This book is an introduction to the classification theory of higher-dimensional algebraic varieties, with a special emphasis on the rôle of rational curves.

Chapter 1 presents some background material on curves and divisors, including the cone of curves, intersection theory, criteria for a divisor to be ample or nef and the Riemann-Roch theorem. The end of the chapter outlines the main ideas of the classification. There are two propositions illustrating the type of results to expect; for example, if $X$ and $Y$ are smooth projective varieties and $\pi : X \rightarrow Y$ is a birational morphism which is not an isomorphism, then there exists a rational curve $C$ on $Y$ contracted by $\pi$ such that $K_Y.C < 0$.

Chapter 2 deals with parametrizing morphisms from one variety to another. The space of morphisms from $\mathbb{P}^1$ to a given variety is, of course, the most important case. There are some interesting examples involving the families of lines contained in Fermat hypersurfaces.

Chapter 3 introduces the ‘bend-and-break’ method, which was Mori’s original idea for producing rational curves on a variety. This is used to prove that through every point of $n$-dimensional Fano variety there exists a rational curve $C$ such that $K_X.C \leq n + 1$. This chapter also contains results on rational curves on varieties whose canonical divisor is not nef, and on the structure of the Albanese map of varieties with nef anticanonical divisor.

Chapter 4 studies uniruled, rationally connected and rationally chain-connected varieties. These are the analogues of ruled and rational surfaces for many purposes. Smooth projective uniruled or rationally connected varieties in characteristic 0 can be characterised by the existence of a rational curve whose tangent bundle is generated by global sections or ample sections, respectively.

Chapter 5 introduces the notion of the rational quotient. Two points on a variety are equivalent if they can be connected by a chain of rational curves. In general, the equivalence classes do not form an algebraic variety, but it is possible to construct an algebraic variety called the rational quotient, which has an appropriate universal property. This can be used to bound the degree of Fano varieties, and to prove that they form a limited family.

The last two chapters form a brief introduction to the Minimal Model program. Chapter 6 contains the proof of the cone theorem for a smooth variety, and gives examples of extremal contractions, including an example of a flip. It also explains why singular varieties also need to be considered.

Chapter 7 is entitled ‘Cohomological methods’. This is the most difficult and least geometric part of the book; it includes the definitions of terminal and canonical singularities, the Kawamata–Viehweg and Nadel vanishing theorems, Shokurov’s
non-vanishing theorem, the base-point-free theorem and the rationality theorem. These lead to a proof of the log version of the cone theorem and the contraction theorem.

The book is based on a course that the author taught at Harvard University. As a result, the amount of material had to be restricted, and not every aspect of the theory is covered; for example, birational maps between minimal models and flops are only mentioned in an exercise and a footnote. Concentrating on rational curves was a good idea, since their theory is not treated to the same extent in other introductory books on higher-dimensional algebraic geometry.

The text is well-written and user-friendly, and contains lots of examples; it is a further good feature that there are exercises at the end of each chapter. The prerequisite knowledge is at the level of R. Hartshorne’s *Algebraic geometry*. The book provides a good introduction to higher-dimensional algebraic geometry for graduate students and other interested mathematicians.

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GABOR MEGYESI

INTRODUCTION TO THE MORI PROGRAM
(Universitext)

By Kenji Matsuki: 478 pp., £52.50 (US$69.95), ISBN 0-387-98465-8
(Springer, New York, 2002).

Mori theory has been one of the most active areas of algebraic geometry in the past twenty years. It led to a breakthrough in the study of three- and higher-dimensional algebraic varieties where little was previously known, and it also provided a new interpretation of classical results about algebraic curves and surfaces.

The first page of the introduction sets out the four main themes of the book:

(1) find a good representative in a given birational equivalence class (for example, a minimal model or a Mori fibre space);
(2) study the properties of the good representatives;
(3) study the relations between the good representatives in the same birational equivalence class;
(4) construct the moduli space of algebraic varieties with certain fixed discrete invariants.

