1. Introduction

In the early 1930s, Wiener proved that if \( f(x) \) is a strictly positive periodic function whose Fourier series is absolutely convergent, then the Fourier series of \( g(x) = 1/f(x) \) is also absolutely convergent [8, pp. 10–14]. This phenomenon can be easily understood nowadays using Banach algebra techniques (see, for example, [4, pp. 202–203]). In fact, these techniques allow us to study the absolute convergence of \( g(x) = F(f(x)) \), where \( F \) is holomorphic in an open subset of \( \mathbb{C} \) that contains the range of \( f(x) \) (for \( x \in \mathbb{R} \)). In this context, Wiener’s original problem corresponds to the choice \( F(z) = 1/z \).

In this work we want to analyse the constraints on the simultaneous rate of vanishing of the Fourier coefficients \( f(n) \) and \( g(n) \) as \( n \to \infty \). We shall focus on \( g = 1/f \), but we shall also study the general case \( g = F(f) \). In either case, there are obviously no constraints when \( f \) is a constant function.

Although this problem does not seem to be directly related to uncertainty inequalities for the Fourier Transform, we observe that there are some analogies, both in the nature of the results and in the proof techniques. The general fact with which we are dealing is that \( f(n) \) and \( g(n) \) cannot vanish too quickly at the same time as \( n \to \infty \), unless \( f(x) \) is constant. The general fact that underlies uncertainty inequalities is that a non-periodic function \( \phi(x) \) and its Fourier Transform \( \phi(u) \) cannot vanish too quickly at the same time as \( x \to \infty \) and \( u \to \infty \), unless \( \phi(x) \) is zero (almost everywhere). For a simple introduction to some aspects of uncertainty inequalities, see [5]; for a thorough and recent introduction to this vast subject, see [3].

2. Results obtained by complex variable techniques

Let us start with the study of a special case. If \( f(x) \) is a strictly positive, non-constant, trigonometric polynomial, then \( g(x) = 1/f(x) \) cannot be a trigonometric polynomial and has infinitely many Fourier coefficients different from zero. These Fourier coefficients \( g(n) \) vanish ‘rapidly’ as \( n \to \infty \), because \( g(x) \) is a smooth function. On the other hand, they cannot decay arbitrarily rapidly, because of the following two theorems.

**Theorem 1.** Let \( f(x) \) be a strictly positive, non-constant, trigonometric polynomial. There then exists a \( \delta > 0 \), which depends only on \( f \), such that the Fourier coefficients of \( g(x) = 1/f(x) \) satisfy the estimate \( g(n) = O(e^{-\delta n}) \) as \( n \to \infty \).
**Proof.** It is sufficient to establish the result in the even, $2\pi$-periodic case, that is, when $f(x) = \sum_{k=-N}^{N} \hat{f}(k)e^{ikx}$ with $\hat{f}(k) = \hat{f}(-k)$. We have

$$f(x) = \sum_{k=-N}^{+N} \hat{f}(k) z^k = z^{-N} p(z),$$

where $z = e^{ix}$ belongs to the unit circle $T$ in the complex plane, and $p(z) = \sum_{k=-N}^{2N} \hat{f}(k-N) z^k$ is a symmetric complex polynomial of degree $2N$. We have $\hat{g}(n) = \hat{g}(-n)$ and, for every fixed $n = 0, 1, 2, \ldots,$

$$\hat{g}(n) = \frac{1}{2\pi i} \int_{T} \frac{1}{f(x)} e^{inx} dx = \frac{1}{2\pi i} \int_{T} \frac{z^N}{p(z)} z^n dz = \frac{1}{2\pi i} \int_{T} \frac{z^{N+n-1}}{p(z)} dz.$$

These integrals can be easily evaluated by the residue theorem. The rational function that we are integrating has a finite number of poles, that correspond to the zeros $z_1, z_2, \ldots, z_M$ of $p(z)$ which are inside the contour $T$. We are not concerned with the zeros of $p(z)$ outside $T$ (since $p$ is symmetric, there are another $M$ of them), and we observe that it is impossible for $p(z)$ to have zeros exactly on $T$, because that would violate our original assumption that $f(x) > 0$. When all these zeros are simple, we have $M = N$ and, for any $n = 0, 1, \ldots$,

$$\hat{g}(n) = c_1 z_1^n + c_2 z_2^n + \ldots + c_M z_M^n,$$

where $|z_k| < 1$ for $k = 1, \ldots, M$, and $c_1, \ldots, c_M$ are suitable constants. The absolute value of this sum is dominated by a constant times $e^{-\delta n}$ for some $\delta > 0$. When $p(z)$ has multiple zeros, we obtain a similar expression with $M < N$ terms, and we see that $|\hat{g}(n)|$ is dominated again by a constant times $e^{-\delta n}$.

