Then \( x \subseteq \alpha \) iff \( x = y \). Let \( \beta > \alpha \). Then \( x \subseteq \alpha \) iff \( x|_\alpha \subseteq \alpha \). But \( x|_\alpha = \bigcup_\alpha \{ \bigcup_\beta X \} = \bigcup_\alpha X \) and similarly for \( Y \).

\[ \square \]

**Lemma 4**

Let \( A \) and \( B \) be basic models. Suppose that \( f : A \to B \) and \( F \subseteq (f)_\beta \) (where \( \beta < \kappa \)) is a set of functions in \([A \xrightarrow{m} B]\). Then \( \bigcup_\beta F \) is also \( \alpha \)-monotonic for all \( \alpha < \kappa \).

**Proof**

Suppose that \( \alpha, \beta < \kappa \) and \( x \subseteq \alpha \) in \( A \). Then \( (\bigcup_\beta F)(x) = \bigcup_\beta \{ f(x) : f \in F \} \) and \( (\bigcup_\beta F)(y) = \bigcup_\beta \{ f(y) : f \in F \} \). We have that \( f(x) \subseteq \alpha f(y) \) for all \( f \in F \). Thus, if \( \alpha = \beta \), then clearly \( \bigcup_\alpha \{ \bigcup_\beta F \}(x) \subseteq \bigcup_\alpha \{ \bigcup_\beta F \}(y) \).

Suppose that \( \beta < \alpha \). Then \( \bigcup_\beta \{ f(x) : f \in F \} = \bigcup_\beta \{ f(y) : f \in F \} \) since \( f(x) = \beta f(y) \) for all \( f \in F \). Thus, \( \bigcup_\beta F \subseteq \bigcup_\alpha (\bigcup_\beta F)(y) \).

Suppose that \( \beta > \alpha \). Then \( \bigcup_\alpha \{ f(x) : f \in F \} \subseteq \bigcup_\alpha \{ \bigcup_\beta F \}(y) \). We equip \([A \xrightarrow{m} B]\) with the order relations \( \subseteq \) and \( \subseteq_\alpha \) inherited from \( A \to B \). We have the following lemma:

**Lemma 5**

If \( A \) and \( B \) are basic models, then so is \([A \xrightarrow{m} B]\) with the pointwise definition of the order of relations \( \subseteq \) and \( \subseteq_\alpha \) for all \( \alpha < \kappa \).

**Proof**

It is proved in (Ésik and Rondogiannis 2014) that the set of functions \( A \to B \) is a basic model with the pointwise definition of the relations \( \subseteq \) and \( \subseteq_\alpha \), so that for all \( f, g : A \to B \) and \( \alpha < \kappa \), \( f \leq g \) iff \( f(x) \leq g(x) \) for all \( x \in A \) and \( f \subseteq_\alpha g \) iff \( f(x) \subseteq_\alpha g(x) \) for all \( x \in A \). It follows that for any \( F \subseteq B^A \) and \( \alpha < \kappa \), \( \bigvee F \) and \( \bigcup_\alpha F \) can also be computed pointwise:

\[
(\bigvee F)(x) = \bigvee \{ f(x) : x \in A \} \quad \text{and} \quad (\bigcup_\alpha F)(x) = \bigcup_\alpha \{ f(x) : f \in F \}.
\]

By Proposition 1 and Lemma 4, for all \( F \subseteq B^A \), if \( F \) is a set of functions \( \alpha \)-monotonic for all \( \alpha \), then \( \bigvee F \) and \( \bigcup_\alpha F \) are also \( \alpha \)-monotonic for all \( \alpha \). Since the relations \( \subseteq \) and \( \subseteq_\alpha \), \( \alpha < \kappa \) on \([A \xrightarrow{m} B]\) are the restrictions of the corresponding relations on \( B^A \), in view of Proposition 1 and Lemma 4, \([A \xrightarrow{m} B]\) also satisfies the axioms in Definition 1, so that \([A \xrightarrow{m} B]\) is a basic model.

The following lemma is shown in (Ésik and Rondogiannis 2014, Subsection 5.2) and will be used in the proof of the basis case of the next lemma:

**Lemma 6**

\((V, \leq)\) is a complete lattice and a basic model.

**Lemma 2**

Let \( D \) be a nonempty set and \( \pi \) be a predicate type. Then, \((\lbrack \pi \rbrack_D, \leq_\pi)\) is a complete lattice and a basic model.
Proof
Let $\pi$ be a predicate type. We prove that $[\pi]_D$ is a basic model by induction on the structure of $\pi$. When $\pi = \alpha$, $[\pi]_D = V$, a basic model. Suppose that $\pi$ is of the sort $\iota \to \pi'$. Then $[\pi_1]_D = D \to [\pi']_D$, which is a basic model, since $[\pi']_D$ is a model by the induction hypothesis. Finally, let $\pi$ be of the sort $\pi_1 \to \pi_2$. By the induction hypothesis, $[\pi_1]_D$ is a model for $i = 1, 2$. Thus, by Lemma 5, $[\pi]_D = [[\pi_1]_D \xrightarrow{m} [\pi_2]_D]$ is also a basic model. 