Chapter 1, almost long enough to be a book in itself, describes the birational geometry of algebraic surfaces in terms of the Mori program, and provides the motivation for the later chapters. It starts with a classical result, Castelnuovo’s contractibility criterion, and then it introduces concepts of Mori theory and gives two-dimensional versions of many of the important theorems. After achieving the Enriques classification of surfaces, the structure of birational maps between Mori fibre spaces (ruled surfaces and the projective plane) is explained in terms of the Sarkisov program.

Chapter 2 is a brief account of log surfaces. According to Iitaka’s philosophy, an open variety $U$ can be studied by taking a completion $X$ such that $X \setminus U = D$ is
a divisor, and for every theorem about $X$ and its canonical divisor $K_X$ there should be a corresponding theorem about the pair $(X, D)$ and its log canonical divisor $K_X + D$. Log varieties also provide a stepping-stone toward the study of varieties in the next higher dimension.

Chapter 3 outlines the key points of the Mori program in dimensions three and higher. To find a good representative of the birational equivalence class of the variety $X$ satisfying certain conditions on singularities, one needs to consider the canonical divisor $K_X$. If $K_X$ is nef, then $X$ is good; it is a minimal model. Otherwise, there exists a morphism $\phi : X \to Y$ which contracts some curves $C$ with $K_X \cdot C < 0$. There are three possibilities. If $\phi$ contracts a divisor, then continue the process with $Y$. If $\phi$ contracts a subvariety of codimension 2, then one needs a birational map called a flip, which gives a variety $X^+$ with a morphism $\phi^+ : X^+ \to Y$, which is ‘better’ than $X$. Flips are discussed in Chapter 9; this is the step that is still only conjectural in dimensions greater than three. The third possibility is that $\dim Y < \dim X$; in this case $X$ is a Mori fibre space, and the process terminates.

In one and two dimensions it is possible to work with just smooth curves and surfaces. In higher dimensions, however, singularities cannot be avoided. Chapter 4 introduces the types of singularities – terminal, canonical, various flavours of log terminal, and log canonical – which occur in the Mori program. It also includes an explicit classification of canonical and log terminal surface singularities.

The next four chapters are devoted to the four types of fundamental theorems used in the Mori program: vanishing theorems, base point freeness theorems, the cone theorem and the contraction theorem.

Chapter 10 describes Mori’s original proof of the cone theorem by the ‘bend and break’ method, leading to the corollary on the uniruledness of the exceptional locus of a contraction.

Chapter 11 deals with Mori program in the logarithmic category. Here, many things are still conjectural, even in dimension three.

The next two chapters study the birational relations between the good representatives in a birational equivalence class. It is a theorem in dimension three, and a conjecture in higher dimensions, that a birational map between minimal models can be factorised into a sequence of special birational maps called flops, which, like flips, only change the variety in codimension 2. The Sarkisov program provides a way of decomposing a birational map between Mori fibre spaces into special kinds of maps called links, at least in dimension three.

The book finishes with a chapter on toric varieties. In this context everything is much easier; for example, the existence of flips can be proved in any dimension, and the proofs involve calculations with cones and lattices.

This book grew out of author’s personal notes. One of its greatest strengths is that it is probably the easiest-to-read book on the subject. It is written in a clear, comprehensible style, and the author’s love of the subject always shines through. It is mostly self-contained, although the introduction gives a list of the longer or more technical results that are not proved, with references to the original research papers.

The book is aimed at graduate students; the prerequisite is a knowledge of R. Hartshorne’s Algebraic geometry or equivalent, but anyone wanting to learn about the Mori program will find it useful.