**Theorem 2.** If $f(x)$ is a strictly positive trigonometric polynomial, and the Fourier coefficients of $g(x) = 1/f(x)$ decay super-exponentially as $n \to \infty$, that is, $\hat{g}(n) = o(e^{-\delta n})$ for any $\delta > 0$, then $f$ must be a constant function.

**Proof.** Let us assume that $f(x)$ is not constant. The Fourier coefficients $\hat{g}(n)$ are not zero for infinitely many values of $n$ and, as we have previously seen, they are given by a finite sum, with each term of this sum decreasing exponentially as $n \to \infty$. By our assumption, we should also have $\lim_{n \to \infty} \hat{g}(n) e^{\delta n} = 0$, for any given $\delta > 0$. This is a contradiction, and $f$ must be constant.

We now want to deal with general pairs of periodic functions $f$ and $g$ whose Fourier coefficients satisfy

$$\hat{f}(n) = O(e^{-\gamma|n|^p}) \quad \text{and} \quad \hat{g}(n) = O(e^{-\delta|n|^q}).$$

Assuming that $f$ is positive and $g = 1/f$ (or, more generally, assuming that $g = F(f)$ with $F$ analytic on the range of $f$), we want to study what quadruples $\gamma > 0$, $p > 0$, $\delta > 0$, $q > 0$ can be chosen in (1) without forcing $f$ and $g$ to be constants. In some sense, the case of a trigonometric polynomial $f$ is a limiting case of (1) as $p \to + \infty$. 

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First, we observe that if \( p \geq 1 \) in (1), then \( f \) is analytic (see [4, pp. 26–27] and also [6, Chapter 4.1]). Analyticity means that \( f(z) \) admits a local Taylor expansion in a neighbourhood of every \( x \in \mathbb{R} \). It also means that \( f(z) \) can be seen as the restriction to \( T \equiv \{ z \in \mathbb{C} : z = e^{ix}, x \in \mathbb{R} \} \) of

\[
h(z) = \sum_{n=-\infty}^{+\infty} \hat{f}(n)z^n,
\]

a function which is holomorphic in an annulus that strictly contains \( T \). It can also be seen as the restriction to \( \mathbb{R} \) of

\[
f(z) = h(e^{iz}) = \sum_{n=-\infty}^{+\infty} \hat{f}(n)e^{inz},
\]

a function which is holomorphic in a horizontal strip. When \( p = 1 \), this strip is exactly \( \Omega_\gamma \equiv \{ z = x + iy \in \mathbb{C} : |y| < \gamma \} \), where \( \gamma \) is the constant appearing in the first estimate of (1). If \( q = 1 \), then the same happens for \( g(x) \), which can be analytically continued to a holomorphic function in the horizontal strip \( \Omega_\delta \equiv \{ z = x + iy \in \mathbb{C} : |y| < \delta \} \). Actually, the converse is also true, in the following sense: if \( f(z) \) and \( g(z) \) are holomorphic in \( \Omega_\gamma \) and \( \Omega_\delta \), and if their restriction to the real axis is \( 2\pi \)-periodic, then (1) holds with \( p = q = 1 \) and with the same constants \( \gamma > 0 \) and \( \delta > 0 \) that define the widths of \( \Omega_\gamma \) and \( \Omega_\delta \).

If \( p > 1 \) and \( q > 1 \) in (1), then \( f(x) \) is the restriction to the real axis of the entire function \( f(z) = \sum_{n=-\infty}^{+\infty} \hat{f}(n)e^{inz} \), while \( g(x) \) is the restriction to the real axis of the entire function \( g(z) = \sum_{n=-\infty}^{+\infty} \hat{g}(n)e^{inz} \). These facts lead to the following result.

