BOOK REVIEWS

RINGS, MODULES, AND ALGEBRAS IN STABLE HOMOTOPY THEORY
(Mathematical Surveys and Monographs 47)


What is the best way to advertise a marvellous new engine which powers a variety of new vehicles that could not even be contemplated without it? Only a few engineers would appreciate an account of its specifications, the smoothness of its running and its efficiency: advertisements should concentrate on what it means for the consumer, what he can now achieve and what he no longer needs to worry about.

The book under review constructs such an engine, and provides a users’ manual for it. Outsiders would probably assume that such an engine had existed for years, whilst experts (including at least two of the authors) have believed for some time that no such engine could possibly be built. I shall not be commenting on the fine engineering quality of the engine, concentrating instead on the feats it makes possible: I write as a satisfied consumer.

A cohomology theory \( E^\ast(X) \) is a suitable contravariant, graded-group valued, homotopy invariant functor on topological spaces \( X \). In the beginning one constructed a cohomology theory by working on the space \( X \): thus ordinary cohomology \( H^\ast(X) \) is formed by taking homology of the singular cochain complex of \( X \). Similarly, \( K \)-theory \( K(X) \) is formed (for compact \( X \)) by considering complex vector bundles over \( X \) as a monoid under direct sum, and forming the associated group. Such geometric constructions are still the best way of constructing cohomology theories, not only because they provide an interpretation of the result, but also because they allow us easily to enrich its structure. For example, \( H^\ast(X) \) is a ring and also admits Steenrod powers, and \( K(X) \) is a ring and admits Adams operations. When we understand the geometry, it is easy to understand the interaction of these extra pieces of structure. However, our geometric understanding often falls short of what is needed. In studying the invariants or in calculations, one wants to construct new cohomology theories from old ones, and one wants the richest structure available.

For reasons such as this, it is convenient to have a category of objects (called spectra) representing cohomology theories, in the sense that \( E^\ast(X) = [X, E]^\ast \). Here \( E \) is the spectrum representing \( E^\ast(\cdot) \), and \([\cdot, \cdot]\) denotes homotopy classes of maps in the category of spectra. Thus spectra are to be thought of as stable based spaces, and we may hope for a category of spectra with all the good formal properties of the category of spaces. Boardman constructed a good homotopy category of spectra in his 1964 Warwick thesis (popularized by Adams’ account [1]), but the category is not satisfactory before passage to homotopy. The main difficulty is in the formation of smash products \( X \wedge Y \) of spectra: this should be commutative and associative, but Boardman’s construction achieves this only up to homotopy.
The achievement of the book under review is to construct a category of spectra with a commutative and associative smash product: it has other good properties too. This means that it makes sense to talk as in algebra. Thus we may discuss monoids (S-algebras) and commutative monoids (commutative S-algebras) in the strict category of spectra. (These turn out to be essentially the same as \(A_{\infty}\) and \(E_{\infty}\) ring spectra in old terminology; ‘brave new rings’ in Waldhausen’s more colourful phrase.) If \(R\) is a commutative S-algebra, then we may consider the subcategory of \(R\)-modules: this turns out to be very well behaved, and it too has its own commutative and associative smash product \(\wedge_R\). Its homotopy category is triangulated, and the homotopy groups of smash products and function spectra are calculated from classical Tor and Ext groups in a way generalizing Robinson’s pioneering results \([12, 13]\) in the \(A_{\infty}\) case. One may also discuss \(R\)-algebras and commutative \(R\)-algebras, and make many of the formal constructions of commutative algebra. One consequence of the new paradise is that new distinctions must be made between conditions that hold strictly (that is, before passage to the homotopy category) and those that hold up to homotopy. Previous terminology had no need of this. The authors have therefore had to introduce terminology without subverting existing usage: thus strict ring spectra are ‘S-algebras’ and ring spectra up to homotopy remain ‘ring spectra’. When working with an S-algebra \(R\), we may then consider the category of \(R\)-modules strictly or up to homotopy, and the strict ring objects in this category are sensibly ‘\(R\)-algebras’, but the ring objects up to homotopy are ‘\(R\)-rings’: the reviewer believes that this will lead to confusion, and that it is safer to refer simply to ‘strict \(R\)-algebras’ and ‘\(R\)-algebras up to homotopy’.

Amongst the applications there are results which could not be proved before, and there are results that previously involved devious circumlocution, and restriction to specially favourable cases. An example of the second sort is that of algebraic K-theory. There are various definitions of the Quillen K-theory of a discrete ring, and Waldhausen has given a definition which applies to brave new rings arising from ‘functors with smash product’, allowing him to define his functor \(A(X)\) relevant to high-dimensional manifolds. The book shows that Waldhausen’s \(S\)-construction applies to arbitrary S-algebras to give a definition of a K-theory spectrum \(K(R)\). If \(R = Hk\) is the representing S-algebra for ordinary cohomology with coefficients in a discrete ring \(k\), then this agrees with the Quillen K-theory of \(k\), and if \(R\) is the suspension spectrum of \(\Omega X\), then it agrees with \(A(X)\). Similarly, when attempting to calculate K-theory, one seeks traces in refinements of Hochschild homology. This book allows one to define the topological Hochschild homology for an S-algebra \(A\) by \(\text{THH}(A; M) = M \wedge_{A, A^{op}} A\), or alternatively by taking \(\text{thh}(A; M)\) to be the geometric realization of a simplicial S-module; it is shown they are equivalent under cofibrancy hypotheses, and calculable by a spectral sequence beginning with ordinary Hochschild homology. More important is that the present approach allows exactly analogous definitions to be used in the relative case when \(A\) is a strict \(R\)-algebra for an S-algebra \(R\): no definition was available before in the relative case.