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Gabor Megyesi
In the year 2000 there were 167 papers published with the word ‘moduli’ in the title. Twenty years earlier, there were 39. Clearly, even allowing for inflation, this is a concept that has gained in importance in recent years, and the book under review offers an insight into why this should be so. The word was coined initially by Riemann in 1857, in discussing the deformation parameters for a Riemann surface expressed as a branched covering of the projective line. The final chapter of the book, on conformal field theory, is in some ways a long distance from those beginnings, concerned as it is with one of the most important current interfaces between pure mathematics and physics, and yet it is essentially about the moduli of stable curves – not far removed from Riemann’s original problem. In between, the authors cover Kodaira–Spencer theory (the extension of Riemann’s concept to higher-dimensional complex manifolds), the Torelli problem (determining when the periods of differentials on a complex manifold determine its modulus) and the monodromy of degenerating algebraic varieties.

A plane curve determined by a single polynomial clearly has parameters – the coefficients of the polynomial – but considered as a Riemann surface in the abstract, it is more difficult to determine what the degrees of freedom are for varying the complex structure of that space. This is what moduli theory initially described, and it was Kodaira and Spencer in the 1950s who used the analytical tools of partial differential equations to establish (under suitable circumstances) the existence of a complex manifold that parametrizes ‘nearby’ complex structures in any dimension. The definitive version of this, where the moduli space is allowed to be singular, is called the Kuranishi space.

The first chapter of the book is devoted to this area. It follows the Kodaira style, on which the authors were presumably brought up, with concrete cocycles and plenty of examples. Lack of space dictates that proofs are only sketched, but this material is quite accessible in a number of other sources, notably Kodaira’s own book [2]. One addition to this standard material is the deformation theory of principal bundles, which returns briefly in the final chapter.

The second chapter is concerned with complex tori, Jacobians and the Torelli problem for curves, and then moves into a higher gear for the treatment, given in Chapter 3, of Griffith’s work in the 1960s on period mapping domains [1]. Here, as in most places, the authors again ‘refer to the literature for the proof of many theorems’. For the basic Hodge theory, one is referred to [3]. The authors go quite a long way beyond this, though, and discuss mixed Hodge structures, the monodromy theorem and the more modern differentiation between three types of Torelli theorem – ‘infinitesimal’, ‘global’ and ‘weak global’. This is good material for a mathematician studying degeneration of algebraic varieties, which nowadays could be anyone from a number theorist to a string theorist.

The final chapter describes conformal field theory in the style of the many papers by the second author and his collaborators on the subject. It is a clean treatment,
translating certain physical assumptions into pure mathematics at the outset, and then going on to discuss the projectively flat connection and the Verlinde formula. There is, however, another side to this story, which – although fleetingly mentioned – is essentially omitted here but is equally a central part of moduli theory. This is the question of the interpretation of conformal blocks in conformal field theory as the space of holomorphic sections of a line bundle on the moduli space of stable bundles. In fact stability of bundles, or Mumford stability and GIT quotients in general, is hardly mentioned. The student who enters the subject through this book is going to get a one-sided view of moduli theory.

But there is a saving grace, for in its original Japanese version, the contents of this book were called ‘Moduli theory 3’. It appears that Parts 1 and 2, by Shigeru Mukai, are being translated into English, and will appear with Cambridge University Press rather than the AMS. A fuller picture will then be available to the reader. And which reader? Well, the cover suggests ‘graduate and upper-level undergraduate students’. The latter is a little optimistic, but certainly this is a place to dig into and get facts and examples about a number of important areas in algebraic geometry. The pace may be rather hurried and the proofs are elsewhere, but for those who need, for example, an essential introduction to the monodromy structure of degenerating varieties, it is a good starting point. Each chapter finishes with a summary of what has been covered, and offers the reader a quick pointer to where to acquire the essential tools of the trade. This physically compact book thus provides a good pocket guide to a subject of increasing importance.

References


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NIGEL HITCHIN

This is an updated and expanded version of the author’s book Graphs, groups and surfaces, which was published as volume 8 in this series in 1973 in the wake of the renewed interest in topological graph theory generated by the proof of the long-standing Heawood map-colouring conjecture. This is a natural extension of the four-colour conjecture, asking how many colours are required for a map on an orientable surface of genus $g > 0$, but it is more conveniently studied in its
dual form, asking for the least genus of a surface in which a complete graph on \( n \) vertices can be imbedded without crossings. Groups enter the story here through the dual notions of current graphs and voltage graphs, which allow complicated graph-imbeddings to be constructed as coverings of simpler ones. In addition to solving the Heawood conjecture, these techniques allow the construction of surface-imbeddings of arbitrary graphs, including Cayley graphs of groups, thus leading to the search for the minimum genus of a graph or a group.

