NONABSOLUTE INTEGRAL ON MEASURE SPACES

NG WEE LENG AND LEE PENG YEE

Abstract

We define a Henstock-type integral on measure spaces with metric topologies, and give an example of a function which is integrable but whose absolute value is not.

In [1], Henstock constructed a division space from an arbitrary non-atomic measure space with a locally compact Hausdorff topology that is compatible with the measure, and defined the Davies–McShane integral. However, the integral defined is absolute in the sense that if \( f \) is integrable, then so is \( |f| \). This is because Henstock considered all measurable sets in his construction of a division. We shall show in this paper how a division can be constructed in order to define a nonabsolute integral measure theoretically.

1. Preliminaries

Let \((X, d)\) be a metric space, and let \((X, \Omega, \iota)\) be a measure space with the metric topology \(\mathcal{T} \subset \Omega\) induced by the metric \(d\) on \(X\).

The measure \(\iota\) is assumed to be non-negative and countably additive. Furthermore, the following condition will be assumed throughout this paper.

(*) For every measurable set \(W \in \Omega\) and every \(\epsilon > 0\), there exist an open set \(U\) and a closed set \(Y\) such that \(Y \subset W \subset U\) and \(\iota(U \setminus Y) < \epsilon\).

All measurable sets were considered in the construction of a division in [1], and as a result an absolute integral was obtained. This shows that in order to define a nonabsolute integral, we must recruit fewer objects as generalised intervals. To motivate our argument, we observe that on the real line, bounded intervals are used, and a bounded interval can be seen as the difference of two bounded intervals such that one does not contain the other. Generalising this concept, we can consider the finite intersection of sets each of which is the difference of two connected sets. We shall now define this formally. Let \(\mathcal{T}_1\) be the set of all open balls. A set of the form \(\{y \in X : d(x, y) < r\}\), where \(x \in X\) and \(r > 0\), denoted by \(B(x, r)\), is called an open ball with centre \(x\) and radius \(r\). We shall also call its closure a closed ball. Throughout this paper we shall assume that \(\iota(U) > 0\) and \(\iota(U) = \iota(\bar{U})\) for all \(U \in \mathcal{T}_1\), where \(\bar{U}\) denotes, as usual, the closure of \(U\).

Consider the following sets:

\[
\mathcal{H}_1 = \{B_1 \setminus B_2 : B_1, B_2 \in \mathcal{T}_1 \text{ where } B_1 \supseteq B_2 \text{ and } B_2 \supseteq B_1\},
\]

\[
\mathcal{H}_2 = \left\{ \bigcap_{i \in \Lambda} X_i \neq \emptyset : X_i \in \mathcal{H}_1 \text{ and } \Lambda \text{ is a finite index set} \right\}.
\]
More precisely, members of $\mathcal{H}$ are either closed balls or scalloped balls. A typical member of $\mathcal{H}$ is a finite intersection of a combination of closed balls or scalloped balls. We note that taking the difference of two scalloped balls such that one does not contain the other ensures that we obtain connected scalloped balls with no ‘holes’.

We shall call members of $\mathcal{H}$ \emph{generalised intervals}, or simply \emph{intervals} where there is no ambiguity. Note that generalised intervals are relatively compact, though not necessarily closed or compact. Also note that generalised intervals are connected. Since we assume that $\mathcal{I}(U) = \mathcal{I}(U)$ for each $U \in \mathcal{F}$, for each generalised interval $I$ we also have $\mathcal{I}(I) = \mathcal{I}(I)$.

**Example 1.** Let $X$ be the real line and $\mathcal{F}$ the family of all open sets. Then $\mathcal{F}$ is the set of all open intervals $(a, b)$. It is easy to see that $\mathcal{H}$ is the set of all intervals of the form $(a, b), (a, b], [a, b)$ or $[a, b]$. In other words, generalised intervals in this case are the usual bounded intervals. Notice that taking the difference of two bounded intervals such that one does not include the other ensures that we obtain a connected interval rather than two disjoint bounded intervals.