Remark 2
Let $C$ denote the category of all basic models and $\alpha$-monotonic functions. The above results show that $C$ is cartesian closed, since for all basic models $A, B$, the evaluation function $\text{eval} : (A \times B) \times A \to B$ is $\alpha$-monotonic (in both arguments) for all $\alpha < \kappa$.

Indeed, suppose that $f, g \in [A \xrightarrow{m} B]$ and $x, y \in A$ with $f \sqsubseteq_\alpha g$ and $x \sqsubseteq_\alpha y$. Then $\text{eval}(f, x) = f(x) \sqsubseteq_\alpha g(x) = \text{eval}(g, x)$ by the pointwise definition of $f \sqsubseteq_\alpha g$. Also, $\text{eval}(f, x) = f(x) \sqsubseteq_\alpha f(y) = \text{eval}(f, y)$ since $f$ is $\alpha$-monotonic.

Since $C$ is cartesian closed, for all $f \in [B \times A \xrightarrow{m} C]$ there is a unique $\Lambda f \in [B \xrightarrow{m} [A \xrightarrow{m} C]]$ in with $f(y, x) = \text{eval}(\Lambda f(y), x)$ for all $x \in A$ and $y \in B$.

Appendix B Proofs of Lemmas 3, 4 and 5

Lemma 3
Let $E : \rho$ be an expression and let $D$ be a nonempty set. Moreover, let $s$ be a state over $D$ and let $I$ be an interpretation over $D$. Then, $[E]_s(I) \in \rho_D$.

Proof
If $\rho = \iota$ then the claim is clear. Let $E$ be of a predicate type $\pi$. We prove simultaneously the following auxiliary statement. Let $\alpha < \kappa$, $\forall : \pi$, $x, y \in [\pi']_D$. If $x \sqsubseteq_\alpha y$ then $[E]_{s[\forall/x]}(I) \sqsubseteq_\alpha [E]_{s[\forall/y]}(I)$. The proof is by structural induction on $E$. We will cover only the nontrivial cases.

Case ($E_1 E_2$): The main statement follows directly by the induction hypothesis of $E_1$ and $E_2$. There are two cases. Suppose that $E_1 : \pi_1 \to \pi$ and $E_2 : \pi_1$. Then $[E_1]_s(I) \in [\pi_1 \to \pi]_D = [[\pi_1]_D \xrightarrow{m} [\pi]_D]$ and $[E_2]_s(I) \in [\pi_1]_D$ by the induction hypothesis. Thus, $[E_1]_s(I) (\{E_2\}_s(I)) \in [\pi]_D$. Suppose now that $E_1 : \iota \to \pi$ and $E_2 : \iota$. Then $[E_1]_s(I) \in [\iota \to \pi]_D = D \to [\pi]_D$ by the induction hypothesis and $[E_2]_s(I) \in [I]_D = D$. It follows again that $[E_1]_s(I) (\{E_2\}_s(I)) \in [\pi]_D$.

Auxiliary statement: Let $x, y \in [\pi']_D$ and assume $x \sqsubseteq_\alpha y$. We have by definition $\{E_1 E_2\}_{s[\forall/x]}(I) = [E_1]_{s[\forall/x]}(I) (\{E_2\}_{s[\forall/y]}(I))$, and similarly for $\{E_1 E_2\}_{s[\forall/y]}(I)$. We have $E_1 : \pi_1 \to \pi$ and $E_2 : \pi_1$ or $E_1 : \iota \to \pi$ and $E_2 : \iota$. In the first case, by induction hypothesis $[E_1]_{s[\forall/x]}(I) \in [\pi_1 \to \pi]_D$, and thus is $\alpha$-monotonic. Also, $[E_1]_{s[\forall/x]}(I) \sqsubseteq_\alpha [E_1]_{s[\forall/y]}(I)$ and $[E_2]_{s[\forall/x]}(I) \sqsubseteq_\alpha [E_2]_{s[\forall/y]}(I)$ by the induction hypothesis. It follows that $[E_1]_{s[\forall/x]}(I) (\{E_2\}_{s[\forall/y]}(I)) \sqsubseteq_\alpha [E_1]_{s[\forall/x]}(I) (\{E_2\}_{s[\forall/y]}(I)) \sqsubseteq_\alpha [E_1]_{s[\forall/x]}(I) (\{E_2\}_{s[\forall/y]}(I))$.

