04.1.1. A Hausman Test Based on the Difference between Fixed Effects Two-Stage Least Squares and Error Components Two-Stage Least Squares

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Consider the following first structural equation of a simultaneous equation panel data model:

\[ y_1 = Z_1 \delta_1 + u_1, \]

where \( Z_1 = [Y_1, X_1] \) and \( \delta_1 = (\gamma_1', \beta_1') \). As in the standard simultaneous equation literature, \( Y_1 \) is the set of \( g_1 \) right-hand-side endogenous variables, and \( X_1 \) is the set of \( k_1 \) included exogenous variables. Let \( X = [X_1, X_2] \) be the set of all exogenous variables in the system. Let \( k_2 \), the number of excluded exogenous variables from the first equation \( (X_2) \), be larger than or equal to \( g_1 \). The disturbance \( u_1 \) follows the one-way error component model

\[ u_1 = Z_\mu \mu_1 + \nu_1, \]

where \( Z_\mu = (I_N \otimes \iota_T) \) with \( I_N \) being an identity matrix of dimension \( N \) and \( \iota_T \) a vector of ones of dimension \( T \). In this case, \( \mu_1' = (\mu_{11}, \ldots, \mu_{N1}) \) is a vector of random individual effects of dimension \( N \) and \( \nu_1' = (\nu_{111}, \ldots, \nu_{NT1}) \) is a vector of remainder disturbances of dimension \( NT \). These vectors have zero means and covariance matrix

\[
E \begin{pmatrix} \mu_1 \\ \nu_1 \end{pmatrix} \begin{pmatrix} \mu_1' \\ \nu_1' \end{pmatrix} = \begin{bmatrix} \sigma^2_{\mu_{11}} I_N & 0 \\ 0 & \sigma^2_{\nu_{11}} I_{NT} \end{bmatrix}.
\]

Baltagi (1981) derives the error component two-stage least squares (EC2SLS) estimator of this model, which is the random effects (RE) counterpart of a classical error components panel data regression. Similarly, the fixed effects two-stage least squares (FE2SLS) estimator of this model is the fixed effects (FE) counterpart of a classical error components panel data regression. Both EC2SLS and FE2SLS allow for the endogeneity of \( Y_1 \).
Hausman (1978) suggests comparing the FE and RE estimators in the classical panel data regression. With endogenous right-hand-side regressors such as $Y_1$ this test can be generalized to test $H_0: E(u_1|Z_1) = 0$ based on $\hat{\delta}_{1,\text{FE2SLS}} - \hat{\delta}_{1,\text{EC2SLS}}$.

(a) Show that under $H_0: E(u_1|Z_1) = 0$, plim $\hat{\delta}_1 = 0$ and the asymptotic $\text{cov}(\hat{\delta}_1, \hat{\delta}_{1,\text{EC2SLS}}) = 0$.

(b) Conclude that var$(\hat{\delta}_1) = \text{var}(\hat{\delta}_{1,\text{FE2SLS}}) - \text{var}(\hat{\delta}_{1,\text{EC2SLS}})$, where var denotes the asymptotic variance. This is used in computing the Hausman test statistic given by

$$m_1 = \hat{\delta}_1^{\prime} \text{[var}(\hat{\delta}_1)]^{-1} \hat{\delta}_1.$$ 

Under $H_0$, $m_1$ is asymptotically distributed as $\chi_r^2$, where $r$ denotes the dimension of the slope vector of the time varying variables in $Z_1$. This can be easily implemented using standard software packages, like STATA. The usual Hausman test can yield misleading inference in the presence of $Y_1$.

