SYMMETRIC INVERSE SEMIGROUPS
(Mathematical Surveys and Monographs 46)

By STEPHEN LIPSCOMB: 166 pp., US$49.00, ISBN 0 8218 0627 0

Inverse semigroups bear the same relation to `partial permutations' as groups do to permutations, and the analogue of the symmetric group $S_n$ is what Lipscomb calls the symmetric inverse semigroup $C_n$, consisting of all partial one-one mappings (`charts', in his terminology) of a set of $n$ symbols. In the introductory chapter, Lipscomb quotes a remark made by Gracinda Gomes and the reviewer in 1987 (see [I]):

Since the theory of inverse semigroups is now extensive enough to have been the subject of a substantial book by Petrich [2], it is perhaps rather surprising that very little has been written on the symmetric inverse semigroup.

Whether as a result of that comment or not, much has been added to our knowledge of $C_n$ since 1987, but many questions remain, and Lipscomb's book, by bringing together many of the known results into a unified whole, makes it more likely that rapid progress will be made.

To some extent, the book is a piece of advocacy for a particular notational approach. His `path' notation, in which, for example, the partial permutation

$$
\begin{pmatrix}
1 & 2 & 4 & 5 & 6 \\
2 & 3 & 5 & 4 & 7
\end{pmatrix}
$$

of $\{1,2,\ldots,7\}$ is denoted by $(123][45)(67)$, is certainly a flexible device, and throughout the book he makes a powerful case for it to become as standard as the Cayley notation it generalizes.

Semigroup theorists are interested also in maps that are not one-one, and Lipscomb extends his path notation to cover more general partial selfmaps of a set. Thus, for example, the partial map

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 1 & 1 & 3 & 3 & 7
\end{pmatrix}
$$

is written $(12)(431\rangle(531\rangle(67)$, where the angle bracket denotes a point of entry to a cycle. This is certainly less cumbersome in print than drawing a digraph, but time will tell whether it is a notation that can be used in a truly effective way.

This book is a most welcome addition to the literature of semigroup theory. The existence of a standard reference and a standard notation should ensure that further work on symmetric inverse semigroups will take place in a more organized and coherent way than has hitherto been possible.

References


J. M. HOWIE
In the last ten years many mathematical physicists, to whom talk of coproducts, antipodes and bialgebras had hitherto been the jabbering of an alien tribe, and maybe many pure mathematicians also, have come to realise that they had been speaking the prose of Hopf algebra theory all their lives without realising it. A coproduct on an algebra $A$, formally defined as a map $\Delta: A \rightarrow A \otimes A$, is nothing but a rule for extending a concept from a single system to a pair of systems: for a physicist, for example, it is the formula for the centre of mass of a pair of particles in terms of their individual positions, or for the total momentum in terms of their individual momenta; for an algebraist, it is the rule for defining the action of a group $G$ or a Lie algebra $\mathfrak{g}$ on the tensor product of a pair of $C^\ast G$- or $U(\mathfrak{g})$-modules. The occasion for learning to cast such familiar ideas in such unfamiliar language has been the advent of quantum groups, giving new versions of the old ideas which bear a relation to them reminiscent of the relation of quantum to classical mechanics. All of the examples mentioned above have a ‘classical’ aspect arising from a commutative operation somewhere in the structure—addition of momenta, pointwise multiplication of functions on configuration space or phase space or a group. In the corresponding quantum group, this commutative algebra is replaced by a non-commutative one. The discovery which launched the theory was that it was possible to do this while retaining the compatibility between, for example, the algebra of functions on a group and the structure of the group—a compatibility expressed in this case by the homomorphism property of the coproduct $\Delta: A \rightarrow A \otimes A \cong A$, where $A$ is the algebra of functions on the group $G$, $A \otimes$ is the algebra of functions on $G \times G$, and $\Delta(g, h) = f(gh)$. In a quantum group the commutative algebra of functions is replaced by a non-commutative algebra whose elements, therefore, cannot logically be functions on anything, but which can be treated psychologically as functions because of the Hopf algebra axioms which they satisfy. The same philosophy informs the idea of non-commutative geometry, whose development has been closely linked to that of quantum groups.