**Example 2.** Let $X$ be the two-dimensional Euclidean space $\mathbb{R}^2$. The metrics $d_1$ and $d_2$ in $X$ are given by

$$d_1(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$
$$d_2(x, y) = (|x_1 - y_1|^2 + |x_2 - y_2|^2)^{1/2};$$

for each $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $X$. It is well known that the $d_1$-open balls are squares without the boundaries, and the $d_2$-open balls are open circular discs. It is easy to see that when the metric $d_1$ is used, a generalised interval looks like a polygon with edges each being vertical or horizontal, and each edge is not necessarily included. When the metric $d_2$ is used instead, a generalised interval is a simply connected domain in the plane with piecewise circular edges, and each arc may or may not be included.

We next present the necessary and standard terminology for defining a Henstock-type integral.

Let $E$ be a finite union of (possibly just one) mutually disjoint intervals, and call it an \emph{elementary set}. Throughout this paper, we shall let an elementary set $E$ with finite measure be fixed, and define integrability on $E$. An elementary set $E$ is said to have finite measure if $\mathcal{I}(E) < +\infty$. Note that since $\mathcal{I}(I) = \mathcal{I}(I)$ for each generalised interval $I$, we also have $\mathcal{I}(E) = \mathcal{I}(E)$.

A set $\langle(I_i, x_i) : i = 1, 2, \ldots, n\rangle$ of interval–point pairs is called a \emph{partial division} of $E$ if $I_1, I_2, \ldots, I_n$ are mutually disjoint subintervals of $E$ such that $E \setminus \bigcup_{i=1}^n I_i$ is either empty or an elementary subset of $E$, and for each $i$, we have $x_i \in \overline{I_i}$. For each $i$, we call $x_i$ the \emph{associated point} of $I_i$. A \emph{division} of $E$ is a partial division $\langle(I_i, x_i) : i = 1, 2, \ldots, n\rangle$ such that the union of the $I_i$ is $E$.

Let $\delta : E \to \mathbb{R}^+$ be a positive function. We call $\delta$ a \emph{gauge} on $E$. Note that we need to consider gauges $\delta$ defined on $E$ and not just $E$, because for each interval–point pair $(I, x)$ in a partial division, the associated point $x$ comes from $I$ and not just $I$.

Let a gauge $\delta$ on $E$ be given. An interval–point pair $(I, x)$ is $\delta$-\emph{fine} if $I \subseteq B(x, \delta(x))$. A partial division $D = \langle(I_i, x_i) : i = 1, 2, \ldots, n\rangle$ of $E$ is $\delta$-\emph{fine} if $(I_i, x_i)$ is $\delta$-fine for each $i = 1, 2, \ldots, n$. Since divisions themselves are partial divisions, the $\delta$-fine divisions of $E$ are similarly defined.
A partial division $D*$ of $E$ refines or is a refinement of another partial division $D$ of $E$ if for each $(I, x) \in D*$, we have $I \subset J$ for some $(J, y) \in D$.

A gauge $\delta_1$ is said to be finer than a gauge $\delta_2$ on $E$ if for every $x \in E$ we have $\delta_1(x) \leq \delta_2(x)$. Let $\delta_1$ and $\delta_2$ be two gauges on $E$. There exists a gauge $\delta$ on $E$ finer than $\delta_1$ and $\delta_2$. Consequently, if $D$ is a $\delta$-fine division of $E$, then $D$ is both $\delta_1$-fine and $\delta_2$-fine.

2. Existence of a division

For each $Y \subset X$, the diameter of $Y$ is given by

$$d(Y) = \sup\{d(x, y) : x, y \in Y\}.$$ 

We say that a subset $Y$ of $X$ is totally bounded if for every $\varepsilon > 0$, the open cover $\{B(y, \varepsilon) : y \in Y\}$ of $Y$ has a finite subcover. It is well known that if $Y$ is compact, then $Y$ is totally bounded. We shall need this concept in the following theorem.

**Theorem.** Given a gauge $\delta_0$ on an interval $I$, there exists a $\delta_0$-fine division of $I$.

**Proof.** We shall prove this in two parts.

1. Note that $I$ is compact and has finite diameter. By Condition (*) above, we can choose an open set $U$ such that $I \subset U$ and $d(U) < + \infty$. Let $\delta$ be a gauge on $I$ such that for each $x \in I$, we have $B(x, \delta(x)) \subset U$ and $B(x, \delta(x)) \subset B(x, \delta_0(x))$. This is to ensure that we obtain a $\delta$-fine division later. We first construct a finite collection of open balls which covers $I$ such that the centre of each ball lies outside all other open balls. For each $x \in I$, we define

