A COUNTEREXAMPLE TO UNIQUENESS IN THE
RIEMANN MAPPING THEOREM FOR UNIVALENT
HARMONIC MAPPINGS

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1. Introduction

Let $f$ be an orientation-preserving univalent harmonic mapping of the unit disk $U$. Then $f = h + g$, where $h$ and $g$ are analytic in $U$. Furthermore, $f$ satisfies the equation

$$f_z = af_z$$

in $U$, where $a(z) = g'(z)/h'(z)$, and $|a(z)| < 1$ in $U$. The function $a(z)$ is the analytic dilatation of $f$.

In [2], Hengartner and Schober proved the following version of the Riemann mapping theorem for univalent harmonic mappings.

**Theorem A.** Let $D$ be a bounded simply connected domain whose boundary is locally connected. Fix $w_0 \in D$, and let $a(z)$ be analytic in $U$, with $a(U) \subseteq U$. Then there exists an orientation-preserving univalent harmonic mapping $f$ with the following properties.

(a) $f$ maps $U$ into $D$, and $f(0) = w_0, f_z(0) > 0$.
(b) $f$ satisfies the equation $f_z = af_z$.
(c) Except for a countable set $E \subset \partial U$, the unrestricted limit $f^*(e^{it}) = \lim_{z \to e^{it}, z \in U} f(z)$ exists and belongs to $\partial D$.
(d) The one-sided limits $\lim_{t \to t^-} f^*(e^{it})$ and $\lim_{t \to t^+} f^*(e^{it})$ through values of $e^{it} \not\in E$ exist and belong to $\partial D$; for $e^{it} \not\in E$ they are equal, and for $e^{it} \in E$ they are different.
(e) The cluster set of $f$ at $e^{it} \in E$ is the straight line segment joining the left and right limits in (d).

It should be noted that in (a) of Theorem A, $f$ is not required to be onto $D$. In fact, as pointed out in [2, p. 481], there is no univalent harmonic mapping with $a(z) = z$ such that $f(U) = U$. If $a(z) \equiv 0$, then in Theorem A, $f$ is analytic, and uniqueness follows from the classical Riemann mapping theorem. In [3], it is proved that if $a(z) = z$ and $D = U$, then $f$ is unique, and $f(U)$ is a triangle contained in $U$. In [5], it is shown that uniqueness holds when $a(z) = z^n$ and $D$ is convex. As we shall see in Section 2, uniqueness fails in general. For the example given there, we shall construct two functions, $f$ and $f_1$, such that $f$ maps onto a domain $D$, and $f_1$ maps to a proper subdomain, but still satisfies the conditions of Theorem A.
The question remains as to whether or not there could be two univalent harmonic mappings onto the same domain \(D\) with the same normalization (a) in Theorem A. If \(D\) is strictly starlike with respect to \(w_0\), then it follows from Lemma 2 of [1] that there is at most one map as in Theorem A which is onto.

2. A counterexample

Let \(D\) be the nonconvex quadrilateral having ordered vertices \(1, i, 1/2, -i\) as in Fig. 1. Now, \(D\) is a rotation of an example of Sheil-Small [4, p. 473]. Then \(f\) is the mapping of \(U\) onto \(D\) given by the Poisson integral of the function which is 1 on the arc of \(\partial U\) from \(-i\) to \(i\), \(i\) on the arc from \(i\) to \((-3 + 4i)/5\), \(1/2\) on the arc from \((-3 + 4i)/5\) to \((-3 - 4i)/5\), and \(-i\) on the arc from \((-3 - 4i)/5\) to \(-i\).

![Fig. 1](image-url)

It follows from [4, p. 473] that

\[
f_{\alpha}(z) = \frac{6(z + 1)}{\pi(z^2 + 1)(5z^2 + 6z + 5)}, \quad f_{\beta}(z) = \frac{-6z(z + 1)}{\pi(z^2 + 1)(5z^2 + 6z + 5)}. \tag{2.1}
\]

and \(1/2 < f(0) < 1\). For \(x \in (-1, 1)\), we have in general that \(f(x)\) is real, and from (2.1),

\[
\frac{df}{dx} = \frac{6(1 - x)(x + 1)}{\pi(x^2 + 1)(5x^2 + 6x + 5)} \quad (-1 < x < 1). \tag{2.2}
\]

We now inscribe an equilateral triangle \(T\) in \(D\) having common vertex at \(1/2\) as in Fig. 1, and map \(U\) onto \(T\) by first taking the Poisson integral of the function which is \(e^{i\pi/3}\) on the arc of \(\partial U\) from \(1\) to \(e^{i2\pi/3}\), \(-1\) on the arc from \(e^{i2\pi/3}\) to \(e^{-i2\pi/3}\), and \(e^{-i\pi/3}\) on the arc from \(e^{-i2\pi/3}\) to \(1\). Dilating this function by an appropriate positive constant and translating by an appropriate \(u_0 > 0\), we obtain the desired function \(f_{1}(z)\).

Computing the Poisson integral as in [4, pp. 460–461], we obtain

\[
(f_{1})_{\alpha}(z) = \frac{c}{(1 - z)(z^2 + z/2 + 1)}, \quad (f_{1})_{\beta}(z) = \frac{-cz^2}{(1 - z)(z^2 + z/2 + 1)}. \tag{2.3}
\]

where \(c\) is a positive constant.
Again, when \( x \in (-1, 1) \), we have that \( f_1(x) \) is real, and by (2.3),

\[
\frac{df_1}{dx} = \frac{c}{x^2 + x/2 + 1}.
\] (2.4)

Now \( f \) ranges from 1/2 to 1 as \( x \) goes from \(-1\) to 1, and \( f_1 \) goes from 1/2 to \( u_1 < 1 \) as \( x \) goes from \(-1\) to 1. Furthermore, by (2.2) and (2.4), \( df_1/dx = 0 < df_1/dx \) when \( x = -1 \). We conclude that there is a point \( z \in (-1, 1) \) such that \( f(z) = f_1(z) \).

It is now easy to see that with

\[
T(z) = \frac{z + x}{1 + xz},
\]

\( f \circ T \) and \( f_1 \circ T \) both satisfy (a) and (b) of Theorem A, with \( D \) as above and \( a(z) = -T(z) \). By Theorem 6 of [4], we have that \( f \) and \( f_1 \) are homeomorphisms onto \( D \) and the triangle of Fig. 1, respectively. Thus \( f \) satisfies the conditions of Theorem A trivially. Regarding \( f_1 \), its boundary values are vertices on the boundary of \( D \), so aside from the three points of discontinuity on \( \partial U \), which make up the set \( E \) of Theorem A, the values of \( f_1(e^{it}) \) are in \( \partial D \). Since the cluster set of \( f_1 \) at these points of discontinuity comprises the line segments joining the right- and left-hand limits, and the segments are in \( D \), it follows that \( f_1 \) also satisfies the conditions of Theorem A for the image domain \( D \).

References


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