Shahn Majid has been one of the most active figures in this field since its inception. In this book he gives a systematic account of the subject, starting by introducing the reader to the classical theory of bialgebras and Hopf algebras. Quantum groups are non-commutative non-cocommutative Hopf algebras in which the non-commutativity is kept under control by the property of quasitriangularity. Majid devotes a chapter to this general concept before turning to the standard examples of quantised enveloping algebras and matrix quantum groups, the latter with its attendant quantum form of linear algebra. At this point the theoretical development pauses to allow two chapters to be devoted to applications of Hopf algebras to combinatorics, probability theory and (speculatively) quantum mechanics and quantum gravity. Returning to the main stream of quantum group theory, Majid explains Drinfel’d’s ‘quantum double’ construction and the infinitesimal (or ‘semiclassical’) theory, in which we see something very like the process of quantising a classical mechanical system, and the reason for the name ‘quantum group’ emerges at last. The last two chapters are concerned with the category theory which is the natural setting for much of quantum group theory, allowing generalisation into
braided categories (which can be seen as a further extension of non-commutativity or, complementarily, revealing the controlled non-commutativity of quantum groups as a form of commutativity in an unfamiliar ‘braided’ world).

Every chapter is equipped with historical notes, which leave the reader in no doubt how much of this theory is, in his favourite phrase, ‘due to the author’. The book does not provide homogeneous coverage of the field—for that one must turn to Chari and Pressley’s Guide [I]. It does, however, trace a coherent, detailed path through the field, with full, followable proofs and clear, readable explanations. It is a pleasure to follow this path with a guide who, unlike many mathematical authors, does not shirk his duty to write, and who shares with his readers his general understanding of and above all his enthusiasm for his subject.

Reference


Tony Sudbery

**MULTIDIMENSIONAL HYPERGEOMETRIC FUNCTIONS**
**AND REPRESENTATION THEORY OF**
**LIE ALGEBRAS AND QUANTUM GROUPS**
(Advanced Series in Mathematical Physics 21)


At the centre of the role of quantum groups in conformal field theory uncovered a few years ago is a deep relation between hypergeometric functions associated to the multivalued function \( \prod (t_i - t_j)^{a_{ij}} \), quantum groups \( U_q(\mathfrak{g}) \) and Kac–Moody Lie algebras with Cartan datum \( a_{ij} \). Coming out of physics, the ideas here could easily take a great deal of infrastructure to explain.

Varchenko’s monograph, which develops his work with V. Schechtman, provides a self-contained account of this relationship in the form of a line of rigorous theorems and minimal digression. Although intentionally not aiming to survey the whole field or even to outline the bigger picture, the monograph succeeds very well as a clear account of this part of the overall subject. It will be very useful for any mathematician wanting a solid fixed point in his or her understanding of the wider literature on conformal field theory and quantum groups.

Before turning to the specific topic of the book, it is necessary to explain some of the items mentioned in the title. First, quantum groups and related topics. These are generalisations of our usual concepts of groups in which the ‘coordinate ring’ of the group is allowed to be noncommutative. Equivalently, they generalise our notion of ‘enveloping algebra’ of a Lie algebra in such a way that the implicit coproduct involved in the derivation rule (by which Lie algebra representations extend to tensor products) is generalised (it is allowed to be noncocommutative). From a mathematical point of view, the key properties may be expressed through the notion of a Hopf algebra. Basically, this is an algebra \( H \) equipped with a homomorphism \( \Delta: H \to H \otimes H \) (the coproduct) which makes \( H \) also into a coalgebra. A coalgebra is like an algebra but with all arrows reversed.
Actually, like most of the recent several books with ‘quantum groups’ in the title, Varchenko’s work is only really concerned with the particular Drinfeld–Jimbo quantum groups $U^\text{DJ}_q(g)$ associated to the root system of a semisimple Lie algebra $g$. In fact, Varchenko uses for the most part a simplified version of these quantum groups, which he calls $U_q(g)$, in which the $q$-analogue of the Serre relations among simple root vectors is left out (and with $a_{ij}$ not necessarily coming from a Cartan matrix). On the other hand, at about the same time as these $U^\text{DJ}_q(g)$ were being discovered in the mid 1980s, other rich classes of quite different quantum groups were also being discovered in mathematical physics from another point of view, that of the ‘noncommutative geometry’ of certain quantum systems. For example, every semisimple $g$ has an associated bicrossproduct quantum group $\mathbb{C}(G)\rhd U(g_c)$ associated to the Iwasawa decomposition $G_c = G_t G_o$, where $g_c$ is the complexification of $g$ and the associated groups are denoted in upper-case. For a wider picture of quantum groups, see the reviewer’s text [2].