One of the most frustrating shortcomings of the old world is that if \(R\) is a ring only up to homotopy, then the mapping cone of a map of \(R\)-modules up to homotopy need not be an \(R\)-module up to homotopy. This meant that to construct various bordism-related theories of central importance in homotopy theory, it was necessary to return to geometry and consider manifolds with singularities; this is very intricate, and neither covers all cases nor gives all the structure one might
want. Since $MU$ is visibly an $E_\infty$-ring spectrum, one may now simply work in the category of $MU$-modules and construct $BP$, $E(n)$, $K(n)$, etc., by copying the obvious constructions on coefficient rings. In many cases this also shows that they are $MU$-algebras up to homotopy. At the prime 2, some further work is necessary, as was also true in earlier treatments. N. P. Strickland has made further progress [14]. The machinery also works to construct equivariant versions of these spectra [11].

Next, we consider results which were not approachable before the introduction of the new machinery. A triangulated category of module spectra is essential in the modelling of commutative algebra. For example, it allows one to implement arguments presented as conjectural in [6], and hence to prove local cohomology and completion theorems in topology for Noetherian equivariant $S$-algebras by a formality. With substantial extra input, one may also prove the completion theorem for equivariant bordism [8]. Similarly, with some equivariant elaboration of the theory of the present work (see [5]), one may correct and prove conjectures of Benson on the ordinary cohomology of classifying spaces of compact Lie groups by proving the relevant local cohomology theorem [2]. This is only an example of the power of the commutative algebra that can be implemented in the brave new context [7].

Finally, Mandell [9] has shown that the present theory can be used to show that the singular cochain complex $S^\ast (X; \mathbb{F}_p)$ with coefficients in the algebraic closure of $\mathbb{F}_p$ gives an algebraic model for the homotopy theory of nilpotent $p$-complete spaces of finite type, as a full subcategory of the category of $E_\infty \mathbb{F}_p$-algebras. This goes by way of Mandell’s result connecting topological and algebraic categories of highly structured objects: the derived category of commutative $Hk$-algebras is equivalent to the derived category of $E_\infty k$-algebras for any discrete ring $k$.

It may be helpful to direct interested readers to the introductory summary given in [4], and to the announcement [3]. In addition, readers familiar with other approaches to highly structured rings will be pleased to know that Mandell, May, Shipley and Schwede have established a framework for comparison including the present approach, Segal’s $\Gamma$-spaces, Smith’s symmetric spectra, and that of functors with smash product [10].

The book itself is written for ease of reference. Thus the first nine chapters give the main definitions and state the main results, but only the more formal proofs are given. The technical heart of the proofs is contained in the last three chapters (beginning on page 179). The reader needs to bear this in mind, but the book is well cross-referenced, so it does not cause real difficulty. On the other hand, it is most unfortunate that the book contains no index of notation.

This will be the standard source, so it may be useful to know there is an errata page at http://zaphod.uchicago.edu/~mandell/; at present, this records only that II.6.1 requires the additional hypothesis (which holds in the applications) that $T$ preserves reflexive coequalizers, and provides a corrected proof.

References


Sheffield University

J. P. C. Greenlees

ELLiptic Boundary VALUE PROBLEMS in Domains with
Point SingularITIES
(Mathematical Surveys and Monographs 52)


Let Ω ⊂ \mathbb{R}^n be a domain with smooth boundary \partial Ω. Consider the boundary value problem

\begin{align}
Lu &= f & \text{in } \Omega, \\
B_k u &= g_k & \text{on } \partial \Omega, \quad k = 1, \ldots, m,
\end{align}

where f and g_k are given functions, u is the unknown function, L is a differential operator of order 2m acting in Ω, and B_k are differential operators acting from Ω to \partial Ω. The coefficients of L are assumed to be infinitely differentiable up to the boundary, and those of B_k are also infinitely differentiable.

The theory of smooth elliptic boundary value problems of the type (1), (2) is well developed, the classical text being the book by J.-L. Lions and E. Magenes. The authors of the book under review are interested in the situation when \partial Ω has a finite number of angular (n = 2) or conical (n ≥ 3) points. Such a situation is quite natural, especially in the two-dimensional case: it corresponds to a piecewise smooth boundary.

Starting with the pioneering works of V. A. Kondrat’ev published in the 1960s, the theory of elliptic boundary value problems in domains with angular and conical points has been developed by various authors, but there are still very few clearly written books on the subject. The book under review is a welcome addition, which will, we hope, expose a wider mathematical audience (in particular, applied mathematicians) to these results.

The main results of the book fall into the following categories.
• Rigorous mathematical statement of the boundary value problem. This involves conditions on $\Omega$, $L$, $B_k$, and the choice of appropriate weighted Sobolev spaces.

• Establishing the Fredholm property of the operator corresponding to the boundary value problem.

• A priori estimates for the solutions in different function spaces.

• Asymptotic expansions for the solutions near the singularities of the boundary.

The book requires the reader to be acquainted with the main facts concerning Sobolev spaces, but otherwise it is more or less self-contained and well-structured. The authors start with the most basic case, when $n = 1$, $\Omega$ is the half-line, and $L$ is an ordinary differential operator with constant coefficients. They gradually proceed to boundary value problems in the half-space, then to problems in domains with smooth boundaries, then to model problems in cylinders and cones, and only by the middle of the book actually start dealing with general problems in domains with angular and conical points. The subject inevitably involves a lot of technical material, and the gradual approach adopted by the authors makes this material digestible.

The last (relatively short) part of the book is devoted to boundary value problems in domains with exterior cusps.

The book does not give the most general results in the subject. There are publications dealing with pseudodifferential boundary value problems, with boundary singularities of other types (such as edges and polyhedral vertices), etc. However, the somewhat restricted nature of this book is its advantage. The authors have struck the right balance by understanding where to stop, otherwise the book would have become unreadable.

The book can be recommended to specialists in partial differential equations as an accessible and up-to-date research monograph. At the same time, it is a good text for graduate students specialising in the subject.

Sussex University  

Dmitri Vassiliev

### SYMMETRIES, TOPOLOGY, AND RESONANCES IN HAMILTONIAN MECHANICS

(Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 31)

By Valerij V. Kozlov: 378 pp., DM.198.–, ISBN 3 540 57039 X  
(Springer, 1996).