**Theorem 3.** Let \( g(x) = F(f(x)) \), where \( f \) and \( g \) are periodic functions whose Fourier coefficients satisfy \( \hat{f}(n) = O(e^{-\gamma|n|^p}) \), with \( \gamma > 0 \) and \( p > 1 \), and \( \hat{g}(n) = O(e^{-\delta|n|^q}) \), with \( \delta > 0 \) and \( q > 1 \). Let \( F(z) \) be holomorphic in an open subset \( \Omega \) of \( \mathbb{C} \) that contains the range of \( f(x) \) (for \( x \) real). Then \( f(x) \) must be constant if one of the two following extra conditions are satisfied:

(a) \( F(z) \) has at least two singularities \( \zeta_1 \) and \( \zeta_2 \) (polar or essential) in \( \mathbb{C} \);

(b) \( F(z) \) has one singularity at \( \zeta \in \mathbb{C} \), and \( \zeta \) belongs to the complex range of the entire function \( f(z) = \sum_{n=-\infty}^{+\infty} \hat{f}(n)e^{inz} \).

**Proof.** Let us assume that \( f(x) \) is not a constant function. The entire function \( g(z) = \sum_{n=-\infty}^{+\infty} \hat{g}(n)e^{inz} \) is also given by \( g(z) = F(f(z)) \), and has no singularities for \( z \) (finite) in \( \mathbb{C} \). It has one singularity (polar \( g(z) \) is a polynomial, essential otherwise) at \( z = \infty \). The entire function \( f(z) \), by Picard’s theorem, attains each complex value with one possible exception. If we assume condition (a), then \( F(f(z)) \) has a singularity either when \( f(z) = \zeta_1 \) or when \( f(z) = \zeta_2 \), and we obtain a contradiction. If we assume condition (b), then \( f(z) = \zeta \) for at least one (finite) value of \( z \), and again we obtain a contradiction.

We observe that the case \( F(z) = 1/z \) is only partially covered by the previous theorem. In fact, \( \zeta = 0 \) might not be in the complex range of \( f(z) \). On the other hand, we have the following.

**Theorem 4.** Let \( f(x) \) be a non-constant, strictly positive (for \( x \in \mathbb{R} \)) periodic function whose Fourier coefficients satisfy \( \hat{f}(n) = O(e^{-\gamma|n|^p}) \), with \( \gamma > 0 \) and \( p > 1 \). Then its analytic continuation \( f(z) \) is an entire function with at least one zero in \( \mathbb{C} \).


Proof. It is well known (because of Hadamard’s factorization theorem) that all the entire functions \( f(z) \) that do not attain the value \( \zeta = 0 \) can be written as \( f(z) = e^{\alpha(z)} \), where \( \alpha(z) \) is any entire function. Since the restriction of our \( f(z) \) to the real axis is \( 2\pi \)-periodic and positive, we can assume that \( \alpha(z) = \sum_{n=-\infty}^{+\infty} \hat{\alpha}(n) e^{i\alpha n z} \), where \( \alpha(z) \) is entire and sends the real axis \( \mathbb{R} \) into itself. Since \( \alpha(z) \) is not a polynomial, \( M_\alpha(r) = \max_{|z|=r} |\alpha(z)| \) grows faster than any power of \( r \) when \( r \to \infty \). Cauchy’s inequality (see [1, pp. 2–3]) implies that the growth of \( \max_{|z|=r} \Re(\alpha(z)) \) is also faster than the growth of any power of \( r \), and this in turn implies that the growth of \( M_\alpha(r) = \max_{|z|=r} |f(z)| \) is faster than the exponential of any power of \( r \). In other words, if we define the order \( \rho \) of \( f(z) \) in the usual way,

\[
\rho = \limsup_{r \to +\infty} \frac{\log \log M_\alpha(r)}{\log r},
\]

then our \( f \) must be an entire function of infinite order. There is a well-known connection between the order of an entire function and its Taylor coefficients (see [1, p. 9]). If \( f(z) = \sum_{k=0}^{\infty} a_k z^k \), then we have

\[
\rho = \lim sup_{k \to +\infty} \frac{k \log k}{-\log |a_k|}.
\]