Associated to a subalgebra $U_q(n_+ \subset U_q(g)$, Varchenko builds two Hochschild complexes. The first is associated to $U_q(n_+)$ with values in the tensor product of Verma modules, while the other is associated to the dual of $U_q(n_+)$ and the dual Verma modules. The two complexes are related by a homomorphism. This occupies the middle portion of the book. On the other hand, the first portion of the book studies the combinatorics associated to the function $\prod (t_i - t_j)^{a_{ij}/\kappa}$ on $\mathbb{C}^n$. Its branches define a local system over the complement of the diagonal hyperplanes, and Varchenko builds two complexes with values in this. Essentially, one computes the homology of the complement, while the other computes the homology of the affine space modulo the complement; the two complexes are related by a homomorphism. Actually, Varchenko explains this for general configurations of affine hyperplanes before specialising to the diagonal hyperplanes and to certain ‘discriminantal’ configurations. Diagonal configurations come up in conformal field theory when one looks at the operator product of fields, which is generally singular on diagonals.

Another important trend since the 1980s (and playing a central role in conformal field theory) has been the emergence of infinite-dimensional Kac–Moody Lie algebras $g$. It is natural to include such objects when Lie algebras and their enveloping algebras are defined through simple generators and a Cartan datum $a_{ij}$. The final portion of Varchenko’s book covers this aspect of the theory. Thus, the function $\prod (t_i - t_j)^{a_{ij}/\kappa}$ provides multivalued differential forms, the integration of which against certain cycles provides multidimensional hypergeometric functions. The latter can be understood as solutions of the Knizhnik–Zamolodchikov (KZ) equation, and make sense for a general Kac–Moody Lie algebra. Varchenko considers the standard complex of the Lie algebra $n_+ \subset g$ with values in tensor products of Verma modules. In fact, $n_+^*$ also has a natural Lie algebra structure and provides a second complex, related by a homomorphism to the first. The operator in the KZ equation is then developed as a connection on bundles associated to each of these complexes.

Of course, Varchenko also establishes precise identifications between these settings, thereby achieving the aim of the book. The overall treatment has a different flavour from Drinfeld’s theory of quasi-Hopf algebras and the KZ equation (see some of the recent introductions to quantum groups, such as the texts of Chari and Pressley [1] or Schnider and Sternberg [3]). In effect, Varchenko’s treatment takes more from conformal field theory and the theory of vertex operator algebras. Either way, it should be evident that this is an advanced book and will take some work to read, but will be correspondingly rewarding. The book is elegantly structured and sticks closely
to the point, and is also fairly down to earth (there are more categorical routes to some of the theorems). This means that as well as serving as an excellent specialist monograph, it should also be useful as a first exposure to these topics for anyone who likes to learn a subject through the study of a concrete problem.

