This is a book with a difference. It is written not as a textbook, but rather to entice readers into thinking about mathematics in general, and, in particular, parts of geometry which are important for the study of low-dimensional manifolds. The reader is encouraged to think not only in abstract terms, but also using paper models and the physics of everyday life.

Almost any mathematician, including those who are well-educated, would learn many fascinating things from this book—see the description below of the contents. For a graduate student, the material could also be very rewarding: however, solitary reading might be frustrating, as parts of it are very hard, particularly the exercises. A better, and probably more enjoyable, strategy would be to read it as a collective endeavour. The book could form the basis of any number of interesting courses at the graduate or advanced undergraduate level.

The book is based on examples. General theory is introduced later and with reference to the examples. This is an increasingly popular approach, at least in theory, and it is widely recognized as sound pedagogy. In this book, the approach is clearly very close to the heart of the author, and the initial examples are not just optional extras, but are core material.

Many of the most interesting points are confined to exercises (with no solutions provided). In fact, some exercises would require considerable work even to formulate them as precise mathematical questions in the normally understood way.

Instead of starting with an abstract definition of a surface or manifold, Chapter 1 shows how to build surfaces by gluing together polygonal pieces along their edges. Many beautiful examples are given. For example, the standard tesselation of the plane by regular octagons leads to particular structures on the torus, and in this context the reader is invited to find a division of the torus into seven regions, which can be coloured so that no two adjacent ones have the same colour.

Inversion in circles is discussed in a thoroughly non-computational but rigorous way. Some of the proofs of this standard material are not standard. The geometry of the hyperbolic plane is introduced through the study of inversions, but in a discursive way—a formal definition could be extracted from the treatment, but isn’t. (A formal definition of hyperbolic space of dimension $n$ is given in Chapter 2.)

Mechanical linking is briefly mentioned, with an example showing how to express inversion using a linkage. A steam engine requires the conversion of a linear movement to a circular movement, so this is potentially a practical application to engineering. However, Thurston tells us that because the linkage has so many parts, the linkages used in practice convert instead between a circular movement and an approximately linear movement.
Gluing allows a natural introduction of surfaces with a hyperbolic structure. For example, you can construct a regular hyperbolic octagon in which each angle is $2\pi/8$, which can be glued to itself so as to yield a compact surface of genus two with a hyperbolic structure.

As with most of the topics in the book, the Euler characteristic for surfaces is rethought from scratch, with productive results. The elegant treatment uses a construction which ‘jiggles’ a vector field to make it transverse to a triangulation in a way which generalizes directly to the more difficult transversality of foliations in all dimensions, a result which Thurston used in some of his famous papers on foliation theory.

When the three-dimensional torus is discussed, the reader is shown pictures of what it would look like to live in such a torus, with some solid objects strewn about. There are illustrations produced at the Geometry Center, regrettably now deceased, at the University of Minnesota. The Poincaré dodecahedral space is then treated, with its relationship to three-dimensional spherical geometry, and also the Seifert–Weber dodecahedral space, with its relationship to three-dimensional hyperbolic geometry. Lens spaces with their spherical geometry and the complement of the figure-eight knot with its hyperbolic geometry are also discussed in the context of gluing.

Thurston’s technique is not to insist on explaining three-dimensional hyperbolic geometry first, but to give an intuitive idea of the shape of the hyperbolic pieces plus an explanation of how the faces are glued to each other to form particular manifolds. So, as it should, a motivation for studying three-dimensional hyperbolic geometry precedes the formal definition.

Although Chapter 1 is only 40 pages long, the above summary gives an idea of how much material is packed into it. However, the style gives the feeling of a leisurely stroll, rather than a forced march. It is only if one insists on proving all statements made and doing all the examples that one feels the weight.

Chapter 2 starts by saying what is meant by ‘geometry’. This is similar in concept to the content of Riemann’s thesis, but more general in that Thurston’s geometries are not quite as homogeneous as Riemann’s. The ‘inside’ and ‘outside’ views of a manifold are explained. These are informal but important terms first introduced by Thurston in Geometry Center videos. The outside view of hyperbolic three-space might be a round ball (the Poincaré disk model), whereas the inside view is what it might look like to live in a hyperbolic universe, when you wouldn’t see any boundary. As is often the case, true understanding comes from being able to look at a problem from many different points of view.

Curvature is introduced from the outside by looking at a surface embedded in Euclidean three-space, and using the Hessian. An inside (intrinsic) definition is also given: if you express the circumference of a small disk of radius $r$ as a power series in $r$, then the curvature is essentially the coefficient of $r^2$. The book does not point out that one needs to prove that the two definitions are equivalent, nor that it is a difficult result to deduce from first principles.

The tractrix and pseudosphere are then discussed, and the curvature of surfaces of revolution in general. I liked the description of two ways of making hyperbolic paper. Students (and professional mathematicians) playing with such paper learn much more quickly what hyperbolicity really means. There is a brief discussion of Hilbert’s Theorem that a complete surface of constant negative curvature cannot be isometrically imbedded in Euclidean three-space, but I couldn’t understand it.

Various models of hyperbolic space are discussed, and inversive geometry makes
another appearance, this time in more than two dimensions. In particular, the beautiful Steiner’s Porism is given as an exercise. All the standard models of hyperbolic space appear, and also some unusual models, such as the paraboloid model. There are some nice pictures of hyperbolic three-space from *Not Knot*, the award-winning Geometry Center video.

Inner products in the Minkowski metric are used as the basic tool in proving the various standard trigonometry formulas in hyperbolic geometry, such as the cosine law and the hexagonal law of sines. This is probably a new approach, though it’s hard to tell. The book contains scattered historical remarks, on a fairly random basis. Maybe this is because Thurston has worked out nearly all the material for himself, and it’s a matter of chance whether he can be confident as to its origins. He gives, without attribution, Gauss’ marvellous proof of the formula for the area of a triangle in the hyperbolic plane.