$$a_1 = \sup\{\delta(x) : x \in I\}.$$ 

Then since $\delta(x) \leq d(U)$ for each $x \in I$, we see that $a_1$ is finite. Let $B_1 = B(x_1, a_1/2)$ for some $x_1 \in I$ such that $\delta(x_1) \geq a_1/2$, and define

$$a_2 = \sup\{\delta(x) : x \in I \setminus B_1\}.$$ 

Let $B_2 = B(x_2, a_2/2)$ for some $x_2 \in I \setminus B_1$ such that $\delta(x_2) \geq a_2/2$, and note that $x_1 \notin B_2$ while $x_2 \notin B_1$. Continuing this process inductively, we further define, for each $k = 3, 4, \ldots$,

$$a_k = \sup \left\{ \delta(x) : x \in I \setminus \bigcup_{i=1}^{k-1} B_i \right\},$$ 

and obtain $B_k = B(x_k, a_k/2)$ for some $x_k \in I \setminus \bigcup_{i=1}^{k-1} B_i$ such that $\delta(x_k) \geq a_k/2$. Note that $x_k \notin B_i$ if $k \neq i$, and $\{a_k\}$ is monotone. Now let $\lim_{k \to \infty} a_k = a_0$. Clearly, $a_k \geq a_0 \geq 0$ for all $k$. We shall show that $a_0 = 0$ by contradiction. Suppose $a_0 > 0$, and consider $\mathcal{B} = \{B(x, a_0/4) : x \in I\}$. Note that we have constructed a countable collection of open balls $B_1, B_2, \ldots$ such that the distance between the centres of any two such balls is more than $a_0/2$. Hence any open ball with radius $a_0/4$ can contain at most one of the centres $x_i$. It follows that $\mathcal{B}$ does not have a finite subcover. But this is a contradiction, as $I$ is compact and so is totally bounded. Therefore $a_0 = 0$. Consequently, we have $\bigcup_{k=1}^\infty B_k = I$. Indeed, if there is $x \in I \setminus \bigcup_{k=1}^\infty B_k$, then $x \in I \setminus \bigcup_{k=1}^{n-1} B_k$ for all $k$, which implies that $\delta(x) \leq a_k$ for all $k$. That is, $\delta(x) = 0$, which is not true. Hence $\{B_1, B_2, \ldots\}$ is an open cover of $I$. By the compactness of $I$, there exist $k_1, k_2, \ldots, k_n$ such that

$$I \subset \bigcup_{i=1}^n B_{k_i}.$$ 

For convenience, we shall still write $B_i = B_{k_i}$ for $i = 1, 2, \ldots, n.$
(2) We shall construct a division of $I$. We first define the generalised intervals
\[ I_i = I \cap \overline{B_i}, \]
\[ I_i = I \cap \bigcup_{k=1}^{i-1} (\overline{B_i} \setminus \overline{B_k}), \quad i = 2, \ldots, n. \]

As mentioned earlier, $x_i \notin B_k$ for $k \neq i$. Hence for each $i = 2, \ldots, n$, the point $x_i$ belongs to the interior of $\bigcup_{k=1}^{i-1} (\overline{B_i} \setminus \overline{B_k})$. We also have $x_i \in I$. Consequently, $x_i \in I_i$. On the other hand, from our construction, it is clear that
\[ \bigcup_{i=1}^{n} I_i = I \cap \left( \bigcap_{i=1}^{n} \overline{B_i} \right) = I, \]
and obviously the intervals $I_i$ are mutually disjoint. Finally, we note that for each $i = 1, 2, \ldots, n$,
\[ I_i \subset \overline{B_i} \subset \overline{B(x_i, \delta(x_i))} \subset B(x_i, \delta_\delta(x_i)). \]

Hence the division $\{(I_i, x_i): i = 1, 2, \ldots, n\}$ of $I$ is $\delta$-fine.

Observe that in the above proof, part (2) is not metric-dependent. As for part (1), the essence is to construct a finite collection of open balls which covers $I$ such that the centre of each ball lies outside all other open balls. So the result can be extended to the case when $\mathcal{F}$ is replaced by suitable topologies. The above proof is also an improvement of that in [3].

We further observe that each $I_i$ is a scalloped ball with its centre as the associated point $x_i$. This property will be used later in the construction of a nonabsolutely integrable function.