REFERENCES


04.1.2. Correcting for Heteroskedasticity of Unspecified Form

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Consider the basic linear regression model with heteroskedasticity, $y_i = x_i' \beta + e_i$, with $i = 1, \ldots, n$, where $e_i$ is a zero-mean random variable distributed independently from all $x_i$ and all $e_j$ with $i \neq j$. The variance of $e_i$ is $\sigma_i^2$, where all $\sigma_i^2, i = 1, \ldots, n$ may be different. The “true” generalized least squares estimator (GLSE) $\hat{\beta}$ of $\beta$ is obtained as the ordinary least squares estimator (OLSE) from the regression of the $y_i/\sigma_i$ on the $x_i/\sigma_i$. When, as usual, the $\sigma_i$ are unknown, we might instead be tempted to use the absolute value of the residual $\hat{\sigma}_i = |y_i - x_i' b|$, with $b$ the OLSE from the regression of the $y_i$ on the $x_i$, and obtain a “feasible” GLSE $\tilde{\beta}$, say, by regressing the $y_i/\hat{\sigma}_i$ on the $x_i/\hat{\sigma}_i$. Intuitively, this is not a sensible approach, but one is hard put to find an argument in the literature. What are the (finite or asymptotic) distributional properties of $\tilde{\beta}$? (One might conjecture that $\tilde{\beta}$ is consistent and asymptotically normal.) Is there any sense in which $\tilde{\beta}$ constitutes an improvement over $b$?
SOLUTIONS

03.1.1. Deriving the Observed Information Matrix in Ordered Probit and Logit Models Using the Complete-Data Likelihood Function—Solution

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The complete-data log likelihood function is

\[
\ln L^*(y^* | x, \beta) = \text{constant} - \frac{1}{2} \sum_{i=1}^{n} (y_i^* - \beta'x_i)^2. \tag{2}
\]

Define the categorical variables

\[ Z_{ij} = \begin{cases} 
1 & \text{if } y_i^* \text{ belongs to the } j \text{th category}, \\
0 & \text{otherwise, } i = 1, 2, \ldots, n, j = 1, 2, \ldots, m. 
\end{cases} \tag{3} \]

Therefore, the missing information matrix is given by

\[
I_m = \text{Var}(\partial \ln L^*(y^* | x, \beta)/\partial \beta | Z_{ij})
= \text{Var}\left( \sum_{i=1}^{n} (y_i^* - (\beta'x_i))x_i | Z_{ij} = z_{ij} \right)
= \text{Var}\left( \sum_{i=1}^{n} (u_i x_i) | Z_{ij} = z_{ij} \right)
= \sum_{j=1}^{m} \sum_{i=1}^{n} z_{ij} [1 - ((\phi_{j-1,i} - \phi_{j,i})/(\Phi_{j-1,i} - \Phi_{j,i}))^2 
+ (w_{j-1,i} \phi_{j-1,i} - w_{j,i} \phi_{j,i})/(\Phi_{j-1,i} - \Phi_{j,i})]x_i x_i', \tag{4}
\]

where \( \phi_{j-1,i} = \phi(w_{j-1,i}), \Phi_{j-1,i} = \Phi(w_{j-1,i}), \phi_{j,i} = \phi(w_{j,i}), \Phi_{j-1,i} = \Phi(w_{j-1,i}), \)
\( w_{j-1,i} = \alpha_{j-1} - \beta'x_i, w_{j,i} = \alpha_j - \beta'x_i, \) and \( \sum_{j=1}^{m} z_{ij} = 1 \) because

\[
\text{Var}(u_i | Z_{ij} = z_{ij}) = \sum_{j=1}^{m} z_{ij} \text{Var}(u_i | \alpha_{j-1} < y_i^* < \alpha_j)
= \sum_{j=1}^{m} z_{ij} [1 - ((\phi_{j-1,i} - \phi_{j,i})/(\Phi_{j-1,i} - \Phi_{j,i}))^2 
+ (w_{j-1,i} \phi_{j-1,i} - w_{j,i} \phi_{j,i})/(\Phi_{j-1,i} - \Phi_{j,i})], \tag{5}
\]

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where $f$ and $F$ are, respectively, the probability density function and cumulative distribution functions of a $N(0,1)$ random variable using formulas for variances of doubly truncated distributions (Maddala, 1983, p. 366). Furthermore, the complete-data information matrix is given by

$$I_c = -E(\partial^2 \ln L^* (y^* | x, \beta) / \partial \beta \partial \beta') = \sum_{i=1}^{n} x_i x_i'.$$  \hfill (6)