This is followed by the classification of isometries of hyperbolic space. Buseman’s definition of negative curvature is mentioned—this connects with modern investigations of CAT($k$) metric spaces. A number of nice approaches to complex coordinates in three-dimensional hyperbolic geometry are introduced. There is a nice discussion of the quaternions, the three-sphere with its group structure and the Hopf fibration.

Having now described a large number of different manifolds in an explicit way, Chapter 3 introduces the general definitions, including pseudogroups, so that various fancy structures can be easily discussed: Thurston calls such additional structures *stiffenings*, a name which is sufficiently evocative that one can only hope that it catches on. A number of structures are treated, including manifolds with a volume form. Sufficient detail is given so that Moser’s result (that if there is a diffeomorphism between two compact manifolds having the same volume, then there is a volume-preserving diffeomorphism between them) can be recovered, but Moser’s name is not mentioned.

Triangulated manifolds and the double suspension of the Poincaré dodecahedral space are briefly mentioned. References to Edwards and Cannon might have been helpful here—their result is totally amazing and unbelievable to anyone meeting the topic for the first time, and the proof is very difficult.

Geometric structures on manifolds are brought in, with the complements of the Whitehead link and Borromean rings as examples. There is a detailed discussion of the Knotted Y, which is a hyperbolic structure on a compact three-manifold with connected geodesic boundary. There is a nice picture of a sculpture by Helaman Ferguson representing the Knotted Y. The book goes on to define developing maps of geometric structures, and the important concept of completeness. Discrete groups acting on manifolds are then introduced. In view of the confusing plethora of related concepts, such as free, discrete, wandering and properly discontinuous, some counterexamples are given to show how these concepts differ.

Principal and flat bundles are then discussed. It is mentioned in passing that the Levi–Civita connection of a Riemannian manifold can be derived from the physical process of rolling a plane along the manifold without allowing slipping. Unfortunately, details are not given for this non-standard insight. Some more unusual structures, such as contact structures, are also discussed. Again there are nice real-world illustrations by means of ice-skating and car parking. The Frobenius Theorem relating foliations to vector fields is also proved.

Thurston’s eight model geometries are then introduced, and also related objects such as the Heisenberg group.
Piecewise integral projective structures have not yet made a significant impact in the mathematical literature. It is stated without proof that Richard Thompson’s example of a finitely presented group which is simple is the group of orientation-preserving, piecewise integral projective transformations of the circle.

It is nice to have a really elementary discussion of the problem of smoothing low-dimensional piecewise linear manifolds. Exercises are the main vehicle used, but a determined reader should be able to reconstruct complete proofs, whereas, as far as the reviewer knows, all previously existing discussions in the literature are very painful, usually because they prove far more than a low-dimensional topologist might need to know.

In Chapter 4, Thurston investigates subgroups of Lie groups generated by small elements. These results, due to Zassenhaus, are nowadays commonly ascribed to Margolis. Crystallographic groups and the results of Bieberbach are then discussed, with new clean proofs. Several examples of such groups are worked out. However, the Thurston–Conway notation is not introduced. Special linear groups over various rings come in here, with some standard results for the finite field case proved by geometric means. Various classification results for two- and three-dimensional manifolds with particular geometric structures are then discussed.

This is followed by the thick–thin decomposition of a hyperbolic manifold, and of cusps. Teichmüller space for surfaces is discussed in terms of Fenchel–Nielsen coordinates. The book ends with fibred geometries.

This view has included a long description of the book’s contents so that potential readers can see what a treasure-trove it contains. The mathematical public owes a considerable debt of gratitude to Silvio Levy, whose devoted work over many years has enabled this work to be made public.

Some versions of the notes for this book have contained far more material. The book also contains a great deal of material which was not in the original 1978 notes (which are available at www.msri.org/publications/books/gt3m). The book does not mention Thurston’s Conjecture, even though all the necessary background concepts are introduced. This conjecture has been a prime motivating force in the theory of three-dimensional manifolds ever since it was made public around 1978. The fact that the notes have been somewhat artificially cut off (in the interests of getting valuable material properly published) is shown by the references that survive, perhaps deliberately, to the first chapter of the not yet existent Volume 2. The book under review is called Volume 1. I’m sure I speak for the whole mathematical community in expressing a strong desire to see Volume 2 and the further volumes which would be necessary to give a complete account of the way in which Thurston has reworked the foundations in this area.

University of Warwick

D. B. A. Epstein
Most mathematicians will be aware that, over the past twenty years, considerable progress has been made in the birational theory of complex projective varieties, the theory developed known as the minimal model program or Mori’s program. This progress has been particularly spectacular in dimension three. The classical Enriques–Severi theory for complex projective surfaces had as one of its key points that for any surface \(X\) there exists a birationally equivalent smooth projective surface \(Y\) which is minimal (in the sense that it does not admit birational morphisms to smooth projective surfaces \(Z\), other than isomorphisms). Furthermore, if \(X\) is not covered by rational curves, then \(Y\) is unique up to isomorphism, and \(Y\) is also characterised by the property that its canonical divisor class \(K_Y\) (the class corresponding to any non-zero rational form of top degree) is non-negative on all curves lying in \(Y\). This property is now generally referred to as \(K_Y\) being nef, a neologism due to Miles Reid, and the more fruitful property to consider in higher dimensions. It had, however, been realised that, even for complex projective threefolds not covered by rational curves, there was not in general any birationally equivalent smooth projective variety \(Y\) for which \(K_Y\) was nef, and that, when there was, it was not in general unique.