This was the main theme of the first edition of White’s book, which contained chapters on introductory graph theory, automorphism groups of graphs, Cayley graphs, surface topology, graph imbeddings, the genus of a group, map-colouring problems, and quotients (of graphs, manifolds and groups). In the second edition (1984), new chapters were added on voltage graphs, nonorientable graph imbeddings, block designs, hypergraph imbeddings, map automorphism groups and change ringing. Most of these are natural developments of the earlier material, for instance extending the range of surfaces or of combinatorial objects to be imbedded, but the last topic may seem out of place here. However, White demonstrates convincingly how campanology can be studied through Hamiltonian cycles in Cayley graphs for symmetric groups, and how seventeenth-century bell-ringers such as Stedman implicitly understood such concepts as cosets and stabilisers in permutation groups; the connection with surfaces is less obvious, but the author (who must be one of the few citizens of Kalamazoo to have had his compositions rung in Oxford’s Carfax Tower) shows how surface imbeddings and coverings can greatly simplify the task of composing.

In addition to further updates of the previous material, this new edition contains extra chapters covering numerous interesting recent developments on surface imbeddings of finite fields and finite geometries, on enumeration of graph imbeddings, and on random topological graph theory. Many sections from the previous edition have been updated, though not always consistently: for instance, the section on symmetrical (or regular) imbeddings of complete graphs is unchanged, ignoring the fact that these maps were classified soon after the second edition appeared [5].

The style throughout is very clear, and the material should be accessible to any postgraduate student with a reasonable background in basic graph theory, group theory and topology. There are plenty of well-chosen illustrative examples, and each chapter ends with a selection of problems, some fairly easy, some (starred) quite challenging, and others (double-starred) open research problems. A bibliography of about 400 items provides a comprehensive guide to the literature, and will be of great use to those working in this field.

White covers his chosen ground very thoroughly, and my only real criticism is that he has little to say about alternative approaches to this subject. Examples include the study of regular maps by Coxeter and Moser [3, Chapter 8], the general theory of hypermaps (imbeddings of hypergraphs), developed by Cori and Machi [1, 2] in the orientable case and by Izquierdo and Singerman [4] for more general surfaces, and the exciting links with Riemann surfaces, algebraic number fields and Galois groups through Grothendieck’s theory of dessins d’enfants [6, 7, 8].

As a specific example, there is an excellent chapter on voltage graphs, which illustrates this technique by constructing certain families of triangular and quadrilateral maps on the torus. In fact, these – together with the duals of the triangular maps – account for all the regular (or symmetrical) maps on this surface, thus
solving a simple but important instance of one of the basic problems in topological graph theory. However, the book makes no mention of this, nor does it make any reference to the sections in [3] where this classification is obtained by other means.

Having said that, let me emphasise that this is a very well-written and readable book, which I recommend to anyone wanting to learn this particular approach to the subject.

References


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SPECTRAL METHODS OF AUTOMORPHIC FORMS (2nd edn)
(Graduate Studies in Mathematics 53)

By HENRYK IWANIEC: 220 pp., US$49.00, ISBN 0-8218-3160-7
(American Mathematical Society, Providence, RI, 2002.)

DOI: 10.1112/S002460930303252085

This book offers a swift introduction to the spectral theory of the Laplace operator on finite volume quotients of the complex upper half plane \( \mathbb{H} \). The central theme of the book is the spectral decomposition of the space \( L^2(\Gamma \backslash \mathbb{H}) \) with respect to the hyperbolic Laplace operator. Here, \( \Gamma \) is a discontinuous group of hyperbolic motions such that \( \Gamma \backslash \mathbb{H} \) has finite volume, also called the finite volume group or the Fuchsian group of the first kind. This spectral decomposition is central to the theory of automorphic forms on \( \text{GL}_2 \). The material and the exposition are well suited for use by undergraduate students. Researchers from other fields and graduate students who are entering the field will also benefit from this well-written book.