In our case, we can express the \( a_k \) through the Fourier coefficients of \( f \). In fact,

\[
a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!} \left( \frac{d^k}{dz^k} \sum_{n=-\infty}^{+\infty} \hat{f}(n) e^{i\alpha n z} \right)_{z=0} = \frac{1}{k!} \sum_{n=0}^{+\infty} (in)^k \hat{f}(n),
\]

and, by our original assumption on the rate of decay of \( \hat{f}(n) \), we obtain that

\[
|a_k| \leq \frac{c_1}{k!} \sum_{n=0}^{+\infty} n^k e^{-\gamma n} \leq \frac{c_2}{k!} \int_0^{+\infty} x^k e^{-\gamma x} dx = \frac{c_3}{k!} \frac{\Gamma((k+1)/\gamma)}{\gamma^{(k+1)/\gamma}}.
\]

Stirling’s formula for the Gamma function tells us that when \( t \) is large and positive, \( \Gamma(t+1) = (t/e)^t \sqrt{2\pi t} (1 + o(1)) \). Using this formula, together with the expression (4) for the order \( \rho \) of \( f \), we finally obtain \( \rho \leq p/(p-1) \). This is a contradiction, because \( f(z) \) should have infinite order, and so we have shown that \( f(z) \) actually has at least one complex zero.

A straightforward consequence of the above results is the following.

**Theorem 5.** Let \( f(x) \) be a non-constant, strictly positive (for \( x \) real) periodic function whose Fourier coefficients satisfy \( \hat{f}(n) = O(e^{-\gamma n}) \) with \( \gamma(n) \to +\infty \) as \( n \to \infty \). Let \( \hat{g}(n) \) be the Fourier coefficients of \( g(x) = 1/f(x) \). One of the following two cases must occur:

(a) \( \hat{g}(n) = O(e^{-\delta n}) \), where the constant \( \delta \) is related to the zeros \( \{z_k\} \) of \( f(z) \) in the complex plane by \( \delta = \inf \{ |\Im(z_k)| \} \);

(b) \( \hat{g}(n) = O(e^{-\delta(n)}), \) where the function \( \delta(n) \) tends to \( +\infty \) as \( n \to \infty \), but slower than any power of \( n \), that is, we have \( \delta(n) = o(n^\varepsilon) \) for every given \( \varepsilon > 0 \).

**Proof.** The function \( f(x) \), because of the super-exponential decay of its Fourier coefficients, can be analytically continued to an entire function \( f(z) \) in the complex plane. If \( f(z) \) has no zeros, then also \( g(z) = 1/f(z) \) is entire, and the coefficients \( \hat{g}(n) \)
decay super-exponentially as \( n \to \infty \). On the other hand, for any given \( \varepsilon > 0 \), the sequence \( \delta(n) \) of part (b) of the statement must satisfy \( \delta(n) = o(n^r) \), otherwise Theorem 4 would lead to a contradiction.

If \( f(z) \) has zeros in \( \mathbb{C} \), then \( g(z) = 1/f(z) \) is not entire, but it is holomorphic in an open set of \( \mathbb{C} \) that contains the horizontal strip \( \Omega_\delta \equiv \{ z = x + iy \in \mathbb{C} : |y| < \delta \} \), where \( \delta \) is the infimum of the distances of the zeros \( z_k \) of \( f \) from the real axis. This in turn implies that \( \hat{g}(n) = O(e^{-\delta n}) \).

We observe at this point that, in contrast with these results, a strictly exponential decay of the Fourier coefficients \( \hat{f}(n) \), that is, \( p \to 1 \) in (1), does not put any constraints on the rate of decay of the Fourier coefficients of \( g(x) = 1/f(x) \). In fact, one can construct non-constant solutions of (1) with \( p \to 1 \) and arbitrary constants \( \gamma > 0 \), \( q > 1 \), \( \delta > 0 \), just by choosing \( g(x) \) equal to some suitable positive trigonometric polynomial, whose zeros in the complex plane satisfy \( \delta \leq \min_k |\text{Im}(z_k)| \).

We shall see in the next section that in the case of sub-exponential decay of \( \hat{f}(n) \) and \( \hat{g}(n) \), even more can be said.