In the early 1980s, Reid proposed that in order to ensure the existence of minimal models, one should allow mild singularities, which he called terminal singularities. One of the properties of these singularities is that there is still a numerical canonical class, which when intersected with any curve lying on the threefold yields a rational number (namely the canonical class is \(\mathbb{Q}\)-Cartier), and so one can still talk about \(K_Y\) being nef. Moreover, these singularities were of the type expected if it was possible to carry out the program, proposed by Mori, for making sequences of birational contractions on threefolds (starting from a smooth threefold), contracting curves on which the canonical divisor was negative. Reid also addressed the problem of the non-uniqueness of minimal models, introducing birational transformations which he called flops.

In the 1980s, further important contributions were made to the minimal model program by Kawamata, Kollár, Mori, Reid, Shokurov and others. The obstruction to continuing the sequence of birational contractions (until reaching a minimal model) was the existence at some stage of a birational contraction which contracted only a finite number of curves (necessarily rational), because then the canonical divisor would cease to be \(\mathbb{Q}\)-Cartier. It was conjectured that instead of making that contraction, one should instead perform a different kind of birational map, known as a flip, in which the curves which would have been contracted are transformed into curves on which the canonical divisor is positive. After much effort, the existence of these flips was finally proved by Mori in a paper published in 1988, and Mori received a Fields Medal for his contributions at the Kyoto ICM in 1990.

The last ten years have seen extensions and refinements of these results, particularly for families of threefolds, and for log threefolds (pairs given by a threefold together with a divisor). The log category is a very convenient one for arguments
involving induction on the dimension. The intervening period has also seen considerable improvements and refinements of both the proofs and the exposition of them. The book under review may be seen as a culmination of these refinements, and one can trace much of the book back to earlier survey articles by Kollár and others. The proof of the existence of flips in dimension three is beyond the scope of the book, but in the last chapter a new proof (based on an unpublished note of Corti) is given for the existence of flips in the case of semi-stable families of surfaces (which should itself be regarded as a weighty result), which avoids the use of facts about quotient surface singularities, and is therefore perhaps more amenable to extension into higher dimensions.

The authors have produced a book which is accessible to anyone who has mastered the rudiments of algebraic geometry from, say, the standard text by Hartshorne, but also one which is remarkable in a number of ways. It is almost entirely self-contained (and therefore topics such as simultaneous resolution in families of Du Val singularities, duality theory, or elliptic surface singularities, all have subsections devoted to them). It is written very precisely—the reader who in the past has had difficulty remembering the difference between the various flavours of the condition log terminal for log varieties (klt, plt, dlt) will find not only these clearly defined, but also an explanation as to which condition is most useful for different kinds of proof. It is fairly comprehensive in its coverage (Mori’s proof of the existence of threefold flips has to be omitted, but most of the rest of the basic theory is there), and corresponding results valid for analytic spaces are also clearly stated (albeit not always proved). On the other hand, the book is very crisply written, unusually easy to read for a book covering advanced material, and is moreover very concise for the amount of material covered, and not over-expensive. It will clearly be an indispensable book for reference, but is also an ideal book on which to base a series of seminars for research students, or indeed for research students to read by themselves.

Cambridge University

P. M. H. Wilson

ALGORITHMIC AND COMPUTER METHODS FOR THREE-MANIFOLDS
(Mathematics and its Applications 425)


I really enjoyed this book; it is great fun and full of enthusiasm for the subject. The choice of material betrays the authors’ preference for those parts of 3-manifold theory which can be explained by simple geometrical arguments, a preference which I share. Furthermore, there are some excellent figures, which bring the book to life and often show really clearly what is going on. The book covers a good selection of topics in 3-manifold theory (details below), including a thorough treatment of normal surfaces and connected sums. It does not, however, include the most important recent advance in this area, namely the Rubinstein–Thompson–Matveev algorithm to detect the 3-sphere, which omission I found particularly disappointing as the ground for this algorithm is well prepared in the book.
The title of the book is rather misleading. Although there is some mention of the algorithmic status of some of the results, this is not in any sense a book about ‘algorithmic and computer methods’. It is about the topology of 3-manifolds and related topics, very much in the spirit of Dale Rolfsen’s classic text Knots and links (Publish or Perish, Wilmington, DE, 1976). Indeed, like Rolfsen’s book, this is ideal for a starting graduate student to learn some of the basic material on 3-manifolds, although, unlike Rolfsen, the authors include no exercises and precious few examples; a text at this level and for this audience ought to contain both. However, I would happily give this book to one of my own students to read, confident that they would gain a great deal from it.

That was the good news; now comes the bad news. This book is quite absurdly overpriced at £109, a price at which no graduate student in England could afford a copy. Apart from the price, I have some other minor criticisms. The typesetting is far from perfect: at first I found the escape from the ubiquitous \TeX liberating, but after a while I found the inconsistencies of spacing and the illegibility of some of the formulae (especially those involving sub/superscripts) very offputting. There are also some odd translation errors: for example, the word ‘precizize’, which is not in any dictionary I consulted; the use of ‘biangle’ where ‘bigon’ is standard in English; and, more seriously, a mathematical error on page 56, which appears to have been caused by a mistranslation, since the text contradicts the figures. (Cutting double curves from branch point to branch point or boundary does change the topological type, as Figure 64 clearly shows. Figure 65 shows how to do this, without changing topological type, when the cut is between two branch points.) Further, it is not always clear when a result which needs proof has been omitted: for example, the disk theorem is never stated, but just regarded as obvious; if there were exercises, they could be used to signpost this easy kind of omitted result. Finally, I found the numbering very unhelpful: for example, we have Theorems 5.1 and 5.2 with Corollaries 5.1 and 5.2 interleaved, which makes it very difficult to find things.