References


S. Majid

**ERGODICITY FOR INFINITE DIMENSIONAL SYSTEMS**

(London Mathematical Society Lecture Note Series 229)


The study of stochastic partial differential equations has been slowly gathering momentum among both mathematicians and practitioners. Professors Da Prato and Zabczyk have been at the forefront of mathematical developments through their systematic approach to such equations considered as stochastic differential equations on Hilbert spaces. As the title of this book suggests, this approach includes other types of stochastic evolutions: as well as semilinear p.d.e. with space time white noise forcing terms, they discuss stochastic delay equations, non-linear heat equations on domains subjected to boundary noise, and evolution equations coming from classical and quantum spin systems. Their Encyclopedia of Mathematics and its Applications volume *Stochastic equations in infinite dimensions* (Cambridge University Press, 1992) laid the foundations of the theory but this, most welcome, volume carries it much further, and brings it to sparkling life by showing both the mathematical richness of the functional analytic techniques and their power in applications to a variety of different equations.

The ergodic theory is basically the ergodic theory of Markov processes going back to Doob, and introduced in the penultimate chapter of Yosida’s *Functional analysis*. The thorough treatment in Part I of the present volume will also be of value to those who have no intention of considering anything infinite dimensional. After a quick discussion of basic concept for general dynamical situations, the relationships with Markov semigroups \( \{P_t : t \geq 0\} \) are established. A sample result here is that if \( \mu \) is an invariant measure for such a semigroup, then with some additional hypotheses \( \mu \) is strongly mixing if and only if \( P_t \phi \) converges in \( L^2(\mu) \) to the integral of \( \phi \) for all \( \phi \) in \( L^2(\mu) \). Parts II and III are based on recent results, particularly by the authors. After a brief review of the theory of stochastic differential equations in Hilbert spaces, Part II goes on to discuss the main theme: the existence, uniqueness and ergodic properties of invariant measures for such equations. Basic to these are the concept of irreducibility and the strong Feller property described in Part I, and the use of dissipativity hypotheses. The Markov semigroups now are semigroups acting on functions on an infinite dimensional Hilbert space, and their generators are partial
differential operators in infinitely many variables. Some of the more remarkable results are that these semigroups often have smoothing properties, similar to those with elliptic generators in finite dimensions.

The final section, Part III, describes how these techniques apply to specific models. These models include stochastic versions of reaction diffusion equations, Burger's equation and 2-dimensional Navier–Stokes equations, as well as those mentioned above. There is a wealth of references, and the reader will soon realise that there is an abundance of mathematically intriguing work to be done in this rather new area. This is a book which both fills an expository gap and serves as a springboard into the latest research into stochastic evolution equations.

K. D. Elworthy

INTRODUCTION TO SYMPLECTIC TOPOLOGY
(Oxford Mathematical Monographs)

By Dusa McDuff and Dietmar Salamon: 425 pp., £46.95, ISBN 0 19 851177 9
(Clarendon Press, 1995).

J-HOLOMORPHIC CURVES AND QUANTUM COHOMOLOGY
(University Lecture Series 6)

By Dusa McDuff and Dietmar Salamon: 205 pp., US$24.00, ISBN 0 8218 0332 8
(American Mathematical Society, 1994).

The classical differential geometer studies the properties of a smooth manifold equipped with a connection and a Riemannian metric. If we abandon the metric, it is still possible to study the topological properties of the manifold by means of algebraic invariants, and, working backwards, to apply the same techniques to geometric as opposed to topological problems. At the infinitesimal level, the resulting obstructions are expressed in terms of characteristic classes, and, assuming local integrability, the passage from local to global requires more delicate invariants both for existence and for classification. Viewed in this light, a Riemannian metric is only one of a large number of possible geometric structures, and one obtains just as rich a theory by replacing a smoothly varying choice of inner product \( g \) in the tangent bundle with a non-degenerate skew-symmetric 2-form \( \omega \) with flatness condition \( d\omega = 0 \). Furthermore, as in the Riemannian case, this symplectic geometry has its roots in physics (Hamiltonian systems), and in recent years there has been valuable give and take between the rapidly developing abstract theory and its applications. Kähler manifolds provide an important class of symplectic manifolds, and although there are now several ways available to construct more general, even non-complex, examples, there is a sense in which Kähler theory provides the subject with its flavour.