From equations (4) and (6), it follows that the observed information matrix is

$$I_o = I_c - I_m$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} z_{ij} / (\Phi_{j-1,i} - \Phi_{j,i})^2 [(\Phi_{j-1,i} - \Phi_{j,i})^2 - (w_{j-1,i} \Phi_{j-1,i} - w_{j,i} \Phi_{j,i})$$

$$\times (\Phi_{j,i} - \Phi_{j-1,i})] x_i x_i',$$  \hfill (7)

which is equal to the observed information matrix, $-\partial^2 \ln L / \partial \beta \partial \beta'$, obtained from the observed-data log likelihood function

$$\ln L(Z | x, \beta) = \sum_{j=1}^{m} \sum_{i=1}^{n} z_{ij} \ln [\Phi_{j,i} - \Phi_{j-1,i}].$$  \hfill (8)

Alternatively, the observed information matrix may be computed by using the result that the observed score function is the conditional expectation of the latent score function given the observed variables (Louis, 1982, p. 227):

$$\partial \ln L(Zx, \beta) / \partial \beta = E(\partial \ln L^* (Z | x, \beta) / \partial \beta | Z)$$

$$= \sum_{i=1}^{n} E((y_i^* x_i - (\beta' x_i) x_i) | Z_{ij} = z_{ij})$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} z_{ij} [(\Phi_{j-1,i} - \Phi_{j,i})/(\Phi_{j,i} - \Phi_{j-1,i})] x_i.$$  \hfill (9)

By differentiation of the observed score function in equation (9) with respect to $\beta'$, we obtain the observed information matrix in equation (7) as the negative of the hessian of the observed log likelihood function.

REFERENCES


03.1.2. Redundancy of Lagged Regressors in a Conditionally Heteroskedastic Time Series Regression — Solution

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Let $g_{i,t} = (y_t - \beta x_t)x_{t-i}$ ($i = 0, 1, \ldots$). For any finite set $I$ of nonnegative integers, let $\hat{\beta}_I$ be the optimal generalized method of moments estimator of $\beta$ based on the moment conditions

$$E_{g_{i,t}} = 0, \quad i \in I.$$ 

By the results of Breusch, Qian, Schmidt, and Wykowski (1999), the ordinary least squares estimator, $\hat{\beta}_{(0)}$, is at least as efficient as any $\hat{\beta}_I$ if, and only if, for all $i \geq 1$, the condition $E_{g_{i,t}} = 0$ is redundant given $E_{g_{0,t}} = 0$; i.e.,

$$D_i = \Omega^{-1}_{i0}\Omega^{-1}_{00}D_0, \quad i \geq 1,$$

(1)

where $D_i = E(\partial g_{i,t}/\partial \beta)$ and $\Omega_{ij} = \sum_{l=-\infty}^{\infty} E(g_{i,t}g_{j,t-l})$. Let $\gamma_i = \text{Cov}(x_t, x_{t-i})$. Then, for $i \geq 0$,

$$D_i = -E(x_t x_{t-i}) = -\gamma_i,$$

$$\Omega_{i0} = E \sum_{l=-\infty}^{\infty} e_t x_{t-i} e_{t-l} x_{t-i} = E[(\omega + \lambda(x_t - \mu)^2)x_t x_{t-i}] = \frac{\gamma_i}{\gamma_0} \Omega_{00},$$

(2)

so (1) is seen to hold. The last equality in (2) follows by writing

$$x_{t-i} = \frac{\gamma_i}{\gamma_0} x_t + u_t$$

with $x_t$ and $u_t$ independent. That this is possible follows from the bivariate normality of $x_{t-i}$ and $x_t$.

NOTE

1. An excellent solution has been independently proposed by the S. Anatolyev, the poser of the problem.

REFERENCE

03.1.2 Redundancy of Lagged Regressors in a Conditionally Heteroskedastic Time Series Regression—Solution

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A solution to the problem is to calculate the greatest lower bound for the asymptotic variance of GMM estimators based on moment function \( h(i; \beta) = (y_i - \beta_0 x_i) x_{t-i}, \ i = 0,1,2,\ldots \). To do so, we apply Hansen (1985). In the sequel, we will denote \( h(i) \) the moment functions \( h(i; \beta_0) \). Let \( L^2(h) \) be the linear space spanned by \( \{h(i): i = 0,1,\ldots\} \); it is made of elements \( G = \sum_{i=0}^{\infty} \alpha_i h(i) \) for any real \( \alpha_i \). Note that \( \{h(i): i = 0,1,\ldots\} \) are martingale difference sequences. Moreover, they are linearly independent and in that sense are complete in \( L^2(h) \).