Here is a summary of the material covered in the book. Chapter 1 comprises a speedy review of basic material: this is far from complete (for example, $D^n$ and $S^n$ are never defined), and has some minor errors (for example, the definition of a manifold with structure—never used in the book—is faulty). Chapter 2 covers surfaces and the basic technique of scissors and glue proofs, including Dehn’s lemma. Chapter 3 covers the Alexander trick, braids and the homeotopy groups of surfaces and handlebodies. Chapter 4 explains how to glue polyhedra to obtain 3-manifolds, and gives a thorough treatment of Lens spaces. Chapter 5 covers Heegaard splittings and diagrams, including stable equivalence and methods for classifying genus 2 diagrams algorithmically. Chapter 6 covers in detail the old (pre Rubinstein–Thompson–Matveev), and now redundant, methods for recognising the 3-sphere. Since one of the authors of this book is one of the originators of the new complete algorithm, this chapter is particularly unsatisfactory; I feel strongly that it ought to have been thoroughly revised for this English edition. (The new algorithm can be explained at no greater length than is required for this old material.) Chapter 7 contains a good, thorough treatment of connected sums, including the necessary material on normal surfaces, and Chapter 8 covers racks/quandles and the various knot polynomials. This chapter again shows the age of the original Russian version of the book, in that the Vassiliev invariants are never mentioned. There is another unhappy translation here: racks and quandles are translated as ‘distributive groupoids’ (Matveev’s Russian name), which is then shortened to groupoids, leading to confusion with the
fundamental groupoid. Chapter 9 covers the (easy parts of) the Kirby calculus, and Chapter 10 covers Seifert manifolds. Chapter 11 covers class $H$ manifolds (usually called graph manifolds), and Chapter 12 covers Haken’s methods, including his algorithm to recognise the unknot.

This is a great book for self-study for graduate students, but is hopelessly overpriced for this niche of the market, which is already occupied by Rolfsen’s classic text, available at less than a quarter of the price. I have the following suggestions for the publishers: get the book updated, and preferably include some exercises in the text, then bring out a paperback version priced at around £25. The book could then be very strongly recommended for graduate students to buy and put on their shelves. This present hardback edition will be bought only by (rich) libraries.

Warwick University

COLIN ROURKE

CODES AND ALGEBRAIC CURVES
(Oxford Lecture Series in Mathematics and its Applications 8)

By OLIVER PRETZEL: 192 pp., £35.00, ISBN 0 19 850039 4

In 1981, there occurred one of those tremors in the mathematical world whose effect was wider than new results generally produce, simply because it provided a link between two previously-disjoint areas, coding theory and algebraic geometry.

Goppa [1] had constructed linear codes from the rational points of a non-singular algebraic curve over a finite field, having noted that the Riemann–Roch theorem gives information on the parameters of the codes. It was then shown by Tsfasman, Vlăduț and Zink [7] that, for a certain family of curves, these codes surpassed the Gilbert–Varshamov bound on the limiting curve of two parameters of a family of codes.

This also brought in the question of the number of rational points that an algebraic curve can have, with all the associated number theory.

The problem, then, for coding theorists, is how to learn sufficient algebraic geometry and number theory to understand this material.

The book under review is in two parts. The first part is a revised version of the last five chapters of a previous book [4]. The aim of this first part is to present enough algebraic geometry to explain the construction of Goppa’s codes (now generally known as algebraic geometry codes). The second part aims to supply some proofs not included in the first part, and to give a more rounded picture of the foundations of the theory of algebraic curves.

Is there a difference between writing a book based on a course of lectures and one giving a systematic account of a mathematical topic? In some branches of mathematics, a first course of lectures may very well simply comprise the initial chapters of a systematic account. The inherent difficulty of algebraic geometry makes this almost impossible.

There are three other books, [3] (for a review, see [2]), [5] and [6], whose aim was to present the theory of algebraic geometry codes; all three are completely different and also very different from this account.
In contrast to these three other books, this is definitely a book whose basis is a course of lectures, and it has been honed by several courses. The first part still makes a coherent course, gives many examples, and describes the theory of the corresponding codes in some detail. The aim of describing the codes provides a clear focus for the algebraic geometry needed. There are many examples, and the practical details of decoding are clearly described.

However, it is not for nothing that the foundations of algebraic geometry have been revised many times. The account in Part I works only for non-singular curves. The essential idea of a place has to wait for Part II.

So, to learn enough algebraic geometry to understand Goppa’s codes, a student could take this book and study it with advantage. For a systematic account of curves over a finite field, this account is not ideal, but gives any student an excellent start.


References


University of Sussex

J. W. P. Hirschfeld

TWELVE SPORADIC GROUPS
(Springer Monographs in Mathematics)

By Robert L. Griess Jr: 169 pp., £49.00, ISBN 3 540 62778 2

The twelve groups referred to in the title of this interesting book are those sporadic simple groups discovered in what the author refers to as the first two generations of the Happy Family. The first generation consists of the five sporadic
simple groups contained in the largest Mathieu group $M_{24}$, and the second generation consists of a further seven which are involved in the largest Conway group $Co_0$. Since $M_{24}$ is itself contained in $Co_0$—indeed, Conway used frequently to refer to his group as ‘$M_{24}$ writ large’—the book may be viewed as an algebraic approach to the exceptional geometrical configurations related to the Leech lattice $A_{24}$.

In that sense, the second generation dates from 1965, when John Leech discovered his remarkable lattice in connection with sphere-packing in 24-dimensional real space. The basic sphere-packing problem consists of arranging non-overlapping, $n$-dimensional spheres of fixed radius in such a way as to maximise the proportion of $n$-space they occupy. Dimension one we can all cope with, and arguably honey bees knew the solution to the 2-dimensional problem thousands of years ago. Sadly, the general 3-dimensional problem remains unsolved, although we are pretty sure we know the answer. However, the problem becomes more tractable if we require the centres of the spheres to lie at the vertices of an $n$-dimensional integral lattice. Thus in 1965, John Leech used the length 24 binary Golay code, which has $M_{24}$ as its group of automorphisms, to construct a 24-dimensional lattice which yielded an extraordinarily efficient sphere-packing.