Nearly a century ago, Poincaré revolutionized science through his pioneering work on celestial mechanics, in which he demonstrated that the three-body problem could not be completely integrable. Since then, it has been a major challenge for mathematicians to develop methods which could determine if a given Hamiltonian system could possibly be classified as being integrable or non-integrable. V. V. Kozlov, who is one of the leading researchers in this rich and active field, has written an impressive and thorough book covering nearly all known aspects of the subject.

Starting in Chapter I with a historical review and a long list of concrete worked examples, he makes it clear how varied the question of integrability can be, often
with surprising answers. The book then essentially splits into two parts. The first, Chapter II, is devoted to detecting various forms of integrability in a system, and the second part is devoted to the converse, that is, to determining obstructions to integrability.

Integrability is invariably associated with symmetry groups, and in Chapter II the author gives a very readable presentation of Lie’s Theorem on integrability of systems with symmetry fields consisting of a solvable Lie algebra as well as a, perhaps too short, survey of the Lax pair representation (called the Heisenberg representation in the book) of a dynamical system. The Abelian functions which relate to Riemann surfaces of various genus play a central rôle in classical examples of integrable systems. These Abelian functions are also central to the notion of algebraically integrability and are generously covered in this presentation. A section on perturbation theory (Diophantine approximations) and (Birkhoff) normal forms concludes this chapter.

The topology of the configuration manifold can impose quite surprising obstructions to complete integrability. The author has been a major contributor to the subject, in particular for surfaces, and he gives a comprehensive survey thereof in Chapter III. In some cases, it turns out that a negative Euler characteristic of the manifold may effectively prevent integrability. The results presented emphasize the very important relationship between physics and topology, which has become of renewed interest in recent years.

Chapter IV considers non-integrability of systems close to integrable ones. The so-called Poincaré set or resonant set plays a crucial role here and in the rest of the book as, of course, it did in Poincaré’s original work. The much celebrated KAM-theory which is at the edge of the subject can be only loosely sketched in this treatment. The presence of homoclinic or heteroclinic orbits (Chapter V), that is, transverse intersections of stable and unstable manifolds, yields a very powerful method for excluding complete integrability. It also leads naturally to the study of completely hyperbolic dynamical systems and symbolic dynamics. Non-integrability near equilibrium positions (Chapter VI) is also a rich subject by itself. Siegel’s famous theorems on generic (in the sense of Baire) non-integrability, though non-constructive, are very instructive and well described in the book. (Birkhoff) normal forms are again essential in the analysis. Chapter VII discusses the obstruction to integrability which occurs when an analytically extended solution undergoes branching on a multi-sheeted Riemann surface. The analysis is closely related to that of Painlevé in his study of solvability of differential equations. Group-theoretical methods involving Coxeter graphs and Dynkin diagrams are given a careful treatment. The book is rounded off (Chapter VIII) by the consideration of when certain integrals polynomial in the momenta can be excluded.

The book gives a terrific survey of examples and theorems in the field. However, the subject/author index does not quite live up to the vast and diverse amount of information presented. Hopefully this, and possibly also the omnipresent misprints, will be amended in a coming version. The reference list, and in particular the Russian literature, is very comprehensive. The book is not for the novice, but can be read with great pleasure and profit by anyone with a fair background in Hamiltonian systems.

Warwick University  

HANS HENRIK RUGH
FINITELY AXIOMATIZABLE THEORIES
(Siberian School of Algebra and Logic)

By MIKHAIL G. PERETYAT’KIN: 294 pp., US$115.00, ISBN 0 306 11062 8
(Plenum Publishing Corporation (Consultants Bureau), 1997).

The Vaught–Morley problem asks if there exists a first-order sentence $\phi$ such that any two models of $\phi$ of cardinality $\aleph_1$ are isomorphic (the theory axiomatized by $\phi$ is uncountably categorical). The positive answer was found by the author of the book under review in 1980. This is a result belonging to foundations of mathematics: the first-order logic can be very expressive! The motivation of the book is the following question: what else can be expressed in the first-order logic?

In his Preface, the author suggests another title ‘Expressibility of finitely axiomatizable theories and their applications in logic (positive aspects)’. This would suit the book much better than the title actually used. The book is entirely devoted to the author’s method of building of finitely axiomatizable theories.

Theorem 0.6.1 of the Introduction (called the Main Theorem) states the existence of an effective procedure which transforms recursively axiomatizable theories into finitely axiomatizable ones, preserving their type with respect to some long list of model-theoretic and recursion-theoretic properties. The proof of the theorem is the matter of the book. Two final chapters (Chapters 7 and 8) contain applications of the theorem, all of which are very interesting. For example, the following problem is discussed in Chapter 8, ‘Complexity of semantic classes of sentences’. Given a property $\Delta$ of the first-order theories, find the algorithmic complexity of the set $\{n: n$ is the number of a sentence axiomatizing a theory satisfying $\Delta\}$. It is shown that for $\Delta = \{\text{complete theories}\}$, the corresponding complexity is $\Pi^0_2$.

Most likely, a typical reader of the book will read only Chapters 0, 1, 2, 7 and 8, omitting the technically involved constructions of Chapters 3–6.

Nevertheless, I recommend the book to logicians. The results are really impressive, and the ideas are very original. Simplified versions are given in Chapters 1 and 2.

Wroclaw University

ALEXANDRE IVANOV

AN ALGEBRAIC APPROACH TO ASSOCIATION SCHEMES
(Lecture Notes in Mathematics 1628)

By PAUL-HERRMANN ZIESCHANG: 189 pp., £23.50, ISBN 3 540 61400 1
(Springer, 1996).