The second case is similar. We have $[E_1]_{s[\forall/x]}(I) \sqsubseteq_\alpha [E_1]_{s[\forall/y]}(I)$ by the induction hypothesis, moreover, $[E_2]_{s[\forall/x]}(I) = [E_2]_{s[\forall/y]}(I)$. Therefore, $[E_1]_{s[\forall/x]}(I) (\{E_2\}_{s[\forall/x]}(I)) \sqsubseteq_\alpha [E_1]_{s[\forall/x]}(I) (\{E_2\}_{s[\forall/y]}(I))$. 

Case \((\lambda V. E)\): Assume \(V : \rho_1\) and \(E : \pi_2\). We will show that \([\lambda V. E]_s(I) \subseteq [\rho_1 \rightarrow \pi_2]_D\). If \(\rho_1 = \iota\) then the result follows easily from the induction hypothesis of the first statement. Assume \(\rho_1 = \pi_1\). We show that \([\lambda V. E]_s(I) \subseteq [\pi_1 \rightarrow \pi_2]_D\), that is, \(\lambda_d.[E]_s[V/d](I)\) is \(\alpha\)-monotonic for all \(\alpha < \kappa\). That follows directly by the induction hypothesis of the auxiliary statement.

Auxiliary statement: It suffices to show that \([((\lambda V. E)]_s[V/x]\{1\}(I) \subseteq [\lambda V. E)]_s[V/x](I)\) and equivalently for every \(d\), \([E]_s[V/x][U/d](I) \subseteq [E]_s[V/x][U/d](I)\) which follows from induction hypothesis.

\[\square\]

Lemma 4
Let \(P\) be a program. Then, \(I_P\) is a complete lattice and a basic model.

Proof
From Lemma 2 we have that for all predicate types \(\pi\), \([\pi]_{U_P}\) is a complete lattice and a basic model. It follows, by Lemma 2, that for all predicate types \(\pi\), \(P_\pi \rightarrow [\pi]_{U_P}\) is also a complete lattice and a model, where \(P_\pi\) is the set of predicate constants of type \(\pi\). Then, \(I_P\) is \(\prod \pi P_\pi \rightarrow [\pi]_{U_P}\) which is also a basic model (proved in (Ésik and Rondogiannis 2014)).

\[\square\]

Lemma 5 (\(\alpha\)-Monotonicity of Semantics)
Let \(P\) be a program and let \(E : \pi\) be an expression. Let \(I, J\) be Herbrand interpretations and \(s\) be a Herbrand state of \(P\). For all \(\alpha < \kappa\), if \(I \subseteq_\alpha J\) then \([E]_s(I) \subseteq_\alpha [E]_s(J)\).

Proof
The proof is by structural induction on \(E\).

Induction Base: The cases \(V, false, true\) are straightforward since their meanings do not depend on \(I\). Let \(I \subseteq_\alpha J\). If \(E\) is a predicate constant \(p\) then we have \(I(p) \subseteq_\alpha J(p)\).

Induction Step: Assume that the statement holds for expressions \(E_1\) and \(E_2\) and let \(I \subseteq_\alpha J\).

Case \((E_1 E_2)\): It holds \([[E_1 E_2]]_s(I) = [E_1]_s(I)[[E_2]]_s(I)\). By induction hypothesis we have \([E_1]_s(I) \subseteq_\alpha [E_1]_s(J)\) and therefore \([E_1]_s(I)[[E_2]]_s(I) \subseteq_\alpha [E_1]_s(J)[[E_2]]_s(I)\). We perform a case analysis on the type of \(E_2\). If \(E_2\) is of type \(\iota\) and since \(I, J\) are Herbrand interpretations, it is clear that \([E_2]_s(I) = [E_2]_s(J)\) and therefore \([E_1]_s(I)[[E_2]]_s(I) \subseteq_\alpha [E_1]_s(I)[[E_2]]_s(I)\). By definition of application we get \([[E_1 E_2]]_s(I) \subseteq_\alpha [[E_1 E_2]]_s(J)\).

Case \((\lambda V. E_1)\): It holds by definition that \([((\lambda V. E_1)]_s(I) = \lambda_d.[E_1]_s[V/d](I)\). It suffices to show that \(\lambda_d.[E_1]_s[V/d](I) \subseteq_\alpha \lambda_d.[E_1]_s[V/d](J)\) and equivalently that for every \(d\), \([E_1]_s[V/d](I) \subseteq_\alpha [E_1]_s[V/d](J)\) which holds by induction hypothesis.

Case \((E_1 \lor \pi E_2)\): It holds \([[E_1 \lor \pi E_2]]_s(s) = \lor([[E_1]_s(I), [E_2]_s(I)]\). It suffices to show that \(\lor([[E_1]_s(I), [E_2]_s(I)] \subseteq_\alpha \lor([[E_1]_s(J), [E_2]_s(J)])\) which holds by induction hypothesis.