Leech knew Conway through the latter’s interest in packing problems, and so suggested that he might like to work out the group of symmetries of the Leech lattice. And so it was that Conway, in a consummate and elegant piece of work, discovered the perfect group $Co_0$, whose quotient by its centre of order two was a new sporadic simple group. Now several of the ‘second generation’ groups had already been discovered: the Hall–Janko group as a rank 3 permutation group on 100 letters; the Higman–Sims group, both as a rank 3 group on 100 letters, and as a doubly-transitive group on 176 letters; the McLaughlin group as a rank 3 group on 275 letters; and the Suzuki group as a rank 3 group on 1782 letters. The Leech lattice treated us to the joy of seeing these actions realised before our eyes in the ‘real world’. In particular, the isomorphism between the rank 3 group found by Donald Higman and Charles Sims, and the doubly-transitive group found by Graham Higman, was visible for all to see. Besides these known groups, however, $Co_0$ contained two new sporadic simple groups, as stabilisers of certain vectors; these are now known as the Conway groups $Co_2$ and $Co_3$.

This text aims to introduce the graduate student with a sound algebraic background, and a knowledge of permutation and matrix groups, to the five Mathieu and the seven further sporadic groups mentioned in the above paragraph. The author starts by reviewing the group-theoretic results that he will use later, and by presenting an introduction to error-correcting codes. He also gives a brief survey of homological algebra, and includes useful tables of Schur multipliers of known finite simple groups. Calculations of the latter were notoriously error-strewn, with the correct and remarkably rich covering group of $M_{24}$ emerging surprisingly recently; see [I]. Indeed, it was uncertainty about the Schur multiplier of the Fischer group $F_{24}$ which delayed the discovery of the Monster group itself.

The meat of this text, however, lies in Chapters 4 to 10: $M_{24}$ is introduced by way of the Miracle Octad Generator/hexacode construction of the binary Golay code; the ternary Golay code and double cover of $M_{12}$ are defined in an analogous manner; and the Leech lattice and its various sublattice stabilisers are described. Valuable material is included in the various appendices to Chapter 10, the highlight perhaps being D. G. Higman’s beautiful treatment of rank 3 permutation groups.

The text concludes with a brief description of the remaining sporadic simple
groups, both those involved in the Monster and the six so-called ‘pariahs’ which are not, and a personal view of matters such as the classification of finite simple groups.

Altogether, the book brings together a great deal of previously scattered material, and is a valuable addition to one’s bookshelves.

Reference


University of Birmingham

ROBERT CURTIS

**SYMBOLIC DYNAMICS: ONE-SIDED, TWO-SIDED AND COUNTABLE STATE MARKOV SHIFTS**

(Universitext)

*By Bruce P. Kitchens*: 252 pp., £22.50, ISBN 3 540 62738 3


I like this book. It is about symbolic dynamics, a curious word for an even more speculative venture initiated in an attempt to describe the asymptotic behaviour of supposedly continuous physical systems by means of discretisations containing ‘full information’. In more recent years, emphasis has (fortunately, in my view) clearly shifted to the discrete systems themselves, with substantial applications in electronic engineering and computer science, and to description of finite time behaviour as well as asymptotics. This has made the subject more difficult as well as more interesting.

The focus of the exposition at hand is teaching, and the goal group consists of graduate students. Kitchens uses a pleasant, no-nonsense hands-on approach, and communicates in a style which, although perhaps too loose for the traditional mathematician, gets to substantials rapidly and handles the important concepts with precision. No spoon-feeding here, but efficient presentation of interesting objects such as subshifts of finite type, with both finite and countable state spaces, topological and almost-topological conjugacies, automorphisms and factor maps, and a final section connecting the basic concepts with further topics such as sofic systems, cellular automata and channel codes.

My conclusion consists of a strong reading recommendation for graduate students in a wide variety of fields: combinatorics, probability, analysis, dynamics and perhaps theoretical computer science. People outside mathematics can fare better with more lengthy and pedestrian literature which is also available. And, of course, a word of caution: symbolic dynamics is mathematically contagious!

CWI/University of Amsterdam

MICHAEL KEANE
Annick Lesne has made a brave attempt to present the use of renormalization methods in a range of physical contexts, from statistical mechanics and dynamical systems to stochastic processes and fractal structures.

Renormalization of a system means looking at it on a different scale, usually a smaller spatial scale and/or longer timescale and a different scale in ‘state space’. Then any feature of the system is preserved (up to scale). It may seem that nothing is gained. The gain comes on analysing the asymptotic behaviour of the resulting dynamical system on the space of systems. If, for example, the orbit of a system under renormalization converges to a fixed point, then the original system exhibits asymptotic self-similarity in the appropriate scaling limit, with power laws whose exponents are related to the scale factors of the renormalization at the fixed point. If the fixed point has some unstable directions, then any transverse family to its stable manifold exhibits asymptotic self-similarity near the point of intersection, with power laws whose exponents depend in addition on the unstable eigenvalues of the fixed point. Generalizations to more complicated recurrent behaviour under renormalization are immediate.

Despite the breadth of this survey, its scholarly references and a glowing foreword by Pierre Collet, I fear that mathematicians will be disappointed with the book. The treatment is mathematically shallow in nearly every topic treated. For example, no explanation is given for the universality classes of momentum space renormalization in statistical mechanics; the impression is given that there is a satisfactory renormalization procedure for turbulence, but I am not aware of any; and it is not made clear whether there is a satisfactory renormalization approach to percolation problems. Furthermore, the application of renormalization to dynamics in the complex plane, which might particularly interest mathematicians in the light of McMullen’s recent Fields Medal and that of Yoccoz four years ago, is mentioned only in a three-line remark.

Even a physicist might be disappointed. In addition to the above deficiencies, the controversial question of the possibility or not of rigorous real space renormalization of statistical mechanics systems in more than one dimension is not mentioned; and the whole topic of renormalization in quantum field theory, where the method first started, is left to other texts.