An association scheme is a set $X$ of points together with a collection $R$ of relations on $X$ satisfying certain regularity conditions. Besides the book under review, the only book devoted to the theory of association schemes is that of Bannai and Ito [1], although [2, 3] also discuss association schemes extensively, and various other books on combinatorics give a brief treatment. This book, however, treats the theory of association schemes from a viewpoint quite different from the one found in the literature. Its motivation is similar to that of [1]—to establish a connection between finite group theory and association schemes. While [1] stresses the similarity to the character theory of finite groups, this book regards association schemes as
combinatorial objects which directly generalize groups. To be more precise, a finite
group \( G \) is taken as the set of points of an association scheme with relations in one-
to-one correspondence with \( G \). With this perspective, the familiar concepts and
theorems in group theory are generalized to association schemes.

This book suggests a research direction in the theory of non-commutative
association schemes which cannot be found elsewhere. The most important
association schemes among the non-commutative ones seem to be those generated by
a set of ‘involutions’, and a large part of the book is devoted to the investigation of
such schemes. Coxeter schemes are probably the simplest of these, and the author has
succeeded in completing a beautiful theory of Coxeter schemes which enables one to
treat Coxeter groups and buildings in a single framework.

This book is self-contained and very carefully written. Logical dependencies on
previously proved results are always indicated clearly, and the author never forgets
to add sentences which clarify the structure of the proofs. Beginning graduate
students will find this writing style helpful. Thus I recommend this book to anyone
interested in group theory and group theoretical aspects of combinatorics.

References

1. E. Bannai and T. Ito, *Algebraic combinatorics I: Association schemes* (Benjamin/Cummings, Menlo
    Park, CA, 1984).

Kyushu University

Akihiro Munemasa

LINEAR ALGEBRA
(Pure and Applied Mathematics)

By Peter D. Lax: 250 pp., £55.00, ISBN 0 471 11111 2
(John Wiley & Sons, 1996).

In an interview published in [1], Peter Lax was asked: ‘Do you consider yourself
an applied mathematician?’ He replied: ‘I’m both pure and applied.’ This book on
linear algebra is also both pure and applied. It takes a more practical view than
traditional pure linear algebra textbooks, but is more abstract than the more
computationally or matrix oriented books. The book is aimed at beginning graduate
students.

The writing is concise and stylish. Practical examples are given to enliven the
theory.

Among the more unusual material are the chapter entitled ‘Kinematics and
dynamics’, which describes how matrices appear in mechanics, and the chapter
entitled ‘Calculus of vector and matrix valued functions’, which has an insightful
discussion of the ‘avoidance of crossing’ of multiple eigenvalues of a matrix
depending on a parameter.

A few minor criticisms can be noted. An apparent notational convention of roman
font for mappings but italic for sets, spaces and functions is used without explanation
and, at times, inconsistently. In Theorem 22, the polar decomposition is introduced
but is named only in the page header and index; much more could have been said of
this and the important singular value decomposition. The Jordan canonical form (JCF) is also listed in the index but not explicitly found on the page indexed, and the author stops short of stating the JCF explicitly.

Overall, this is a stylish addition to the mass of textbooks on linear algebra, distinguished by its unusual balance of pure and applied viewpoints.

Reference


Manchester University Nicholas J. Higham

REPRESENTATION THEORY AND COMPLEX GEOMETRY

By Neil Chriss and Victor Ginzburg: 495 pp., SFr.108.00, ISBN 0 8176 3792 3 (Birkhäuser, 1997).

Consider the problem of finding the irreducible complex representations of the permutation group on \( n \) letters, \( S_n \). The answer, which goes back to Frobenius and Schur, is that there are as many as the partitions of \( n \), and that one can describe each representation explicitly in terms of the combinatorics of partitions.

Now, \( S_n \) is the Weyl group of the Lie group \( GL_n \), and one may ask such a question for the other Weyl groups. Ideally, the answer should be uniform for all Lie groups, and reduce to the well-known combinatorics for \( GL_n \).

Such a construction was found by Springer and Deligne in 1976. To each reductive group, one associates the space of flags. Now pick a unipotent element \( u \) in the group, and look at the flags fixed by \( u \). This is a connected algebraic variety with lots of components, all of the same dimension. The Springer construction makes these components the basis of a vector space on which the Weyl group acts. This representation depends only on the conjugacy class of the unipotent element, and each irreducible representation of the Weyl group occurs once in precisely one such representation.

For \( GL_n \), unipotent conjugacy classes are indexed by partitions of \( n \), and we recover the classical result indexing representations of \( S_n \).

For other groups, these representations are not quite irreducible; they are indexed by some, but not all, of the representations of the group of components of the centraliser of the unipotent element in \( G \). This strange-sounding statement is not a technical complication, but rather reflects profoundly the existence of cuspidal representations of the finite Chevalley group (hence counterexamples to the generalised Ramanujan conjectures, etc.).

The vector space described above is just the top homology of the space \( \mathcal{B}_u \) of flags fixed by \( u \), and the Springer construction makes the Weyl group act on the entire homology group, preserving each \( H^i \). It is natural to ask what is the meaning of these other representations.

The answer involves looking at the ‘full symmetries’ of the space \( \mathcal{B}_u \). This turns out to be not just the Weyl group, but the semidirect product of the Weyl group with the Picard group of the flag variety—the affine Weyl group; and, in fact, not just this but the ‘quantisation’, or \( q \)-deformation of it, the affine Hecke algebra.
This was defined by Iwahori and Matsumoto in the 1960s, and irreducible representations of it parametrise irreducible spherical representations of the $p$-adic group which is Langlands dual to $G$. The Springer construction can be generalised to describe its representation theory, and that was done by Kazhdan and Lusztig in 1987, thus proving that the representations are parametrised as the Langlands program would predict (the ‘Deligne–Langlands–Lusztig conjecture’).

This story is the subject of the book under review. Some 300 pages are a fairly direct route through the above mathematics, assuming little more than a first acquaintance with Lie algebras and groups. The remaining pages are background and digressions, some of which are a lot of fun.

There is good taste in the choice of material, and the presentation is enthusiastic. The book is a nice companion to the original papers of Kazhdan and Lusztig, and a good general introduction to modern representation theory.