Case \((E_1 \land \pi E_2)\): It holds \([[E_1 \land \pi E_2]]_s(I) = \land([[E_1]_s(I), [E_2]_s(I)]\). Let \(\pi = \rho_1 \rightarrow \cdots \rightarrow \rho_n\).
Let $\rho_n \to \alpha$, it suffices to show for all $d_i \in [\rho_n]_{\mathcal{U}}$, $\bigwedge\{[E_1]_s(I) \cdot d_1 \cdots d_n, [E_2]_s(I) \cdot d_1 \cdots d_n\} \subseteq \alpha \bigwedge\{[E_1]_s(J) \cdot d_1 \cdots d_n, [E_2]_s(J) \cdot d_1 \cdots d_n\}$. We define $x_i = [E_1]_s(I) \cdot d_1 \cdots d_n$ and $y_i = [E_1]_s(J) \cdot d_1 \cdots d_n$ for $i \in \{1, 2\}$. We perform a case analysis on $\nu = \bigwedge\{x_1, x_2\}$. If $\nu < F_\alpha$ or $\nu > T_\alpha$ then $\bigwedge\{x_1, x_2\} = \bigwedge\{y_1, y_2\}$ and $\bigwedge\{x_1, x_2\} \subseteq \bigwedge\{y_1, y_2\}$. If $\nu = F_\alpha$ then $F_\alpha \leq \bigwedge\{y_1, y_2\} \leq T_\alpha$ and therefore $\bigwedge\{x_1, x_2\} \subseteq \bigwedge\{y_1, y_2\}$. If $\nu = T_\alpha$ then $\bigwedge\{y_1, y_2\} = T_\alpha$ and thus $\bigwedge\{x_1, x_2\} \subseteq \bigwedge\{y_1, y_2\}$. If $F_\alpha < \nu < T_\alpha$ then $F_\alpha < \bigwedge\{y_1, y_2\} \leq T_\alpha$ and therefore $\bigwedge\{x_1, x_2\} \subseteq \bigwedge\{y_1, y_2\}$.

Case ($\sim E_1$): Assume $\text{order}([E_1]_s(I)) = \alpha$. Then, by induction hypothesis $[E_1]_s(I) \subseteq \alpha\bigwedge\{E_1\}_{s}(J)$ and thus $\text{order}([E_1]_s(J)) \geq \alpha$. It follows that $\text{order}([E_1]_s(I)) > \alpha$ and $\text{order}([E_1]_s(J)) > \alpha$ and therefore $\text{order}([E_1]_s(I)) \subseteq \alpha\bigwedge\{E_1\}_{s}(J)$.

Case ($\exists V.E_1$): Assume $V$ is of type $\rho$. It holds $\bigwedge\{E_1\}_{s}(I) = \bigwedge\{E_1\}_{s}(V/d\hat{d}(J))$. It suffices to show $\bigwedge\{E_1\}_{s}(V/d\hat{d}(I)) \subseteq \bigwedge\{E_1\}_{s}(V/d\hat{d}(J))$ which holds by induction hypothesis and Axiom 4.

**Appendix C Proof of Theorem 2**

We start by providing some necessary background material from (Ésik and Rondogiannis 2014) on how the $\bigcap$ operation on a set of interpretations is actually defined.

Let $x \in V$. For every $X \subseteq (x)_\alpha$ we define $\bigcap_\alpha X$ as follows: if $X = \emptyset$ then $\bigcap_\alpha X = T_\alpha$, otherwise

$$\bigcap_\alpha X = \begin{cases} \bigwedge X & \text{order}(\bigwedge X) \leq \alpha \\ T_{\alpha + 1} & \text{otherwise} \end{cases}$$

Let $P$ be a program, $I \in \mathcal{I}_P$ be a Herbrand interpretation of $P$ and $X \subseteq (I)_\alpha$. For all predicate constants $p$ in $P$ of type $\rho_1 \to \cdots \to \rho_n \to \alpha$ and $d_i \in [\rho_n]_{\mathcal{U}}$ and for all $i = \{1, \ldots, n\}$, it holds $\bigcap_\alpha X = (\bigcap_\alpha X)(p) \cdot d_1 \cdots d_n = \bigcap_\alpha \{I(p) \cdot d_1 \cdots d_n : I \in X\}$.

Let $X$ be a nonempty set of Herbrand interpretations. By Lemma 4 we have that $\mathcal{I}_P$ is a complete lattice with respect to $\subseteq$ and a basic model. Moreover, by Lemma 1 it follows that $\mathcal{I}_P$ is also a complete lattice with respect to $\subseteq$. Thus, there exist the least upper bound and greatest lower bound of $X$ for both $\subseteq$ and $\subseteq$. We denote the greatest lower bound of $X$ as $\bigwedge X$ and $\bigcap X$ with respect to relations $\subseteq$ and $\subseteq$ respectively. Then, $\bigcap X$ can be constructed in an symmetric way to the least upper bound construction described in (Ésik and Rondogiannis 2014). More specifically, for each ordinal $\alpha < \kappa$ we define the sets $X_\alpha, Y_\alpha \subseteq X$ and $x_\alpha \in \mathcal{I}_P$, which are then used in order to obtain $\bigcap X$.