More seriously for both types of reader, the book is not accurate. For example, the discussion of the topic which I know best, renormalization for the breakup of invariant circles of area-preserving twist maps, is totally misleading. Lesne claims that it is solved by studying the loss of smooth conjugacy for circle homeomorphisms when a point of zero slope appears. It is true that the problems of breakup of invariant circles and loss of smooth conjugacy for circle homeomorphisms are related: the induced map on a ‘critical’ golden invariant circle is a circle homeomorphism which I believe to be an interesting $C^{1,79}$ fixed point of circle map renormalization with $C^{6,71}$ conjugacy to rotation. From a theorem of Birkhoff, however (for a thorough presentation, see M. R. Herman, ‘Sur les courbes invariantes par les difféomorphismes de l’anneau’, *Asterisque* 103–4 (1983) Chapter 1), the
induced map on an invariant rotational circle for an area-preserving twist map is a
Lipomorphism, so has no critical point. Renormalization is believed to apply to this
problem (see, for example, R. S. MacKay, *Renormalisation in area-preserving maps*,
World Scientific Publishing, Singapore, 1993), but not as a corollary of the case of
circle homeomorphisms with a critical point.

In conclusion, the idea of the book is good, but it only scratches the surface of the
subject.

University of Cambridge

ROBERT MACKAY

**BILINEAR ALGEBRA: AN INTRODUCTION TO THE ALGEBRAIC
THEORY OF QUADRATIC FORMS**

(Algebra, Logic and Applications 7)

*By Kazimierz Szymiczek* : 486 pp., £55.00, ISBN 90 5699 076 4

(Gordon and Breach Science Publishers, 1997).

This is a great mathematical story to be told. A young and brilliant mathematician,
by the name of Ernst Witt, was able to develop a fully original and elegant theory of
quadratic forms over any field of characteristic not two [3]. His paper shed new light
on the classical theory of quadratic forms, which was developed and nurtured by such
notable mathematicians as Fermat, Euler, Gauss, Minkowski, Hasse and others. In
just a few pages, Witt defines a ring of classes of similar quadratic forms over a field $F$.
Each theorem (Satz 1 to Satz 7) is firmly carved in the history of mathematics. These
theorems include the well-known cancellation theorem, the decomposition of a
quadratic form into its anisotropic and hyperbolic parts, and the now famous chain
equivalence theorem, which realizes any quadratic form equivalence as a chain of
equivalences of binary subforms.

As in the fairy tale of *Sleeping beauty*, algebraic quadratic form theory was sound
asleep for almost thirty years, when it was awakened with the wonderful theorems of
Cassels and Pfister, who succeeded in proving, among many other results, that for
each $n \in \mathbb{N}$, the non-zero sums of $2^n$ squares form a subgroup of the multiplicative
group of a field. Then followed gem after gem: the theory of Pfister forms, the local–
global principle for recognizing torsion elements of Witt rings, and some important
structural results for Witt rings. Quadratic form theory has blossomed ever since
Pfister’s work, culminating with the recent results of Merkurjev, Rost, Suslin,
Voevodsky and others. They have succeeded in proving that graded Witt rings,
Milnor $K$-theory mod 2 and Galois cohomology mod 2 all coincide. There are already
some wonderful books which describe Witt’s and Pfister’s theory of quadratic forms,
and much more. Certainly, the books by Lam [1] and Scharlau [2] are read, studied
and loved. Other excellent books are available.

There is, however, still room for other welcome contributions and expositions
on algebraic quadratic form theory. Kazimierz Szymiczek’s book is certainly one of
them. This is a carefully written work, by the well-known researcher and expositor,
on the theory of quadratic forms. It is more elementary than the books quoted above.
The important case of fields with characteristic two is included. In addition, there are
some very interesting comments and references to the literature, and a large number
of well-chosen problems is offered. This book stops at Harrison’s well-known criterion for deciding when two Witt rings are isomorphic, but the reader may go on in many possible directions. For many people, including advanced undergraduate students, Szymiczek’s book will open up a beautiful path to the magical country of quadratic forms and its relations with other areas of mathematics.

References


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Ján Mináč

RANDOM MATRICES, FROBENIUS EIGENVALUES, AND MONODROMY

(American Mathematical Society Colloquium Publications 45)


A proof of the Riemann Hypothesis is the major missing element in the theory of the Riemann zeta-function and related zeta-functions. The Riemann Hypothesis would say that every non-trivial zero \( \beta + i \gamma \) of the Riemann zeta-function lies on the line \( \beta = \frac{1}{2} \). However, there are many important problems for which the distribution of the imaginary parts \( \gamma \) of the zeros also plays a significant rôle. Examples include the Mertens conjecture, questions on the fine distribution of primes in short intervals, the Birch–Swinnerton-Dyer conjecture, and class-number problems.

Theoretical work of Montgomery, and numerical calculations by Odlyzko, have led to the GUE hypothesis. This states that the normalized differences

\[
(\gamma' - \gamma)(\log \gamma)/2\pi
\]

of consecutive zeros should have the same limiting distribution as the normalized differences of the eigenvalues of a large random unitary matrix. A second question of interest concerns the distribution of the smallest zero, as one goes through a family of zeta-functions. Thus, for example, one might consider all Dirichlet \( L \)-functions with quadratic characters, and investigate the normalized zeros \( \gamma_q(\log q)/2\pi \), where \( \frac{1}{2} + i\gamma_q \) is the smallest zero of the quadratic \( L \)-function to modulus \( q \). Here one appears to obtain a distribution quite different from the GUE hypothesis.