The author starts with a short review of harmonic analysis on the Euclidean plane \( \mathbb{R}^2 \) and on the torus \( \mathbb{R}^2 / \mathbb{Z}^2 \), culminating in the Poisson summation formula. This is followed by an introduction to the harmonic analysis of the hyperbolic plane \( \mathbb{H} \) and its group of motions \( G \equiv \text{PSL}_2(\mathbb{R}) \). Eigenfunctions of the Laplacian are discussed, as well as invariant integral operators and the Green function on \( \mathbb{H} \).
In Chapter 2, finite volume groups are introduced. Examples of these are finite index subgroups of $\text{PSL}_2(\mathbb{Z})$. As examples, explicit fundamental domains are given. This helps the reader to understand the geometry of the quotient $\Gamma \backslash \mathbb{H}$. The chapter closes with a discussion of Kloosterman sums, which show up among other instances in the Fourier coefficients of Eisenstein series, the latter giving the spectral projections of the Laplacian on $\Gamma \backslash \mathbb{H}$.

Automorphic forms are defined in Chapter 3 as $\Gamma$-invariant functions on $\mathbb{H}$ that are eigenfunctions of the Laplace operator. Their Fourier expansion is discussed in general terms, and all the important examples of Eisenstein series and cusp forms are given. The Fourier coefficients of Eisenstein series are then shown to be Dirichlet series whose coefficients are Kloosterman sums.

Cusp forms are defined as automorphic forms with vanishing zeroth Fourier coefficient at every cusp. In Chapter 4 it is shown that they span the space of functions with vanishing zeroth Fourier coefficient; that is, the Laplacian has pure point spectrum, consisting solely of eigenvalues, on that space.

In order to complete the spectral decomposition, it must be shown that the orthocomplement of the cusp forms is spanned by Eisenstein series. Since Eisenstein series are not themselves members of the $L^2$-space, this assertion has to be interpreted as follows: firstly, the poles of the Eisenstein series form eigenvectors of the Laplacian that are orthogonal to cusp forms. Secondly, the Eisenstein series evaluated on the imaginary axis give spectral projectors to the continuous spectrum of the Laplacian similar to the power functions $x \mapsto e^{ixy}$, giving spectral projectors of the Laplacian on $L^2(\mathbb{R})$ without being members of the space. These three components – the space of cusp forms, the poles of Eisenstein series, and the continuous spectrum with Eisenstein projectors – are finally shown to constitute the whole space $L^2(\Gamma \backslash \mathbb{H})$. The main problem to be solved is to obtain the analytic continuation of the Eisenstein series, which are defined a priori only for $s \in \mathbb{C}$ with real part large enough. In Chapter 5, we find some preparatory computations involving the Green’s function, and Chapter 6 is devoted to the analytic continuation of the Eisenstein series. Several methods are given in the literature to achieve this continuation; the author follows one of Selberg’s methods, which uses Fredholm theory of integral equations. The spectral decomposition is then finalised in Chapter 7.

The most interesting part of the decomposition is the space of cusp forms. Chapters 8–11 of the book are devoted to further spectral techniques that serve the investigation of this space and its connection to the geometry of the quotient $\Gamma \backslash \mathbb{H}$. Of particular importance is the trace formula, which comes about by computation of the trace of an invariant integral operator on the space of cusp forms in geometric terms. In the case when $\Gamma \backslash \mathbb{H}$ is compact, this is just the principle that the trace of an integral operator on a compact manifold equals the integral over the diagonal of its kernel. If $\Gamma \backslash \mathbb{H}$ is not compact, one has to truncate the kernel at the cusps to obtain a similar – though more complicated – formula. One finds various profound applications, such as an asymptotic expression for the number of closed geodesics in $\Gamma \backslash \mathbb{H}$, or an asymptotic formula on the number $N_\Gamma(T)$ of eigenvalues below a bound $T > 0$ plus a certain contribution $M_\Gamma(T)$ from the continuous spectrum. It is an unsolved problem as to which of these two summands dominates, and the answer seems to depend heavily on the group $\Gamma$.