Let $Y_0 = X$ and $x_0 = \bigcap_\alpha Y_0$. For every $\alpha$, with $0 < \alpha < \kappa$ we define $X_\alpha = \{x \in X : \forall \beta \leq \alpha x = x_\alpha \}$, $Y_\alpha = \bigcap_\beta < \alpha X_\beta$; moreover, $x_\alpha = \bigcap_\alpha Y_\alpha$ if $Y_\alpha$ is nonempty and otherwise.

Finally, we define $x_\alpha = \bigwedge_\beta < \alpha x_\beta$. In analogy to the proof of (Ésik and Rondogiannis 2014) for the least upper bound it can be shown that $x_\alpha = \bigwedge X$ with respect to the relation $\subseteq$. Moreover, it is easy to prove that by construction it holds $x_\alpha = x_\alpha x_\beta$ and $x_\beta \geq x_\alpha$ for all $\beta < \alpha$.

**Lemma 7**

Let $P$ be a program, $\alpha < \kappa$ and $M_\alpha$ be a Herbrand model of $P$. Let $M \subseteq (M_\alpha)_\alpha$ be a nonempty set of Herbrand models of $P$. Then, $\bigcap_\alpha M$ is also a Herbrand model of $P$. 


Proof
Assume $\prod_\alpha M$ is not a model. Then, there exists a clause $p \leftarrow E$ in $P$ and $d_i \in [p_i]_D$ such that $[E](\prod_\alpha M) d_1 \cdots d_n > (\prod_\alpha M)(p) d_1 \cdots d_n$. Since for every $N \in M$ we have $\prod_\alpha M \subseteq_\alpha N$, using Lemma 5 we conclude $[E](\prod_\alpha M) \subseteq_\alpha [E](N)$. Let $x = \prod_\alpha \{N(p) d_1 \cdots d_n : N \in M\}$.

If order $x = \alpha$ then $x = \bigwedge \{N(p) d_1 \cdots d_n : N \in M\}$. If $x = T_\alpha$ then for all $N \in M$ we have $N(p) d_1 \cdots d_n = T_\alpha$. Moreover, $[E](\prod_\alpha M) d_1 \cdots d_n > T_\alpha$ and by $\alpha$-monotonicity we have $[E](N) d_1 \cdots d_n > T_\alpha$ for all $N \in M$. Then, $N(p) d_1 \cdots d_n < [E](N) d_1 \cdots d_n$ and therefore $N$ is not a model (contradiction). If $x = F_\alpha$ then there exists $N \in M$ such that $N(p) d_1 \cdots d_n = F_\alpha$ and since $N$ is a model we have $[E](N) d_1 \cdots d_n \leq F_\alpha$. But then, it follows $[E](\prod_\alpha M) d_1 \cdots d_n \leq F_\alpha$ and $[E](\prod_\alpha M) d_1 \cdots d_n < x$ (contradiction).

If order $x < \alpha$ then $x = M_\alpha(p) d_1 \cdots d_n$. If $x = T_\beta$ then $[E](\prod_\alpha M) d_1 \cdots d_n > T_\beta$ and $[E](M_\alpha) d_1 \cdots d_n > T_\beta$. Then, we have $M_\alpha(p) d_1 \cdots d_n \not< [E](M_\alpha)$ and thus $M_\alpha$ is not a model of $P$ (contradiction). If $x = F_\beta$ then $[E](M_\alpha) d_1 \cdots d_n \leq F_\beta$ and by $\alpha$-monotonicity $[E](\prod_\alpha M) d_1 \cdots d_n \leq F_\beta$. Therefore, $[E](\prod_\alpha M) d_1 \cdots d_n < x$ (contradiction).

If order $x > \alpha$ then $x = T_{\alpha+1}$ and there exists model $N \in M$ such that $N(p) d_1 \cdots d_n < T_\alpha$. Moreover, we have $[E](\prod_\alpha M) d_1 \cdots d_n \geq T_\alpha$ and by $\alpha$-monotonicity we conclude $[E](N) d_1 \cdots d_n \geq T_\alpha$. But then, $[E](N) d_1 \cdots d_n > N(p) d_1 \cdots d_n$ and therefore $N$ is not a model of $P$ (contradiction).

In the following, we will make use of the following lemma that has been shown in (Ésik and Rondogiannis 2014, Lemma 3.18):

Lemma 8
If $\alpha \leq \kappa$ is an ordinal and $(x_\beta)_{\beta < \alpha}$ is a sequence of elements of $L$ such that $x_\beta =_\beta x_\gamma$ and $x_\beta \leq x_\gamma$ ($x_\beta \geq x_\gamma$) whenever $\beta < \gamma < \alpha$, and if $x = \bigvee_{\beta \leq \alpha} x_\beta$ ($x = \bigwedge_{\beta \leq \alpha} x_\beta$), then $x_\beta =_\beta x$ holds for all $\beta < \alpha$.