Cambridge University

IAN GROJNOWSKI

**HOMEOMORPHISMS IN ANALYSIS**

*(Mathematical Surveys and Monographs 54)*

*By Casper Goffman, Togo Nishiura and Daniel Waterman: 216 pp., US$69.00, isbn 0 8218 0614 9 (American Mathematical Society, 1997).*

The mathematician who described his opponent’s argument as a homeomorphic image of the truth did not intend to be complementary. At first sight, the study of behaviour of functions under homeomorphism of the underlying space seems unlikely to produce interesting results. But consider the question of which bounded functions $f: (−1, 1) \to \mathbb{R}$ can be the derivative of some differentiable function. The example of the derivative of $x^2 \sin x^{-1}$ shows that the question is non-trivial. A little thought shows that any ‘derivative function’ $f$ must be the pointwise limit of continuous functions, and a little more thought shows that $f$ must have the intermediate value property. Yet more reflection shows that these necessary conditions are not sufficient. Most mathematicians would, I think, conclude that this line of investigation is totally unpromising. However, they would be wrong, for we have following beautiful theorem of Maximoff.

**Theorem 1.** Let $f: (−1, 1) \to \mathbb{R}$ be a bounded function. The following two conditions are equivalent.

(i) There is a homeomorphism $\phi: (−1, 1) \to (−1, 1)$ such that $f \circ \phi$ is the derivative of a differentiable function.

(ii) The function $f$ is the pointwise limit of continuous functions, and has the intermediate value property.

Maximoff’s theorem is easy to state but hard to prove. The book under review gives a ‘relatively simple’ proof based on that of Preiss.

Homeomorphisms also appear in a rather mysterious way in Fourier analysis. It is well known that the Fourier sums $\hat{f}(r) = \sum_{n=1}^{\infty} f(r) \exp(irt)$ of a continuous function $f$ may diverge at a point. The Pál–Bohr theorem states that this phenomenon can be removed by a homeomorphism of the underlying space.
Theorem 2. If \( f: \mathbb{T} \to \mathbb{R} \) is continuous, then there exists a homeomorphism \( \phi: \mathbb{T} \to \mathbb{T} \) such that writing \( g = f \circ \phi \), we have \( \sum_{n=0}^{\infty} g(r) \beta^n \) uniformly convergent on the circle \( \mathbb{T} \).

On the other hand, Olevskii answered a 50-year-old question of Lusin by showing that even homeomorphism of the underlying space can improve things only to a certain extent.

Theorem 3. There exists a continuous function \( f: \mathbb{T} \to \mathbb{R} \) such that if \( \phi: \mathbb{T} \to \mathbb{T} \) is a homeomorphism and \( g = f \circ \phi \), then \( \sum_{n=0}^{\infty} g(r) \beta^n \) fails to converge absolutely.

If you are ideologically opposed to the kind of mathematics exemplified by the theorems above, then this book is not for you. If you could not resist trying to prove the theorems for yourself, and now wonder 'how do they do that?', then you will enjoy browsing through this book. If you already knew these results, then you will probably wish to read the book from cover to cover.

This is an excellent introduction to a specialised but very pretty corner of analysis by three major contributors to the field. The exposition is clear, and the care with which the book was written makes it surprisingly easy to cope with the specialist vocabulary ('the UGW condition', \( \text{npH}^p(E) \), and so on). For the non-specialist, the appendix is an education in itself.

Cambridge University

T. W. Körner

FRACUTRED FRACTALS AND BROKEN DREAMS: SELF-SIMILAR GEOMETRY THROUGH METRIC AND MEASURE

(Oxford Lecture Series in Mathematics and its Applications 7)

By Guy David and Stephen Semmes: 212 pp., £35.00, ISBN 0 19 850166 8

(Clarendon Press, 1997).

Over the past several decades, there has been an explosion of interest in self-similar sets, partially through the work of Mandelbrot, and partially through the realization that such objects are to be found in many different areas of mathematics. Though there have been many examples of self-similar sets, the most famous perhaps being the Mandelbrot set, there has not as yet been a good mathematical definition of what it means for a set to be self-similar. The book under review proposes and explores such a definition.

The authors begin by giving the basic definition of what it means for a metric space to be a BPI space ('big pieces of itself'). This definition has two parts. First, the space is required to be homogeneous in a weak sense; specifically, it is required to be an Ahlfors regular space, which loosely means that the \( d \)-dimensional Hausdorff measure of a ball of radius \( r \) is comparable to \( r^d \). Second, any two balls in the space are required to be roughly conformally bi-Lipschitz, with a scale factor depending on the radii of the balls. Together, these give a reasonable definition for what it means for a space to be self-similar. The authors also define what it means for two metric spaces to be BPI equivalent. Examples of BPI spaces which are discussed include: Euclidean spaces; some of the standard fractal sets such as the Sierpinski carpet and
the Sierpinski gasket; limit sets of convex co-compact Kleinian groups, and more generally of fundamental groups of compact negatively curved manifolds; and the Heisenberg group. It is also possible to make sense of convergence of BPI spaces, and it is shown that, under suitable conditions, the limits are themselves BPI spaces. The authors go on to explore the basic properties of BPI spaces, including comparisons with more well known classes of spaces such as rectifiable sets.

While they give a good introduction to the subject, the authors themselves admit, in the Introduction: ‘There are many examples and concepts and basic facts, but no crisp theorems. The subject remains a wilderness, with no central zone, and many paths to try.’ The second sentence of this quote is important. One of the beauties of this book is that it is not definitive, in the sense that it does not look back over a mature field of mathematics. Rather, the authors describe a new way of looking at self-similarity, give numerous examples of spaces which do and do not have this property, discuss the basic properties of their definition and the connections it has with a wide variety of different parts of mathematics, and then leave readers with room to explore for themselves.

The level of the book is suitable for an advanced undergraduate student, a graduate student, or an established researcher. The book is largely self-contained, requiring of the reader only a basic knowledge of metric spaces and measure theory. It is, overall, an excellent book, of a type which, in this reviewer's opinion, is sorely lacking in mathematics these days.