Our next task is to define the H-integral. Note that since intervals are measurable subsets of $E$, the latter assumed to be of finite measure, $\iota(I)$ is well-defined for all subintervals $I$ of $E$. Also, with our Theorem, it is now meaningful to define Riemann sums. For brevity and where there is no ambiguity, $D = \{(I_i, x_i)\}$ will denote a finite collection of interval–point pairs $(I, x)$, and the corresponding Riemann sum will be denoted by $(D) \sum f(x) \iota(I)$. All functions $f$ considered in this paper are real-valued point functions defined on $E$.

**Definition.** Let $f$ be a real-valued function on $E$. Then $f$ is said to be H-integrable on $E$ to a real number $A$ if for every $\varepsilon > 0$, there exists a gauge $\delta$ on $E$ such that for any $\delta$-fine division $D = \{(I_i, x_i)\}$ of $E$, we have
\[ \left| (D) \sum f(x) \iota(I) - A \right| < \varepsilon. \]

We write $(H) \int_E f = A$. The integrability of $f$ on any elementary subset $E_i$ of $E$ is similarly defined.

It is easy to see that the H-integral is uniquely determined, and closed under addition, scalar multiplication, monotone convergence and controlled convergence. Furthermore, the Cauchy criterion of integrability and Henstock’s lemma also hold [3].

Obviously, the Kurzweil–Henstock integral defined on the real line [2, p. 5] is a special case of the H-integral. More importantly, the H-integral includes the Davies–McShane integral defined in [1] if the underlying topology there is metrisable [3].
By our assumption, \( t(I) = t(\tilde{I}) \) for each interval \( I \). Thus using mutually disjoint or just non-overlapping intervals in a division does not affect the value of the corresponding Riemann sum. Two intervals are said to be non-overlapping if they have disjoint interiors. We choose to consider divisions comprising mutually disjoint intervals, for simplicity of presentation.

3. A nonabsolute example

We shall next prove that the H-integral is nonabsolute by constructing a function which is H-integrable but not absolutely H-integrable.

Let \( B_k \) be the closed ball in \( X \) centred at \( x_0 \) and with radius \( 1/k \), and let \( Y_k = B_k \setminus B_{k+1} \). Note that the \( Y_k, k = 1, 2, \ldots \), are mutually disjoint. Define

\[
f(x) = \frac{(-1)^{k+1}}{k \cdot t(Y_k)}
\]

when \( x \in Y_k \), \( k = 1, 2, \ldots \), and \( f(x_0) = 0 \). Clearly, the function \( f \) is Lebesgue integrable and hence H-integrable on \( Y_k \) for each \( k \). Next, for each interval \( I \subset B_1 \), we define

\[
F(I) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot t(Y_k)} t(Y_k \cap I),
\]

if the right-hand side exists. Obviously, \( F \) is additive.

Now for every \( \varepsilon > 0 \) and every positive integer \( k \), there exists a gauge \( \delta_k \) on \( E \) such that for any \( \delta_k \)-fine partial division \( D = \{(I, x)\} \) of \( E \), we have

\[
(D) \sum_{x \in Y_k} |f(x) t(I) - F(I)| < \varepsilon 2^{-k}.
\]

We may assume each \( \delta_k \) to have the property that if \( x \in Y_k \) we have \( B(x, \delta_k(x)) \subset Y_{k-1} \cup Y_k \). Let \( \delta > 0 \) be such that \( |F(I)| < \varepsilon \) whenever \( I \subset B(x_0, \delta) \). We then put \( \delta(x) = \delta_k(x) \) when \( x \in Y_k \), and \( \delta(x_0) = \delta \). Consequently, for any \( \delta \)-fine division \( D = \{(I, x)\} \) of \( B_1 \), we have

\[
\left| (D) \sum_{x \in Y_k} f(x) t(I) - F(B_1) \right| \leq \sum_{k=1}^{\infty} \left| (D) \sum_{x \in Y_k} f(x) t(I) - F(I) \right| + |F(B_1)|,
\]

where \((I, x) \in D \). It follows that \( f \) is H-integrable to \( F(B_1) \) on \( B_1 \). Obviously, its absolute value is not H-integrable on \( B_1 \).

The above example provides a function \( f \) which is H-integrable but whose absolute value is not. Hence we have shown that the H-integral is a nonabsolute one. We remark that the above example can be made metric-independent.

References


Division of Mathematics
School of Science
Nanyang Technological University
Singapore 259756