Lemma 9
Let $(M_\alpha)_{\alpha < \kappa}$ be a sequence of Herbrand models of $P$ such that $M_\alpha =_\alpha M_\beta$ and $M_\beta \leq M_\alpha$ for all $\alpha < \beta < \kappa$. Then, $\bigwedge_{\alpha < \kappa} M_\alpha$ is also a Herbrand model of $P$.

Proof
Let $M_\infty = \bigwedge_{\alpha < \kappa} M_\alpha$ and assume $M_\infty$ is not a model of $P$. Then, there is a clause $p \leftarrow E$ and $d_i \in [p_i]_D$ such that $[E](M_\infty) d_1 \cdots d_n > M_\infty(p) d_1 \cdots d_n$. We define $x_\alpha = M_\alpha(p) d_1 \cdots d_n$, $x_\infty = M_\infty(p) d_1 \cdots d_n$, $y_\alpha = [E](M_\alpha) d_1 \cdots d_n$ and $y_\infty = [E](M_\infty) d_1 \cdots d_n$ for all $\alpha < \kappa$. It follows from Lemma 8 that $M_\infty =_\alpha M_\alpha$ and thus $x_\infty =_\alpha x_\alpha$ for all $\alpha < \kappa$. Moreover, using $\alpha$-monotonicity we also have $[E](M_\infty) =_\alpha [E](M_\alpha)$ and thus $y_\infty =_\alpha y_\alpha$ for all $\alpha < \kappa$. We distinguish cases based on the value of $x_\infty$.

Assume $x_\infty = T_\delta$ for some $\delta < \kappa$. It follows by assumption that $y_\infty > T_\delta$. Then, since $x_\infty =_\delta x_\delta$ it follows $x_\delta = T_\delta$. Moreover, since $y_\infty =_\delta y_\delta$ and order($y_\delta$) < $\delta$ it follows $y_\delta = y_\infty > T_\delta$. But then, $y_\delta > x_\delta$ (contradiction since $M_\delta$ is a model by assumption).

Assume $x_\infty = F_\delta$ for some $\delta < \kappa$. Then, since $x_\infty =_\delta x_\delta$ it follows $x_\delta = F_\delta$. Then, since $M_\delta$ is a model it follows $y_\delta \leq x_\delta$ and thus $y_\delta \leq F_\delta$. But then, since $y_\infty =_\delta y_\delta$ it follows $y_\delta = y_\infty \leq F_\delta$. Therefore, $y_\infty \leq x_\infty$ that is a contradiction to our assumption that $y_\infty > x_\infty$.

Assume $x_\infty = 0$. Then, $y_\infty > x_\infty = 0$. Let $y_\infty = T_\beta$ for some $\beta < \kappa$. Then, since
Let \( P \) be a program and \( M \) be a nonempty set of Herbrand models of \( P \). Then, \( \bigcap M \) is also a Herbrand model of \( P \).

**Proof**

We use the construction for \( \bigcap M \) described in the beginning of this appendix. More specifically, we define sets \( M_\alpha, Y_\alpha \subseteq M \) and \( M_\alpha \in T_P \). Let \( Y_0 = M \) and \( M_0 = \bigcap_0 M_0 \).

For every \( \alpha > 0 \), let \( M_\alpha = \{ M \in M : \forall \beta \leq \alpha \ M =_\alpha M_\beta \} \) and \( Y_\alpha = \bigcap_{\beta < \alpha} M_\beta \); moreover, \( M_\alpha = \bigcap_\alpha Y_\alpha \) if \( Y_\alpha \) is nonempty and \( M_\alpha = \bigwedge_{\beta < \alpha} M_\beta \) if \( Y_\alpha \) is empty. Then, \( \bigcap M = \bigwedge_{\alpha < \kappa} M_\alpha \). It is easy to see that \( M_\alpha =_\alpha M_\alpha \) and \( M_\beta \supseteq M_\alpha \) for all \( \beta < \alpha \).

We distinguish two cases. First, consider the case when \( Y_\alpha \) is nonempty for all \( \alpha < \kappa \).

Then, \( M_\alpha = \bigcap Y_\alpha \) and by Lemma 7 it follows that \( M_\alpha \) is a model of \( P \). Moreover, by Lemma 9 we get that \( M_\infty = \bigwedge_{\alpha < \kappa} M_\alpha \) is also a model of \( P \).

Consider now the case that there exists a least ordinal \( \delta < \kappa \) such that \( Y_\delta \) is empty. It holds (see (Esik and Rondogiannis 2014)) that \( \bigcap M = \bigwedge_{\alpha < \delta} M_\delta \). Suppose \( \bigcap M \) is not a model of \( P \). Then, there is a clause \( \rho \in E \), a Herbrand state \( s \) and \( d_i \in [\rho]_D \) such that \( \langle E \rangle(M_\infty) d_1 \cdots d_n > \bigcap M(p) d_1 \cdots d_n \). We define \( x_\infty = M_\alpha(p) d_1 \cdots d_n, x_\infty = \bigcap M(p) d_1 \cdots d_n, y_\infty = \langle E \rangle(M_\infty) d_1 \cdots d_n \) for all \( \beta \leq \alpha \) and \( \delta < \kappa \).