Southampton University

JAMES W. ANDERSON

CAUCHY AND THE CREATION OF COMPLEX FUNCTION THEORY

By Frank Smithies: 216 pp., £35.00 (US$59.95), ISBN 0 521 59278 X
(Cambridge University Press, 1997).

Of all the many contributions Cauchy made to mathematics, few can rival his creation of complex function theory, yet this is perhaps the first book ever devoted exclusively to a historical analysis of what he did. Cauchy is to blame: his many papers were scattered over 40 years and across numerous journals, and, as his French successors complained (see [1]), they are neither mutually consistent nor indicative of steady progress.

Cauchy first sought to explain how the widely used technique of ‘passing from the real to the imaginary’ in integration problems could be justified. In a memoir of 1814 (published only in 1827), he studied how exchanging the order of integration in a double integral might not be allowed when the function is infinite inside the corresponding rectangle. Our modern concepts are there only beneath the surface. Thus Smithies comments (p. 56): ‘it was not until 1851 that [Cauchy] came to appreciate the significance of the Cauchy–Riemann equations in distinguishing analytic and non-analytic functions’. This memoir contains the seeds of many future discoveries, and the reader will enjoy Smithies’ careful account of this rich and yet confusing work.

Cauchy first proved the Cauchy Integral Theorem for at least some paths in the complex plane, by a variational argument, in an obscurely published pamphlet of 1825. For a function that becomes infinite at some points, he analysed the cases of
simple and higher-order poles. In the next year he wrote prolifically on what he called the calculus of residues. In 1827 he returned to the tentative use of geometrical language that he had employed in 1825, and studied integrals around closed curves, usually circles, which begin to replace the rectangles he had previously preferred.

In voluntary exile in Turin in 1831, Cauchy wrote on the convergence of series and the method of majorants. Then, in a further memoir, he wrote about complex integrals on general simple closed curves, and established the residue theorem in that setting. This memoir only ever circulated in lithographed form (see Oeuvres (2), 15 (1974) 182–261). With these two memoirs, broaching so much of the subject, Smithies concludes his account.

This book will be an invaluable guide to anyone interested in the work of a great mathematician. It is clear and careful, and the significance of what Cauchy did shines through. It is unfortunate that, like Cauchy’s contemporaries, we are left to make sense of much more of what he did before his death in 1857. For Cauchy’s own hesitations ran deep. He was not clear how a complex number should be regarded, and offered three definitions: formal, algebraic and geometric. He was seldom sure how a complex function should be defined so as to fit with his much more precise (and better published) real analysis (see Bottazzini’s essay in [2]). Many-valued functions were always a problem for him. When he returned to Paris in 1838, he often claimed that the results of others followed from his own theory: Laurent series expansions and Liouville’s theorem (see [3]) are cases in point. It would have been even better to have had Smithies guide us to the end, but we must congratulate him for what we have: Cauchy better than he ever understood himself.

References

The Open University

Jeremy Gray

OPERADS: PROCEEDINGS OF RENAISSANCE CONFERENCES (Contemporary Mathematics 202)


An algebra comes with products $A^n \to A$ for all $n \geq 0$. We may specify various types of algebra (for example, commutative, associative, Lie, Poisson, Leibniz) by specifying certain relationships between these products. We may also want to allow the algebra identities to hold only up to homotopy. An operad $\mathcal{O}$ encodes the information specifying such a class of algebras. It consists of objects $\mathcal{O}(n)$, for $n \geq 0$, parametrizing the products $A^n \to A$ in the sense that there is a map $\mathcal{O}(n) \times A^n \to A$; of course, there is additional structure to relate the products for different $n$. Similarly, a module over such an algebra $A$ is given by maps $\mathcal{O}(n) \times A^{n-1} \times M \to M$. To obtain
the axioms, consider the special case of the endomorphism operad $\text{End}(A)$ with $\text{End}(A)(n) = \text{Hom}(A^n, A)$. It is thus helpful to think of an element of $\mathcal{C}(n)$ as giving a means for combining $n$ objects to give one object (a tree with $n$ inputs and one output), and this shows that the main piece of data is a map $\mathcal{C}(k) \times (\mathcal{C}(j_1) \times \cdots \times \mathcal{C}(j_k)) \to \mathcal{C}(n)$, where $n = j_1 + j_2 + \cdots + j_k$ (combining the trees by feeding the outputs from each of the $k$ last trees into the $k$ inputs of the first). The main axiom is the associativity statement that if we make a tree with $n$ inputs by sticking together subtrees, then the order in which a tree is assembled does not affect the resulting element of $\mathcal{C}(n)$. Evidently, we may consider operads in any symmetric monoidal category (that is, with an operation like $\times$ or $\otimes$), and we may then consider $\mathcal{C}$-algebras in this category, and modules over them. It turns out that the language of operads is helpful in many such contexts.

Operads were first considered in homotopy theory, to identify spaces $X$ that are homotopy equivalent to the space $\Omega Y$ of (based) loops on some space $Y$, or to the $m$-fold loop space $\Omega^m Z$ for some space $Z$. If $X = \Omega Y$, then $X$ has a product given by composition of loops, but it is associative only up to homotopy; however, the natural associating homotopy is itself very well-behaved (for example, the two ways of showing that $x(y(zt))$ is homotopic to $(xy)zt$ are themselves homotopic). Stasheff’s $A_\infty$-operad encodes the relevant structure, and a space $X$ is homotopy equivalent to $\Omega Y$ for some $Y$ if and only if $X$ is an $A_\infty$-algebra. The little $m$-balls operad $D^m$ of Boardman and Vogt is defined by taking a point of $D^m(n)$ to consist of $n$ little $m$-balls with disjoint interiors in the unit $m$-ball. This operad allows a similar characterization of when $X$ is an $m$-fold loop space. The formal notion of operad was introduced by May in 1972.