But not only large eigenvalues mystify researchers – small ones do so as well. For a special class of groups, the congruence groups, Selberg conjectured in 1965 that the smallest non-vanishing eigenvalue $\lambda_1$ should have $\lambda_1 \geq \frac{1}{4}$. There has been
continuous progress in stating $\lambda_1 \geq c$ for improving constants $c$, but the conjecture is still open.

The short Chapter 12 gives the hyperbolic lattice point problem, which asks for the number of points of a given $\Gamma$-orbit in $\mathbb{H}$ that lie in a ball of radius $R$, and queries how that number grows as $R$ tends to infinity. The answer is given by trace formula methods, and involves the eigenvalues of the Laplacian. The book closes in Chapter 13 with new pointwise bounds on cusp forms. These bound the growth of the $j$th cusp form as function of the eigenvalue. There are two appendices, on classical results of analysis, and on special functions.

This comprehensive book on the spectral theory of $GL(2)$-automorphic forms is surprisingly short, with only 220 pages. I can highly recommend it to students and researchers who are interested in number theory.

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ANTON DEITMAR

THE SYMMETRY PERSPECTIVE: FROM EQUILIBRIUM TO CHAOS IN PHASE SPACE AND PHYSICAL SPACE
(Progress in Mathematics 200)

By Martin Golubitsky and Ian Stewart: 325 pp., CHF113.00
(75.00 euro), ISBN 3-7643-6609-5 (Birkhäuser, Basel, 2002).

DOI: 10.1112/S0024609303262081

The authors of this book take the view that symmetries of dynamical systems used to model applications should be identified and built into the model a priori. They illustrate with examples how this allows one to understand, model and predict many qualitative phenomena in biological and other application areas. By identifying generic behaviour and generic bifurcations of dynamical systems with a particular symmetry group, they find that a lot of the observed behaviour in such systems can be seen as being symmetry-dependent rather than model-dependent. The subtitle 'From equilibrium to chaos in phase space and physical space' emphasises that they go beyond their previous work [1] in both the dynamical phenomena and the applications that they deal with. The book was awarded the Ferran Sunyer i Balaguer prize for 2001.

In the preface, the authors state that they are not aiming for a formal text book; in consequence, they prove only a selection of the results, referring to the literature for details of many others. Rather, it is a collection of case studies that serve to illustrate, and indeed that have motivated, recent developments of the theory of dynamical systems with symmetry. Non-experts will find that some sections of the book are self-contained, but others are not, and will require reference to other sources (in particular, the previous work [1] of the authors) for a full understanding.

The topics covered include steady and Hopf bifurcation with symmetry and applications to speciation, animal gait patterns, coupled cell systems, spiral wave meandering and molecular vibrations, among others. The main mathematical tools used are group representation theory for Lie groups, singularity theory, ergodic theory and dynamical systems constructions adapted to a symmetric context.
In particular, extensions of results from [I] to non-compact groups are discussed with application to primary visual cortex patterns, bifurcation from group orbits and hidden symmetry problems. Other material that goes beyond applications of the material in [I] includes the chapters on heteroclinic cycles (Chapter 8) and symmetric chaos (Chapter 9).

The book represents what I believe will be a very useful resource in this area, summarizing much of the work of the authors over the last decade. It is unashamedly a personal perspective, and there is evidence at some points of concatenation of papers. Introductory material is found at several stages of the book, although this has the advantage of making several of the chapters more or less self-contained. I believe that the authors have done a considerable service to the research community in collecting these topics into one book.

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