### 3. Results obtained by Gevrey class techniques

In general, even when \( p < 1 \), the condition (1) implies that \( f \) is smooth and belongs to some Gevrey class, that is, there are uniform estimates on the growth of its \( k \)th derivatives as \( k \to \infty \). A very precise version of these estimates is given by the following.

**Theorem 6.** If for \( n \neq 0 \) the Fourier coefficients of \( f(x) \) satisfy \( |\hat{f}(n)| \leq c_4 |n|^\beta e^{-\gamma |n|^p} \), with \( \beta \in \mathbb{R} \), \( \gamma > 0 \) and \( p > 0 \), then the \( k \)th derivatives of \( f \) satisfy, for \( k = 1, 2, \ldots \),

\[
|f^{(k)}(x)| \leq c_2 \cdot b^k \cdot \left( \frac{k}{\varepsilon p} \right)^{k/p} ,
\]

where \( b = \beta/p \) when \( p > 2 \), and \( b = \beta/p + 1/p - 1/2 \) when \( p \leq 2 \). The constants \( c_1 \) and \( c_2 \) depend only on \( f \).

**Proof.** See [7].

The converse of the previous theorem holds for \( p \geq 2 \), and it ‘almost’ holds for \( 0 < p < 2 \). In fact, we have the following.

**Theorem 7.** If the \( k \)th derivatives of \( f \) satisfy, for \( k = 1, 2, \ldots \),

\[
|f^{(k)}(x)| \leq c_2 \cdot b^k \cdot \left( \frac{k}{r} \right)^{k/p} ,
\]

with \( b \in \mathbb{R} \), \( r > 0 \) and \( p > 0 \), then for \( n \neq 0 \) the Fourier coefficients of \( f(x) \) satisfy

\[
|\hat{f}(n)| \leq c_3 |n|^\beta e^{-\gamma |n|^p/r} .
\]

The constants \( c_2 \) and \( c_3 \) depend only on \( f \).

**Proof.** See [7].
The most significant parameter in Theorems 6 and 7 is \(a = 1/p\). In fact, one often encounters the following.

**Definition.** The Gevrey class \(G_a\) is the set of all smooth functions \(f(x)\) such that, when \(x\) belongs to a compact subset \(A\) of \(\mathbb{R}\) and \(k = 1, 2, \ldots\),

\[
|f^{(k)}(x)| \leq c (ck^a)^k,
\]

with a constant \(c = c(f, A)\) that does not depend on \(k\).

The previous two theorems show, among other things, that the Fourier coefficients of a periodic function \(f(x)\) satisfy \(f(n) = O(e^{-\gamma|n|^a})\) if and only if \(f\) belongs to \(G_a\) with \(a = 1/p\). The next theorem gives us a useful property of Gevrey classes \(G_a\) with \(a > 1\).

**Theorem 8.** Let \(F(z)\) be holomorphic in an open subset \(\Omega\) of \(\mathbb{C}\). Let \(f(x)\) be a function in the Gevrey class \(G_a\) with \(a > 1\). If the range of \(f(x)\) (for \(x\) real) is contained in \(\Omega\), then \(g(x) = F(f(x))\) also belongs to \(G_a\).

**Proof.** We must show that for every compact set \(A \subset \mathbb{R}\), there is a constant \(c = c(g, A)\) such that, for \(k = 1, 2, \ldots\),

\[
|g^{(k)}(x)| \leq cc^k k^a
\]

when \(x \in A\).

Without loss of generality, we can assume that \(x = 0\). By the Taylor expansion,

\[
f(x) = \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(0) x^n + R_k(x) = f_k(x) + R_k(x).
\]

The first \(k\) derivatives of the polynomial \(f_k\) coincide at 0 with the first \(k\) derivatives of the function \(f\), so we have \(g^{(k)}(0) = D^k F(f_k)(0)\). Since \(F(f_k(z))\) is holomorphic in \(\Omega\), we can apply the Cauchy formula and obtain

\[
g^{(k)}(0) = \frac{k!}{2\pi i} \oint_{\partial \Omega} \frac{F(f_k(z))}{z^{k+1}} dz,
\]