Evidently, if $\mathcal{C}$ is an operad in the category of spaces, then its singular homology $H_\bullet(\mathcal{C}; k)$ is an operad in the category of graded $k$-algebras if $k$ is a field. Furthermore, if $X$ is an $\mathcal{C}$-algebra, then its homology $H_\bullet(X; k)$ is an $H_\bullet(\mathcal{C}; k)$-algebra. Many interesting examples arise in this way, or by applying some other monoidal functor. Specifically, if $\mathcal{C}(n)$ is a point for each $n$ (giving an operad whose algebras are topological abelian groups), then we obtain the algebraic operad $\text{Comm}$ (with $\text{Comm}(n) = k$ for all $n$), whose algebras are graded commutative $k$-algebras. If $\mathcal{C}(n) = \Sigma_n$, then we obtain the algebraic operad $\text{Ass}$ (with $\text{Ass}(n) = k\Sigma_n$ for all $n$), whose algebras are graded associative $k$-algebras. Similarly, the operad $\text{Lie}$ may be constructed from the little discs operad by taking $\text{Lie}(n) = H_{\Sigma_n}(D^2(n))$: this has Lie algebras as its algebras. The operad with $n$th term $G(n) = H_\bullet(D^2(n))$ has Gerstenhaber algebras as its algebras. If little discs are replaced by little $(m+1)$-balls, then we obtain the $m$-Lie algebras and the $m$-braid algebras. If little discs are framed, then we obtain the Batalin–Vilkovisky algebras. To obtain examples where various identities hold up to homotopy, one considers operads in the category of differential graded $k$-algebras, for example, those obtained from topological operads by taking singular chains.

The tree point of view makes it natural to attempt to define operads $\mathcal{C}$ so that the elements of $\mathcal{C}(n)$ are surfaces with $n$ input boundary circles and one output boundary circle. There are various choices of additional structure (for example, smooth, conformal, complex, markings), giving different operads and different technical problems in defining the structure maps. If one uses only surfaces of genus 0, then one is said to be working ‘at the tree level’. Operads of this sort play a role in conformal field theories, closed string field theories, and vertex operator algebras. The work of Ginzburg and Kapranov shows that one such example plays a universal role amongst all operads.
Although operads have been constantly useful since their introduction, the renaissance of the book’s title is due to recent powerful applications in algebra, representation theory and physics, together with striking advances in homotopy theory because of the operadic tensor product. The book contains surveys by May and Stasheff, together with fascinating examples from very diverse applications. The coverage does not pretend to be uniform or systematic, but the book provides a good starting point for an exploration of the new multiple incarnations of operads. The above examples give only a selection of the possible reasons why you might want to look at this book yourself, so your library should certainly have a copy.

Sheffield University

J. P. C. GREENLEES

THE ERGODIC THEORY OF DISCRETE SAMPLE PATHS
(Graduate Studies in Mathematics 13)

By Paul C. Shields: 249 pp., US$39.00, ISBN 0 8218 0477 4

This is an interesting and well-written account of the ergodic theory of stationary processes from the viewpoint of entropy theory. The introduction of entropy, a basic concept from information theory, to dynamical systems theory by Kolmogorov in the 1950s led to a revolution in the subject. Using entropy, Kolmogorov and Sinai showed that not all Bernoulli shifts are isomorphic, and twenty years later Ornstein showed that in fact Bernoulli shifts are completely classified by their entropy, that is, two Bernoulli shifts are measure theoretically isomorphic if and only if they have the same entropy.

Shields’ earlier book, The theory of Bernoulli shifts [2], is probably the clearest exposition of the fundamental results of the Bernoulli theory of Ornstein and Weiss, in particular the proof of Ornstein’s isomorphism theorem. This book was written soon after Ornstein’s work in the 1970s and was aimed at the dynamical systems community. The book under review is more comprehensive and returns to the roots of the subject: the models for finite alphabet stationary processes that have been developed in probability theory, ergodic theory and information theory. The quality of writing is uniformly high. Shields has an informal but succinct writing style which makes the book very enjoyable to read. The book has a particularly good introduction which shows how ergodic theorists view a stationary process as a measure preserving transformation of a probability space, together with a partition of the space. It would be an excellent text for a postgraduate reading course (there are well thought-out problems at the end of each section).

Finite alphabet stationary processes are of interest to a wide range of mathematicians and scientists, especially those working in abstract ergodic theory, information theory or data compression. Despite lengthy discussion of the application of this circle of ideas to information and coding theory, this is definitely a mathematical text. If one wishes to read more about applications to information theory then Entropy and information theory [1] by R. Gray is a good source. In fact, the present book fills a gap between [2] and [1].

An important theme of the book is the use of partitioning, covering and packing lemmas pertaining to the set of integers to study sample paths and to provide proofs
of the main theorems in ergodic theory—the Birkhoff ergodic theorem and the Shannon–McMillan Breiman Theorem—with minimum machinery or probabilistic formalism. These ideas and techniques yield more than elementary proofs of the main ergodic theorems, and this book shows the prominent role they play in ergodic and information theory.

Chapter 1 provides a clear account of measure theoretic entropy and some applications of entropy to information theory. It quickly introduces the notion of entropy (in a limited context), and continues with an excellent treatment of the standard topologies given to stationary processes: the weak topology and the $d$ metric. The advantages and disadvantages of working in either topology are clearly explained; in particular, the fact that the $d$ limit of ergodic processes is ergodic is proved. Chapter 2 is a discussion of the role of entropy in coding theory. Entropy is shown to be the almost-sure bound on per-symbol data compression. The existence of a universal code which compresses to entropy is proved by a counting argument together with some results about entropy for Markov processes and also by entropy estimates developed by Ornstein and Weiss. The Lempel–Ziv algorithm for data compression is also discussed in this framework.

Highlights of Chapter 3 include results on rates of convergence for entropy and frequencies, and the estimation of joint distributions in the $d$ metric.