In their book, Katz and Sarnak investigate the corresponding questions for the zeta-functions of varieties over finite fields. In this situation, they are actually able to prove distribution laws for the zeros of zeta-functions of ‘almost all’ curves of large genus, in the limit as both the genus and the size of the finite field tend to infinity. For the spacings between consecutive zeros, they establish the GUE hypothesis. However,
the situation for the distribution of the smallest zero, as one runs through a family of varieties, is rather more interesting, in that the result obtained depends on the associated monodromy group. The authors therefore propose that there should be some type of monodromy group associated to a family of global zeta-functions, and dependent on the form of the functional equation. This would explain very well the differing behaviour of $\gamma_1$ for various families of zeta-functions. A fuller description of all this is given by the authors in their survey article [I].

To a large extent, this book is an extended research paper, and much of the material is new. The first eight chapters give a detailed treatment of the spacing of eigenvalues for groups of large matrices. The book then goes on to give the necessary results on varieties over finite fields. This leads up to the equidistribution of certain Frobenius conjugacy classes in the geometric monodromy group. Finally, the two aspects of the theory are combined to handle the distribution of differences of consecutive zeros, and of the smallest zero, of the zeta-function attached to the variety.

This book is not for the faint-hearted. The material is wide ranging and difficult. However, for research workers interested in the Riemann Hypothesis, or in the arithmetic of varieties over finite fields, this work has important messages which may help to shape our thinking on fundamental issues on the nature of zeta-functions.

Reference


Cambridge University

D. R. Heath-Brown

WHITE NOISE DISTRIBUTION THEORY

(Probability and Stochastics Series)


White noise distribution theory is part of general Gaussian analysis, that is, analysis on linear spaces equipped with Gaussian measures. A typical example is related to a given Gelfand triple $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$, where $\mathcal{H}$ denotes a real separable Hilbert space, and $\mathcal{N}$ is a nuclear space densely topologically embedded in $\mathcal{H}$. Then we can introduce a Gaussian measure $\mu$ on the dual space $\mathcal{N}'$ by its characteristic functional

$$\int_{\mathcal{N}'} \exp(i\langle x, f \rangle) \, d\mu(x) = \exp(-\frac{1}{2} \| f \|_2^2), \quad f \in \mathcal{N}'. $$

(The case of so-called white noise measure $\mu$ corresponds to $\mathcal{H} = L^2(\mathbb{R}^d)$.) The concept of generalized functions on Gaussian space $(\mathcal{N}', \mu)$ supposes first of all a construction of riggings for the space of square-integrable functions $L^2(\mu)$ by proper spaces of test and generalized functions:

$$\mathcal{A} \subset L^2(\mu) \subset \mathcal{A}'.$$
The book under review contains an important development with respect to the previous book by the same author and T. Hida, J. Potthoff and L. Streit [2], in as much as in the present book a continuous scale of test function spaces \( \mathcal{X} = (A^\alpha)_{\beta}, \beta \in [0, 1) \), is considered together with the corresponding scale of dual spaces \( \mathcal{X}' = (A^\alpha)' \). (The classical case of Hida spaces in white noise analysis, discussed in [2], corresponds to \( \beta = 0 \).) Starting with this framework, the author gives an easy presentation of white noise distribution theory which is (as pointed out in the Preface) ‘accessible to anyone with a first-year graduate course in real analysis’. Among applications of the distribution theory, the author discusses integral kernel operators, Fourier transforms, infinite-dimensional Laplacians and Feynman integrals. Unfortunately, the author excluded from consideration the case \( \beta = 1 \) which is especially important in applications, as has become clear recently in many works on stochastic differential equations with Wick-type nonlinearity and the theory of Feynman integrals. We refer to the book [3] for related discussions and many concrete examples.

The author mentions in the preface that the book is based on a series of lectures held at several universities. This fact has a natural reflection in the style of the book, and especially in the historical comments in the Preface and Appendix A. As a result of this ‘lecture notes style’, many original monographs and papers remain outside the scope of the book, and references to sources are based on the author’s preferences rather than on an objective picture. A similar situation appears with respect to applications: the presentation here is oriented to problems of stochastics, and almost nothing is written about the extremely important relations with many domains of modern mathematical physics. We refer the reader to the books [1, 2] for more references, historical comments and applications of the theory of infinite-dimensional generalized functions to models of quantum field theory, statistical physics, the theory of differential operators with an infinite number of variables, and many other topics.

References


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Yuri G. Kondratiev
THE CAUCHY PROBLEM FOR HYPERBOLIC OPERATORS: MULTIPLE CHARACTERISTICS. MICRO-LOCAL APPROACH
(Mathematical Topics 12)

By KAREN YAGJIDAN: 397 pp., £45.00., ISBN 3 05 501739 0
(Akademie Verlag (distributed by John Wiley & Sons), 1997).

PARTIAL DIFFERENTIAL EQUATIONS WITH MULTIPLE CHARACTERISTICS
(Mathematical Topics 13)

By MARIA MASCARELLO and LUIGI RODINO: 352 pp., £55.00, ISBN 3 05 501764 1
(Akademie Verlag (distributed by John Wiley & Sons), 1997).

The books being reviewed deal with central problems of the theory of partial differential equations (PDEs):

1. whether a given PDE has a (local) solution,
2. whether this solution is unique, and
3. how to construct it approximately (modulo smooth or analytic functions).

Needless to say, these problems have been actively studied since last century, and satisfactory answers have been obtained for many classes of equations.

From the modern point of view, the notion of a ‘solution’ is almost meaningless unless we specify a function space where we look for solutions. Clearly, the choice of an appropriate space depends on a PDE’s coefficients. In both books, it is assumed that coefficients are infinitely smooth and that solutions belong to the Sobolev function spaces.

In the smooth case, the main object associated with a PDE is the principal symbol \( p(x, \xi) \) of the corresponding partial differential operator \( P \). It is a function which is obtained by dropping the terms with lower-order derivatives and replacing the highest-order derivatives \( \partial^\alpha f / \partial x^\alpha \) with \( \xi^\alpha \), where \( x = (x_1, \ldots, x_n) \) are the spatial variables and \( \xi = (\xi_1, \ldots, \xi_n) \) are the so-called dual variables. The set

\[ \Sigma = \{ (x, \xi); \xi \neq 0, p(x, \xi) = 0 \} \]

is called the characteristic manifold of the operator \( P \).