where \(\omega_k\) is any sequence of circles centred at \(z = 0\) and all contained in \(\Omega\). We can choose \(\omega_k \equiv \{z:|z| = e/c k^{a-1}\}\), with \(e = c(f, A)\) the constant such that \(|f^{(k)}(x)| \leq c(ck^a)^k\) for \(x \in A\), and where \(e\) is so small that \(\omega_k \subset \Omega\). For a technical reason that will become immediately clear, we also assume that \(e < 1/e\). When \(z \in \omega_k\), we have

\[
|f_k(z)| \leq \sum_{n=0}^{k} \frac{e}{n!} c^n n^{a-1} \left(\frac{e}{ck^{a-1}}\right)^n \leq c_1 \sum_{n=0}^{k} n^{a-1} n^{\alpha n}\left(\frac{ck}{e}\right)^n,
\]

with \(c_1\) independent of \(k\). We have used Stirling’s formula

\[
n! = (n/e)^n \sqrt{2\pi n} (1 + o(1)).
\]

Therefore \(|f_k(z)| \leq c_1 \sum_{n=0}^{k} (n/k)^{a-1} n^{a-1} (n\epsilon)^n\) \(\leq c_2\), and since \(F(z)\) is holomorphic in \(\Omega\), it follows that \(|F(f_k(z))| \leq c_2\). We have

\[
|g^{(k)}(0)| \leq \frac{c_2 k!}{2\pi} \left(\frac{e}{ck^{a-1}}\right)^k \left(\frac{ek}{ck^{a-1}}\right)^{k-1} = c_3 \left(\frac{k}{e}\right)^k \sqrt{(2\pi k)(1 + o(1))} \left(\frac{ck^{a-1}}{e}\right)^k \leq c_4 k^n e^{ck^a},
\]

which is what we wanted to show, since \(c_4\) is independent of \(k\).
Corollary 1. Let \( f(x) \) be a periodic function with \( \hat{f}(n) = O(e^{-\gamma|n|^p}) \) for some \( \gamma > 0 \) and \( 0 < p < 1 \). Let \( F(z) \) be holomorphic in an open subset \( \Omega \) of \( \mathbb{C} \) that contains the range of \( f(x) \) (for \( x \) real). Then \( g(x) = F(f(x)) \) is a periodic function with \( \hat{g}(n) = O(e^{-\beta|n|^p}) \) for some \( \delta > 0 \).

In particular, if we consider the case \( F(z) = 1/z \), then we have proved not only that these sub-exponential rates of decay in (1) do not force \( f \) to be constant, but also the following.

Corollary 2. Let \( f(x) \) be a strictly positive periodic function with \( \hat{f}(n) = O(e^{-\gamma|n|^p}) \) for some \( \gamma > 0 \) and \( 0 < p < 1 \). Let the Fourier coefficients of \( g(x) = 1/f(x) \) satisfy \( \hat{g}(n) = O(e^{-\delta|n|^p}) \) for some \( \delta > 0 \) and \( 0 < q < 1 \). If \( q < p \), then the stronger estimate \( \hat{g}(n) = O(e^{-\gamma|n|^p}) \) holds for some \( \gamma' > 0 \). If \( q > p \), then the stronger estimate \( \hat{f}(n) = O(e^{-\gamma'|n|^p}) \) holds for some \( \gamma' > 0 \).

4. A mock Heisenberg inequality

Let \( \phi(x) \) be a function whose \( L^2 \)-norm on \( \mathbb{R} \), that is, \( \| \phi \|_2 = \left\| \int_{-\infty}^{\infty} |\phi(x)|^2 \, dx \right\|^{1/2} \), exists and is positive. Let \( \hat{\phi}(u) = \int_{-\infty}^{\infty} e^{-2\pi iux} \phi(x) \, dx \) be its Fourier Transform. Heisenberg’s inequality (see, for example, [2, pp. 116–121]) tells us that

\[
\frac{\| x\hat{\phi}(x) \|_2}{\| \hat{\phi}(x) \|_2} \leq \frac{1}{4\pi}.
\]

Analogous inequalities, involving \( L^2 \)-norms, hold in the case of our original problem. For example, if \( f(x) \) is 2\( \pi \)-periodic and square-summable together with its derivative \( f'(x) \), we then have

\[
\frac{1}{2\pi} \int_0^{2\pi} |f'(x)|^2 \, dx = \sum_{n=-\infty}^{+\infty} n^2 |\hat{f}(n)|^2 \\
\quad \geq \sum_{n=-\infty}^{+\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx - \left( \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx \right)^2.
\]