What should an introductory book on symplectic topology contain? Modelling our ideal text on one of the better introductions to differential topology, say that by Th. Bröcker and K. Jänich [1], we would expect to find the following:
1. Definitions and examples, including products and submanifolds
2. Linear symplectic structure in the tangent bundle
3. Local properties, such as Moser's theorem and its consequences
4. Dynamical systems  
5. Extension of symplectic diffeotopies  
6. Connected sums (along codimension 2-submanifolds)  
7. Properties of tubular neighbourhoods  
8. Gromov’s h-principle in its symplectic setting (immersions and embeddings)  

This last should prepare the reader for the more full-blooded treatment in [2]. A slightly more ambitious text would also contain a chapter on Morse theory, perhaps leading up to the construction of Floer homology groups. This ‘perhaps’ betrays my unease with the larger of the two books under review – some reordering of the material in Chapters 1 to 3, together with parts of Chapters 4, 6 and 7, would satisfy the requirements suggested above. And the numerous attractive examples with which McDuff and Salamon break up the text could be grouped as exercises-with-hints, as in one of the reviewer’s favourite textbooks [3]. But, as it stands, the book is overloaded; many of the arguments are too rapid for a true introduction, and the authors are too keen to include everything. In order to include the latest results (‘latest’ being circa December 1994), they are forced to resort to the sequence text, outline, remark, footnote. Much of the book is a conflation of at least two lecture courses, and will be a useful mine for any mathematician wishing to give a course of her own. As such it may be strongly recommended, but it will be a brave soul who seeks to use it to learn the subject from scratch unaided.

Parts III and IV contain enough material to fill a second book. Part III, on symplectic diffeomorphisms, starts very nicely with the Poincaré–Birkhoff theorem on area-preserving maps, and goes on to prove some general structural results. Part IV, on invariants, contains chapters on capacities, with applications to rigidity and non-squeezing, and on Floer homology, with its application to the Arnol’d Conjecture on symplectic fixed points. (Commercial break: a T-shirt illustrating non-squeezability is still available from the Isaac Newton Institute in Cambridge.) The proofs tend to become ever more loosely organised—for example, the Hofer relation on the Lie algebra of compactly supported Hamiltonian functions (Section 12.3) is easy to define, but hard to exhibit as a metric. The nub is to show that distinct functions are at positive distance apart. This is achieved in Theorem 12.14, but at the price of postponing one construction to a separate, following section.

A major and deliberate omission from the larger book is the subject of pseudo-holomorphic curves, that is, almost-complex structure-preserving maps with domain a Riemann surface. These are introduced, and the basic properties of their moduli spaces proved, in the smaller book (AMS Lecture Notes) being considered. They are then used to deform the cup product in cohomology, giving an invariant which, for example, can be used to show that two symplectic structures on the same 6-dimensional manifold are not deformation equivalent. By way of an easy illustration, consider \( QH^*(CP^n) = \mathbb{Z}[p,q]/\langle p^{n+1} = q \rangle \) with \( q \) as dummy deformation variable. The book concludes with an outline proof that the quantum product agrees with the ‘pair of pants’ product in Floer theory. This has been overtaken by events—see the paper by Pliunikhin, Salamon and Schwarz [4]. Nonetheless, as a guide to two important subjects, these AMS notes are valuable.

In the reviewer’s opinion, these two books, taken together, constitute a source book for, rather than an introduction to symplectic topology. In a rapidly developing subject, such as this is, the temptation is inevitably to include as much material as possible. Even if the authors have succumbed, both books have earned their passage to any self-respecting departmental library.
The treatment of second-order elliptic partial differential equations in books is often confined to the Laplace equation or more general linear equations; one may hope to see a decent treatment of maximum principles, Harnack inequalities and the proof of existence of solutions of the standard boundary-value problems. More advanced works may deal with quasi-linear equations, such as that connected with minimal surfaces, in which the second-order derivatives appear linearly, any nonlinearities being confined to first-order derivatives or the unknown function itself. Boundary-value problems for these more complicated equations can be handled in various ways, for example by variational techniques or by more classical procedures which use the Leray–Schauder fixed-point theorem plus a priori estimates of possible solutions and their gradients. However, most books are silent about fully nonlinear equations, in which even the top-order derivatives may appear in a nonlinear way: work on this notoriously difficult topic is largely to be found in research papers. These can be traced back to, among others, S. Bernstein at the beginning of the century and J. Leray in the late 1930s, but it is really only in the last 20 years or so that the pace of advance has quickened and the body of results become substantial.