We distinguish cases based on the value of \( x_\infty \).

Assume \( x_\infty = T_\infty \). It follows by assumption that \( y_\infty > x_\infty = T_\infty \).

Then, by Lemma 8 it holds that \( M_\infty =_\beta M_\beta \) and we get \( x_\infty =_\beta x_\infty \) and therefore \( x_\infty = T_\infty \). Moreover, by \( \alpha \)-monotonicity we get \( \langle E \rangle(M_\infty) d_1 \cdots d_n =_\beta \langle E \rangle(M_\beta) d_1 \cdots d_n \) and it follows that \( y_\infty =_\beta y_\infty \). Moreover, since \( y_\infty > T_\infty \) it follows \( y_\infty =_\beta y_\infty \) and \( y_\infty >_\beta y_\infty \). Since \( Y_\beta \) is not empty by assumption we have that \( M_\beta = \bigcap_\beta Y_\beta \) and by Lemma 7 we get that \( M_\beta \) is a model of \( P \) (contradiction since \( y_\infty > x_\infty \)).

Assume \( x_\infty = F_\infty \) for some \( \beta < \delta \). Then, by Lemma 8 it holds \( M_\infty =_\beta M_\beta \) and therefore \( x_\infty =_\beta x_\infty \). It follows \( x_\infty = F_\beta \). Moreover, since \( Y_\beta \) is nonempty by assumption and by Lemma 7 it follows that \( M_\beta = \bigcap_\beta Y_\beta \) is a model of \( P \) and thus \( y_\beta \leq x_\beta = F_\beta \). By \( \alpha \)-monotonicity we get \( \langle E \rangle(M_\infty) =_\beta \langle E \rangle(M_\beta) \) and therefore \( y_\infty =_\beta y_\infty \). It follows \( y_\infty \leq F_\beta = x_\infty \) (contradiction to the initial assumption \( y_\infty > x_\infty \)).

Assume \( x_\infty = T_\infty \). By assumption we have \( y_\infty > x_\infty = T_\infty \). Then, let \( y_\infty = T_\infty \) for some \( \gamma < \delta \). By Lemma 8 it holds \( M_\infty =_\gamma M_\gamma \) and by \( \alpha \)-monotonicity it follows \( \langle E \rangle(M_\infty) =_\gamma \langle E \rangle(M_\gamma) \) and thus \( y_\infty =_\gamma y_\gamma \). It follows that \( y_\gamma = T_\gamma \). Moreover, since \( \gamma < \delta \) we know by assumption that \( Y_\gamma \) is nonempty and therefore \( M_\gamma = \bigcap Y_\gamma \). By Lemma 7 \( M_\gamma \) is a model of \( P \). It follows \( T_\gamma = y_\gamma \leq x_\gamma \), that is, \( x_\gamma = T_\infty \) for some \( \beta \leq \gamma < \delta \). Moreover, since \( x_\infty =_\gamma x_\gamma \) it follows \( x_\infty = T_\infty \) that is a contradiction (since by assumption \( x_\infty = T_\infty \)).

Assume \( x_\infty = F_\infty \). This case is not possible. Recall that \( Y_\alpha \) is not empty for all \( \alpha < \delta \) and thus \( M_\alpha = \bigcap Y_\alpha \). By the definition of \( \bigcap \) we observe that either \( \bigcap_\alpha Y_\alpha \leq F_\alpha \) or \( \bigcap_\alpha Y_\alpha \geq T_\alpha + 1 \). Then, since \( M_\infty = \bigwedge_{\alpha < \delta} M_\alpha \) it is not possible to have \( x_\infty = F_\infty \).

Assume \( x_\infty = 0 \). This case does not arise. Again, \( Y_\alpha \) is not empty for all \( \alpha < \delta \) and
thus $M_\alpha = \bigcap_\alpha Y_\alpha$. Moreover, by definition of $\bigcap_\alpha$, $x_\alpha \neq 0$ for all $\alpha < \delta$. Moreover, since $M_\infty = \bigwedge_{\alpha < \delta} M_\alpha$ and since $\delta < \kappa$ it follows that the limit can be at most $T_\delta$. 

Appendix D Proofs of Lemmas 6, 7 and Theorem 3

Lemma 6
Let $P$ be a program. For every predicate constant $p : \pi \in P$ and $I \in \mathcal{I}_P$, $T_P(I)(p) \in \llbracket \pi \rrbracket_{U_P}$.

Proof
It follows from the fact that $\llbracket \pi \rrbracket_{U_P}$ is a complete lattice (Lemma 2).