In Chapter 4 those processes isomorphic to a Bernoulli shift, the $B$-processes, are discussed and are characterised by the finitely determined property. These are precisely those processes which are stationary codings of independent identically distributed processes, and hence are of central importance to probability theory and ergodic theory. Two more recent characterisations, almost block-independence and almost blowing-up, are defined, and proofs are given of the equivalence of these properties with the finitely determined property.

This book is a beautiful and very readable treatment of an important part of ergodic theory. It is written by an expert in the field who has made very significant contributions and who has an excellent knowledge of the great relevance of the theory to other fields such as information theory. I strongly recommend this book to anyone who is interested in ergodic theory, stochastic processes or information theory.

References

UMIST

MATTHEW NICOL

INTEGRABILITY, SELF-DUALITY, AND TWISTOR THEORY
(LMS Monographs (N.S.) 15)


Mathematicians are notorious for their addiction to precision, yet some concepts in common use defy a precise definition. One such is the notion of an integrable system. A system of differential equations which is integrable has, by common acclaim, some recognizable properties such as a multitude of conserved quantities and
a concrete sense in which these can be incorporated as constants in some well-defined integration process, but as yet there is no accepted mathematical definition. Yet the subject is of great interest and a source of important mathematical results. What is it that enables us to label as integrable the motion of a spinning top, or the curves described by stretching a string over the surface of an ellipsoid, or the equation that determines the shallow water waves that led Scott Russell to gallop along the canal towpath into the pages of mathematical history?

One answer is provided by the philosophy of the book under review: twistor theory. This theory was originated by Roger Penrose some thirty years ago, with a rather different end in view: to provide a setting for a possible unification of quantum theory and gravity by introducing complex numbers at a fundamental level. This approach meant that the points of space–time were no longer the primary objects of study—twistors were, instead. Over the ensuing years, twistor theory became a powerful tool, not only for solving some of the equations of mathematical physics, but also for providing a philosophical glue which sticks together whole groups of equations. It is this adhesion, applied to equations by now quite far removed from general relativity, which the authors convincingly argue lies behind integrability. In fact, it is really one such nonlinear equation, the self-dual Yang–Mills equation (and its near neighbours), which appears to give birth to the well-known integrable systems.

The self-dual Yang–Mills equation is an equation in four space–time dimensions, which is gauge-theoretical: it involves the choice of a Lie group, and has an infinite-dimensional group of gauge symmetries. Integrable systems are usually obtained by taking a solution to these equations which exhibits some symmetry. Using both spatial and gauge symmetries, the equations can often be dimensionally reduced to recognizable forms in two space–time dimensions. A very persuasive argument for the twistor philosophy is provided by the result (due originally to the first author and George Sparling) that if the space–time symmetry is generated by two translations, one null, and the Lie group is the group of $2 \times 2$ invertible matrices, then there are essentially only two reductions of the self-dual Yang–Mills equation. These are the nonlinear Schrödinger equation and the KdV equation: two canonical models of integrability, the latter being the cause of Scott Russell’s famous ride.

Classification is not the only achievement of twistor theory: Penrose’s approach, as shown by Richard Ward, provided a means of solving the self-dual Yang–Mills equation by using the holomorphic geometry of vector bundles. With a suitably generous interpretation, this can also be seen as the inverse scattering method, the traditional approach to solving integrable systems.

All these themes are taken up in the book under review, and it also provides a self-contained account of twistor methods, tailored to the specific goals of dimensional reduction. The treatment is quite dense but also comprehensive, and includes new material. One of the more recent and interesting topics covered is the relationship between the Painlevé equations and twistor theory. The ‘Painlevé property’ which originally motivated the nineteenth century search which led to Painlevé’s six equations is now one accepted characteristic of an integrable system. The reader will find in this most informative book that, with the viewpoint of twistor theory, this classification corresponds to the classification of three-dimensional abelian subgroups of the special linear group SL(4, C).

The philosophical case for twistor theory underlying integrability is compellingly presented by the mathematical arguments, but there are exceptions, as the book
recognizes. The reader might be tempted to view the whole approach as Procrustean. Be that as it may, the reader should also remember that in ancient Attica, Procrustes gave his guests a hearty meal before stretching them or curtailing them to fit his bed. This book provides ample food for both mathematical thought and practice, and is recommended reading for all who have some interest in the intriguing notion of integrability.

Oxford University

N. HITCHIN

INDEX OF BOOK REVIEWS

A. D. ELMENDORF, I. KRIZ, M. A. MANDELL and J. P. MAY, Rings, modules, and algebras in stable homotopy theory [reviewed by J. P. C. Greenlees] 367

V. A. KOZLOV, V. G. MAZ’YA and J. ROSSMANN, Elliptic boundary value problems in domains with point singularities [Dmitri Vassiliev] 370

VALERIJ V. KOZLOV, Symmetries, topology, and resonances in Hamiltonian mechanics [Hans Henrik Rugh] 371

MIKHAIL G. PERETYAT’KIN, Finitely axiomatizable theories [Alexandre Ivanov] 373

PAUL-MER-MANN ZIESCHANG, An algebraic approach to association schemes [Akihiro Munemasa] 373

PETER D. LAX, Linear algebra [Nicholas J. Higham] 374

NEIL CHRISS and VICTOR Ginzburg, Representation theory and complex geometry [Ian Grojnowski] 375

CASPER GOFFMAN, TOGO NISHIURA and DANIEL WATERMAN, Homeomorphisms in analysis [T. W. Körner] 376

GUY DAVID and STEPHEN SEMMES, Fractured fractals and broken dreams: self-similar geometry through metric and measure [James W. Anderson] 377

FRANK SMITHIES, Cauchy and the creation of complex function theory [Jeremy Gray] 378

JEAN-LOUIS LODAY, JAMES D. STASHEFF and ALEXANDER A. VORONOV (eds), Operads: proceedings of renaissance conferences [J. P. C. Greenlees] 379

PAUL C. SHIELDS, The ergodic theory of discrete sample paths [Matthew Nicol] 381

L. J. MASON and N. M. J. WOODHOUSE, Integrability, self-duality, and twistor theory [N. Hitchin] 382