The work under review helps to fill the gap in the book literature mentioned above. It deals with equations of the form
\[ F(D^2u, x) = f(x) \]

in a bounded domain \( \Omega \) in \( \mathbb{R}^n \). Here \( D^2u \) is the Hessian matrix of \( u \), and \( F: S \times \Omega \to \mathbb{R} \) (\( S \) is the space of real \( n \times n \) symmetric matrices) is uniformly elliptic in the sense that there exist positive constants \( \lambda \) and \( \Lambda \) such that for all \( x \in \Omega \) and all \( M, N \in S \) with \( N \) non-negative definite,
\[ \lambda \|N\| \leq F(M + N, x) - F(M, x) \leq \Lambda \|N\|. \]

In typical examples, such as the Bellman equation for the optimal cost in a stochastic control problem, \( F(\cdot, x) \) is concave on \( S \) for each \( x \in \Omega \). Following earlier work by M. G. Crandall, P.-L. Lions and L. C. Evans, the authors introduce the notion of a viscosity sub-solution of (1) in \( \Omega \), by which is meant a function \( u \in C(\Omega) \) such that
\[ F(D^2\phi(x_0), x_0) \geq f(x_0) \]

whenever \( x_0 \in \Omega \), \( \phi \in C^2(\Omega) \) and \( u - \phi \) has a local maximum at \( x_0 \). Viscosity supersolutions and solutions are defined in the obvious way. This weak type of solution...
turns out to be very useful, just as weak (sub- and super-) harmonic functions have an important role to play in the theory of the Laplace operator. If $u \in C^2(\Omega)$, it is a viscosity solution of (1) if, and only if, it is a classical solution. For viscosity solutions of (1), a basic estimate, the Alexandroff–Bakelman–Pucci (ABP) estimate, and a maximum principle are obtained. From the ABP estimate, a Harnack inequality is derived by techniques similar to those used by Krylov and Safanov in their celebrated work on second-order linear elliptic equations with measurable coefficients and in non-divergence form. Interior a priori estimates of $C^{2,\gamma}$ and $W^{2,p}$ type are given for solutions of (1) when $F$ is concave. Further specialisation to the equation

$$F(D^2u) = 0$$

enables estimates of $C^{2,\gamma}$ form up to the boundary to be proved, and from these the existence of a solution of the Dirichlet problem for (2) is established by the standard method of continuity.

The book, which is based on lectures given at the Courant Institute, is clearly written and gives an excellent impression of progress with the regularity theory for equations of types (1) and (2); it highlights in a most useful way some significant recent advances (in which the authors have been very active) which play a dominant part in the theory. It has its limitations, of course: (1) is of a restricted form, not allowing the appearance of $u$ or its gradient. Moreover, the important Monge–Ampère equation

$$\det(D^2u) = f(x)$$

does not fall within the scope of the book as it is not elliptic on $S$ but only on the positive-definite matrices. However, the authors are aware of this and point out that many of their methods can be adapted to deal with, for example, the Monge–Ampère equation and the related equation of prescribed Gaussian curvature. All in all, the book marks an important stage in the theory of nonlinear elliptic problems. Its timely appearance will surely stimulate fresh attacks on the many difficult and interesting questions which remain.

D. E. Edmunds

THEORY OF COMMUTING NON-SELF-ADJOINT OPERATORS
(Mathematics and its Applications 332)

By M. S. Livšic, N. Kravitsky, A. S. Markus and V. Vinnikov: 313 pp., £109.00, isbn 0 7923 3588 0 (Kluwer Academic Publishers, 1995).