If \( \Sigma = \emptyset \), then the corresponding operator is said to be elliptic. An elliptic PDE always has a (local) solution which can be constructed modulo \( C^\infty \) with the use of pseudodifferential operators. If \( p(x, \xi) \) is real and its gradient \( \nabla p \) does not vanish on \( \Sigma \), then the equation \( Pu = f \) is also solvable for any function \( f \), and an approximate solution can be constructed by means of Fourier integral operators. The situation is much more complicated if \( P \) has multiple characteristics, that is, \( \Sigma \neq \emptyset \) and \( \nabla p(x, \xi) = 0 \) for some \( (x, \xi) \in \Sigma \). A PDE with multiple characteristics may not have a solution even locally, the solution is normally not unique, and there is no canonical way of constructing approximate solutions.

The book by M. Mascarello and L. Rodino is an excellent introduction to the subject. It gives a comprehensive exposition of results on PDEs with multiple characteristics, including PDEs with analytic coefficients and solutions from Gevrey classes. The authors not only prove theorems but also illustrate general results with simple examples, which makes the book quite accessible and easy to read. It also
includes a self-contained presentation of the methods of microlocal analysis—the most powerful technique for studying PDEs with smooth coefficients. It is a very useful and enjoyable book, which can serve as a reference book to experts as well as a textbook to those willing to learn the subject (including graduate students).

The book by K. Yagdjidan mostly deals with the third problem (constructing a solution modulo a smooth function) for the Cauchy problem for hyperbolic PDEs with multiple characteristics. The operator which maps initial data into approximate solutions is called a parametrix. For classical hyperbolic PDEs (in particular, for the wave equation), the parametrix is a Fourier integral operator whose Schwartz kernel is represented as a sum of oscillatory integrals of the form

\[ \int e^{i\phi(t,x,y,\xi)} a(t,x,y,\xi) d\xi. \]

Here \( t \) is the time variable, \( x \) and \( y \) are the spatial variables, \( \phi \) is a phase function positively homogeneous in \( \xi \) of degree 1, and \( a \) is given by an asymptotic series \( \sum a_j \) of positively homogeneous in \( \xi \) functions \( a_j \). Substituting an oscillatory integral into the equation and taking into account initial conditions, one obtains a system of ordinary differential equations for \( \phi \) and \( a_j \), which can be easily solved. The author shows that a similar approach can be used for hyperbolic operators with multiple characteristics. In this case, the phase function \( \phi \) is no longer homogeneous, which leads to essential complications. This book is not as easy to read as the previous one, but the author does obtain more advanced results. In particular, the construction of the parametrix in the form of an oscillatory integral may be useful in spectral theory. The book can be recommended for specialists working in the theory of PDEs and intending to extend their knowledge of PDEs with multiple characteristics.

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INTERPOLATION, IDENTIFICATION, AND SAMPLING
(London Mathematical Society Monographs N.S. 17)

By Jonathan R. Partington: 267 pp., £60.00, ISBN 0 19 850024 6

This original book is an admirable blend of pure and applied, of concrete and abstract analysis. Here one may find not only treasures of function theory from the great names of the early decades of the century, but also pointers to the role of their discoveries in robotics, space research and the study of tokamak fusion reactors.

The backdrop to the book is the function theory of the disc, the half-plane, the circle and the line. The importance of this material in much modern technology is widely recognised, and it is a well-worked landscape. The present monograph has overlaps not only with texts of mathematical analysis, but also with treatises on signal processing, control theory and modelling. It is, nevertheless, a very individual treatment. The author’s doctoral training in Banach space theory was modified in a practical direction by a spell in the Engineering Department at Cambridge University, and he is therefore in an unusually good position to illustrate the oft-claimed continuity between pure mathematics and applied science.
The organising principle of the book is the notion of the recovery of functions: recover an unknown function from a prescribed class from suitable measurements (which are themselves contaminated by noise). This question arises in many different contexts, and accordingly goes by several names. Approximation theory and regression are two large bodies of mathematics which address forms of the question, but the author’s approach has most in common with the engineering view, known as the problem of identification, as in the title of the book. The engineering literature is mainly devoted to the stochastic aspect of the problem, but there is also a deterministic or ‘worst case’ view of the matter on which both classical analysis and abstract analysis can throw much light.

Suppose that one wishes to model a physical system by means of a function from a particular class, or, alternatively, to reconstruct a function from a known class after its transmission through a communications channel. A plausible idealisation is that we may perform an arbitrarily large number of imperfect measurements, and a reasonable goal is to approximate the unknown function as closely as desired as the number of measurements tends to infinity and the measurement error tends to zero. To solve various formulations of this problem for natural function classes, the author ranges widely. Some classical preoccupations of function theory are directly relevant, for example, interpolating sequences, zero sequences (if all the measured values are exactly zero, must the unknown function be zero?). Bases of Banach spaces with good properties provide a useful tool, so old and new results on special polynomials, wavelets, etc., are brought in. Some interesting negative results call for a modest amount of Banach space theory: roughly speaking, there is no linear robust identification algorithm for the disc algebra. There is, however, a reasonably practical non-linear algorithm, with analysable error for approximate measurement points. Other topics which have a part to play are Hankel and Toeplitz operators, Paley–Wiener theory, \( n \)-widths, information-based complexity and even Galois theory.

This book can be recommended simply as a clear and well-judged exposition of some beautiful mathematics, but its special virtue is to afford an opportunity to pure mathematicians to gain an understanding of some technological applications of analysis. Doubtless, many analysts are intrigued by these applications but lack the dedication needed to penetrate the very different vocabulary and assumptions of the engineering literature; here is a short-cut.
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