This is, in fact, just one version of Wirtinger’s inequality. Assuming that \( f \) is non-constant, we can rewrite it as

\[
\frac{\| \hat{f}(n) \|_2^2}{\| f(x) \|_2^2} \geq 1,
\]

and it is easy to check that the constant 1 on the right-hand side is attained if and only if \( f \) is a trigonometric polynomial of degree 1. When \( f \) is continuous, strictly positive and non-constant, we can write the same inequality for \( g(x) = 1/f(x) \), and thus obtain

\[
\left( \frac{\| \hat{f}(n) \|_2^2}{\| f(x) \|_2^2 - |f(0)|^2} \right) \left( \frac{\| \hat{g}(n) \|_2^2}{\| g(x) \|_2^2 - |g(0)|^2} \right) > 1.
\]

Notice that the terms \( |\hat{f}(0)|^2 \) and \( |\hat{g}(0)|^2 \) in the denominators of the left-hand side of (6) are both positive under our assumptions. The inequality must be strict, because if one of the two factors on the left-hand side is 1, then the other is not.

Another inequality of the same kind, which makes better use of the fact that \( f(x) g(x) = 1 \), is provided by the following.
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\textbf{Theorem 9.} Let \( f(x) \) be a strictly positive, continuous, \( 2\pi \)-periodic, non-constant function. If \( g(x) = 1/f(x) \), then we have

\[
\frac{\|nf(n)\|_{L^2} \|ng(n)\|_{L^2}}{(\log \|f(n)\|_{L^2} \|g(n)\|_{L^2})^2} \geq \frac{1}{\pi^2}.
\]

\textit{Proof.} We can assume that \( f'(x) \) and \( g'(x) \) are both in \( L^2(T) \), otherwise the numerator of the left-hand side of our inequality would be \( +\infty \), and there would be nothing to prove. We can write \( f(x) = e^{\alpha(x)} \) and \( g(x) = e^{-\alpha(x)} \), where \( \alpha(x) = \log f(x) \) is a periodic and real-valued function with Fourier series

\[
\alpha(x) = \sum_{n=-\infty}^{\infty} \hat{a}(n)e^{inx}.
\]

We have

\[
1 = \int_{-\pi}^{\pi} f(x)g(x) \frac{dx}{2\pi} \leq \|f\|_{L^2} \|g\|_{L^2} \leq \max f(x) \frac{1}{\min f(x)},
\]

therefore we obtain

\[
0 \leq \log(\|f\|_{L^2} \|g\|_{L^2}) \leq \log \max f - \log \min f = \alpha - \min \alpha.
\]

The oscillation \( \max \alpha - \min \alpha \) is dominated by one half of the total variation of \( \alpha(x) \); in other words, we have

\[
\max \alpha - \min \alpha \leq \frac{1}{2} \int_{-\pi}^{\pi} |\alpha'(x)| \, dx.
\]

Taking squares and applying the Schwarz inequality, we obtain

\[
(\log(\|f\|_{L^2} \|g\|_{L^2}))^2 \leq \frac{1}{4} \left( \int_{-\pi}^{\pi} |\alpha'(x)| \, dx \right)^2
\]

\[
\leq \frac{\pi}{2} \left( \int_{-\pi}^{\pi} |\alpha'(x)|^2 \, dx \right) = \frac{\pi}{2} \left( \frac{\int_{-\pi}^{\pi} |\alpha'(x)|^2 \, dx}{2\pi} \right). \tag{8}
\]

We now observe that \( |\alpha'(x)|^2 = |\alpha'(x)e^{i\alpha'(x)} - \alpha'(x))e^{-i\alpha'(x)}| = |f'(x)g'(x)| \), so if we apply the Schwarz inequality again to the right-hand side of (8), we finally obtain

\[
(\log(\|f\|_{L^2} \|g\|_{L^2}))^2 \leq \pi^2 \|f'\|_{L^2} \|g'\|_{L^2}. \tag{9}
\]

This holds even in the trivial case when \( f \) is constant, but then both sides vanish. Otherwise, (9) is clearly equivalent to our original statement, and it is easy to check that the inequality must be strict.

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References


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