Self-adjoint operators on separable Hilbert space are tamed by the Spectral Theorem, a beautiful and immensely powerful result of which the first version originated with Hilbert himself in 1906. Non-self-adjoint operators, on the other hand, are such diverse objects that only the most optimistic mathematician would expect a convincing classification of all of them. Even in finite dimensions there is no straightforward canonical form for matrices under unitary equivalence; in infinite dimensions, the study even of rank 1 perturbation of self-adjoint operators exhibits a high degree of analytic richness and subtlety. There have been many attempts to chart regions of the non-self-adjoint wonderland; one approach is the theory of operator models. This theory exists in several versions, of which the first was developed over 50 years ago by M. S. Brodskii and M. S. Livšic. They gave a
canonical model (with respect to unitary equivalence) of operators which are finite-rank perturbations of self-adjoint operators. The main constituent of the model operator is an integral operator of fairly simple form on $L^2$. In a closely related and perhaps somewhat better known version of the theory due to B. Sz.-Nagy and C. Foiaș, the model operator is simply a restriction of a ‘backward shift’ operator. Both versions of the theory depend heavily on the factorization of meromorphic matrix- or operator-valued functions.

Livšič is still contributing highly original ideas, and the present book describes the extension of the theory of model operators to several commuting operators. It is a substantial body of theory which has been developed by Livšič and his collaborators over the past 15 years. The extension is far from automatic: it takes some surprising turns, and brings the reader into contact with some unexpected branches of classical mathematics.

A first guess might be that in progressing from one to several commuting operators, one would need to replace the theory of analytic functions of one variable by that of several complex variables. There is indeed progress in this direction, but it faces one great difficulty: there is no factorization theory for functions of several variables analogous to the one-variable theory. Livšič adopted a different path; he observed that if $A_1$, $A_2$ are commuting operators on $H$ which are finite-rank perturbations of self-adjoint operators, then there is a non-trivial polynomial $p$ in two variables such that $p(A_1, A_2) = 0$ on the subspace of $H$ invariant under $A_1, A_1^*, A_2$ and $A_2^*$ that is generated by the ranges of $A_1 - A_1^*$ and $A_2 - A_2^*$. The development in the present book is based on analytic functions defined not on $C^2$ but on the algebraic curve $p(x, y) = 0$ (or the corresponding Riemann surface). A fascinating aspect of this profound work is the contact it makes with algebraic geometry, elliptic functions, Jacobian varieties and theta functions. All these play an essential role in the classification of pairs $(A_1, A_2)$ for which the corresponding ‘discriminant curve’ $p(x, y) = 0$ is a real smooth cubic; this is the principal destination of the final quarter of the book (written by V. Vinnikov). A ‘guiding light’ for this section is declared to be the construction, up to unitary equivalence on the principal subspace, of all pairs of commuting non-self-adjoint operators which are finite-rank perturbations of self-adjoint operators, with given discriminant curve, given input determinantal representation and given joint spectrum. Notes point to papers in which this programme is taken further.

The first section of the book, giving the basic definitions and ideas, is by Livšič. The second, on joint spectra and discriminant varieties, is based on work of Livšič and Markus. The third section, by N. Kravitsky, develops the corresponding theory for operators on Banach spaces. In addition to the connections highlighted above, there are results about linear partial differential equations, linear systems and the wave–corpuscle duality.

The authors’ formalism looks at first sight a little daunting. For example, they analyse pairs of operators in terms of a ‘two-operator vessel’ represented as

$$(A_1, A_2; H, \Phi, E; \sigma_1, \sigma_2, \sigma_{\text{in}}, \sigma_{\text{out}}).$$

However, there is good reason for it; the whole is clearly written and well motivated. The standard of production is quite good, though perhaps less so than one might expect for £109; different sections are in subtly different fonts, and the final section, while already the toughest for legitimate mathematical reasons, has some forbiddingly overcrowded pages with some small symbols hardly decipherable by the middle-aged...
eye. Why should we suffer thus in the age of baseline stretch? Still, this is an excellent research monograph, giving an account of an impressive body of work by an innovative school of operator theorists. It demands more than ocular effort, but repays it well.

N. J. Young

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