We are looking for the solution of equation (4.8) of Hansen (1985). That is, we are looking for the element \( G \) in \( L^2(h) \) (if it exists) such that

\[
E \left[ \frac{\partial h}{\partial \beta} (i) \right] = E[h(i)G] \quad \text{for all } i = 0,1,\ldots.
\]

Or, equivalently, we are looking for the constants \( \alpha_j \) solutions of

\[
-E(x_t, x_{t-i}) = E \left[ e_t x_{t-i} \left( \sum_{j=0}^{\infty} \alpha_j e_t x_{t-j} \right) \right].
\]

Note that the solution may not exist in \( L^2(h) \), but there is always a solution in the closure of \( L^2(h) \). If a solution in \( L^2(h) \) exists, it is unique because \( \{h(i): i = 0,1,\ldots\} \) is complete. Let \( \alpha_j = 0 \), for \( j = 1,2,\ldots \); equation (2) becomes

\[
-E(x_t, x_{t-i}) = \alpha_0 E[e_t^2 x_{t-i} x_t] = \alpha_0 E[(\omega + \lambda (x_t - \mu)^2) x_{t-i} x_t] = \alpha_0 (\omega + \lambda \mu^2) E(x_t, x_{t-i}) + \alpha_0 \lambda E(x_t^2 x_{t-i}) - \alpha_0 2 \lambda \mu E(x_t^2 x_{t-i}).
\]

Using the fact that \( \eta_t \) are i.i.d. standard normal, we have

\[
E(x_t^2 x_{t-i}) = E \left[ \left( \sum_j \varphi_j \eta_{t-j} \right) \left( \sum_j \varphi_j \eta_{t-j} \right) \right] = 0,
\]

\[
E(x_t^3 x_{t-i}) = E \left[ \sum_j \varphi_j^3 \eta_{t-j}^3 + 3 \sum_{k \neq j} (\varphi_j^2 \eta_{t-j}^2) (\varphi_k \eta_{t-k}) + 6 \sum_{k \neq j \neq l} (\varphi_j \eta_{t-j}) (\varphi_k \eta_{t-k}) (\varphi_l \eta_{t-l}) \right] \left( \sum_j \varphi_j \eta_{t-j} \right)
\]

\[
= 3 \sum_j \varphi_j^3 \varphi_j \eta_{t-j} + 3 \sum_{k \neq j} \varphi_j^2 \varphi_k \eta_{t-k} = 3 \left( \sum_j \varphi_j^2 \right) \left( \sum_k \varphi_k \varphi_{k-i} \right) = 3 E(x_t^2) E(x_t, x_{t-i}).
\]
Hence, Equation (2) becomes
\[-E(x_i x_{i-1}) = \alpha_0 E(x_i x_{i-1}) (\omega + \lambda \mu^2 + 3\lambda E(x_i^2)),\]
which is satisfied for
\[\alpha_0 = -\frac{1}{\omega + \lambda \mu^2 + 3\lambda E(x_i^2)}.\] (3)

Therefore, we found a solution to (1), which is \(G_0 = \alpha_0 x_i e_t\) with \(\alpha_0\) as in (3). By Lemma 4.3 of Hansen (1985), the greatest lower bound is given by
\[\left[ E(G_0^2) \right]^{-1} = \left[ \alpha_0^2 E(x_i^2 e_t^2) \right]^{-1}.\]

Note that \(\alpha_0\) can be rewritten as
\[\alpha_0 = -\frac{E(x_i^2)}{E(x_i^2 e_t^2)}.\]

Therefore, the GMM efficiency bound is
\[\frac{E(x_i^2 e_t^2)}{[E(x_i^2)]^2},\]
which corresponds to the asymptotic variance of the OLS estimator.

We have shown that the OLS estimator is not only as efficient as any GMM estimator that uses an arbitrary fixed number of instruments from \(\{x_t, x_{t-1}, \ldots\}\) but also as efficient as any GMM estimator that uses an infinity of instruments from \(\{x_t, x_{t-1}, \ldots\}\).

REFERENCES