Lemma 7
Let $P$ be a program. Then, $T_P$ is $\alpha$-monotonic for all $\alpha < \kappa$.

Proof
Follows directly from Lemma 5 and Proposition 1.

Lemma 10
Let $P$ be a program. Then, $M \in \mathcal{I}_P$ is a model of $P$ if and only if $T_P(M) \leq \mathcal{I}_P M$.

Proof
An interpretation $I \in \mathcal{I}_P$ is a model of $P$ iff $\llbracket E \rrbracket(I) \leq \pi I(p)$ for all clauses $p \leftarrow \pi E$ in $P$ iff $\bigvee_{(p \leftarrow \pi E) \in P} \llbracket E \rrbracket(I) \leq \mathcal{I}_P I(p)$ iff $T_P(I) \leq \mathcal{I}_P I$. 

Proposition 11
Let $D$ be a nonempty set, $\pi$ be a predicate type and $x, y \in \llbracket \pi \rrbracket_D$. If $x \leq_\pi y$ and $x =_\beta y$ for all $\beta < \alpha$ then $x \sqsubseteq_\alpha y$.

Proof
The proof is by structural induction on $\pi$.

Induction Basis: If $x =_\beta y$ for all $\beta < \alpha$ then either $x = y$ or $\text{order}(x), \text{order}(y) \geq \alpha$. If $x = y$ then $x \sqsubseteq_\alpha y$. Suppose $x \neq y$. If $\text{order}(x), \text{order}(y) > \alpha$ then $x =_\alpha y$. If $x = F_\alpha$ then clearly $x \sqsubseteq_\alpha y$. If $x = T_\alpha$ then $T_\alpha \leq y$ and therefore $y = T_\alpha$. The case analysis for $y$ is similar.

Induction Step: Assume that the statement holds for $\pi$. Let $f, g \in \llbracket \rho \rightarrow \pi \rrbracket_D$ and $\alpha < \kappa$. For all $x \in \llbracket \rho \rrbracket_D$ and $\beta < \alpha$, $f(x) \leq g(x)$ and $f(x) =_\beta g(x)$. It follows that $f(x) \sqsubseteq_\alpha g(x)$. Therefore, $f \sqsubseteq_\alpha g$. 

Proposition 12
Let $P$ be a program and $I, J$ be Herbrand interpretations of $P$. If $I \leq \mathcal{I}_P J$ and $I =_\beta J$ for all $\beta < \alpha$ then $I \sqsubseteq_\alpha J$.

Proof
Let $I, J \in \mathcal{I}_P$ and $\alpha < \kappa$. For all predicate constants $p$ and $\beta < \alpha$, $I(p) \leq J(p)$ and $I(p) =_\beta J(p)$. It follows by Proposition 11 that $I(p) \sqsubseteq_\alpha J(p)$ and therefore, $I \sqsubseteq_\alpha J$. 

Lemma 13
Let $P$ be a program. If $M$ is a model of $P$ then $T_P(M) \subseteq M$.

Proof
It follows from Lemma 10 that if $M$ is a Herbrand model of $P$ then $T_P(M) \leq_{I_P} M$. If $T_P(M) = M$ then the statement is immediate. Suppose $T_P(M) <_{I_P} M$ and let $\alpha$ denote the least ordinal such that $T_P(M) =_{\alpha} M$ does not hold. Then, $T_P(M) =_{\beta} M$ for all $\beta < \alpha$. Since $T_P(M) <_{I_P} M$, by Proposition 12 it follows that $T_P(M) \sqsubseteq_{\alpha} M$. Since $T_P(M) =_{\alpha} M$ does not hold, it follows that $T_P(M) \sqsubseteq \alpha M$. Therefore $T_P(M) \sqsubseteq M$. 

Theorem 3 (Least Fixed Point Theorem)
Let $P$ be a program and let $\mathcal{M}$ be the set of all its Herbrand models. Then, $T_P$ has a least fixed point $M_P$. Moreover, $M_P = \bigcap \mathcal{M}$.

Proof
It follows from Lemma 7 and Theorem 1 that $T_P$ has a least pre-fixed point with respect to $\sqsubseteq$ that is also a least fixed point. Let $M_P$ be that least fixed point of $T_P$, i.e., $T_P(M_P) = M_P$. It is clear from Lemma 10 that $M_P$ is a model of $P$, i.e., $M_P \in \mathcal{M}$. Then, it follows $\bigcap \mathcal{M} \subseteq M_P$. Moreover, from Theorem 2 it is implied that $\bigcap \mathcal{M}$ is a model and thus from Lemma 13, $\bigcap \mathcal{M}$ is a pre-fixed point of $T_P$ with respect to $\subseteq$. Since $M_P$ is the least pre-fixed point of $P$, $M_P \subseteq \bigcap \mathcal{M}$ and thus $M_P = \bigcap \mathcal{M}$. 