THE NORM OF POWERS OF THE INDEFINITE INTEGRAL OPERATOR ON (0, 1)

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Abstract

In this paper we find the norm of powers of the indefinite integral operator \( V \), acting on \( L^2(0, 1) \). This answers a question raised by Halmos, and supplements some recent results of Manakov in [9]. Using results of Stepanov in [13], we show that the operator norm of \( V^n \) is asymptotically equal to the Hilbert–Schmidt norm as \( n \to \infty \).

1. Introduction

For \( f \in L^2(0, 1) \) and \( n \in \mathbb{N} \), let
\[
(Vf)(x) = \int_0^x f(t) \, dt \quad \text{and} \quad V^{n+1}f = V(V^nf),
\]
so that, on interchanging the order of integration,
\[
(V^nf)(x) = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1}f(t) \, dt. \tag{1.1}
\]

It is clear that \( V \), and hence \( V^n \), is a bounded linear operator from \( L^2(0, 1) \) to \( L^2(0, 1) \). In a lecture given several years ago to the St Andrews Mathematical Colloquium, Professor Halmos raised the question of finding the operator norm of \( V^n \). He mentioned that \( \|V\| = 2/\pi \) (see page 147 of [3]), and that A. Brown had told him that \( \|V^2\| = t^{-2} \), where \( t \) is the smallest positive root of the equation
\[
\cos x \cosh x + 1 = 0.
\]

Soon afterwards, D. Borwein and I found \( \|V^2\| \) and \( \|V^3\| \) as special cases of a very general inequality of Boyd (see page 382 of [1]). His proof involved the calculus of variations and applied to compact operators between possibly different \( L^p \) spaces. In the Hilbert space case there is an easier approach which gives \( \|V^n\| \), and this is contained in Section 2.

If we replace \( n \in \mathbb{N} \) by \( z \) in (1.1), where \( z \) is any positive real number, then we obtain the Riemann–Liouville fractional integral operator, which we still denote by \( V^z \). It is well known (see page 664 of [4]) that \( V^z \) maps \( L^2(0, 1) \) to itself for \( p \geq 1 \). Although this operator has been extensively studied (see, for example, [4, 9, 12, 13]), no expression for \( \|V^z\| \), even in the simplest case as an operator from \( L^2(0, 1) \) to itself, seems to have been given. Using some estimates of Stepanov in [12] and [13], we show that, asymptotically, as \( z \to \infty \), \( \|V^z\| \) is equal to its Hilbert–Schmidt norm.

In a paper in preparation, I hope to look at the more difficult case \( 0 < z < 1 \), showing in particular how to calculate the \( L^2 \)-norm of \( V^{1/2} \).

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2. The main results

It follows from (1.1) that $V^\circ: L^2((0, 1) \to L^2((0, 1)$ is a Hilbert–Schmidt operator and hence is compact. Also, its adjoint $(V^\circ)^*: L^2((0, 1) \to L^2((0, 1)$ is given by $(V^\circ)^*(f) = g$ for $f \in L^2((0, 1)$, where

$$g(x) = \frac{1}{\Gamma(n)} \int_x^1 (t-x)^{n-1} f(t) \, dt.$$ 

Moreover, it is easy to check that $(V^\circ)^* = (V^\circ)^*$, and that the operator $U \overset{\text{def}}{=} V^\circ(V^\circ)^*$ is a positive, compact, self-adjoint operator with domain equal to $L^2((0, 1)$, and $\|U\| = \|V^\circ\|\|(V^\circ)^*\| = \|V^\circ\|^2$. It follows from page 232 of [11] that $\|U\|$ is equal to the largest eigenvalue of $U$. In order to find this largest eigenvalue, we show that $U$ is the inverse of a self-adjoint linear differential operator $M$ with constant coefficients, and so we want to find the reciprocal of the smallest positive eigenvalue of $M$

**Theorem 1.** For each $n \in \mathbb{N}$, let $M: L^2((0, 1) \to L^2((0, 1)$ be the differential operator $M(f) = (-1)^n f^{(2n)}$ with domain

$$\text{dom}(M) = \{ f \in L^2((0, 1) : f, f', \ldots, f^{(2n-1)} \in AC_{\text{loc}}((0, 1), f^{(2n)} \in L^2((0, 1))\}$$

satisfying the boundary conditions

$$f(0) = f'(0) = \ldots = f^{(n-1)}(0) = f^{(n)}(1) = f^{(n+1)}(1) = \ldots = f^{(2n-1)}(1) = 0.$$ 

Then

(i) $M$ is an injective self-adjoint operator mapping onto $L^2((0, 1)$;
(ii) for each $f \in L^2((0, 1)$, $MUF = f$;
(iii) for each $g \in \text{dom}(M)$, $UMg = g$.

**Proof.** (i) If $g \in \text{ker}(M)$, then $g^{(2n)} = 0$, so $g$ is a polynomial of degree $2n - 1$ whose coefficients have to vanish if $g$ satisfies the boundary conditions above. Hence $M$ is injective.

The fact that $M$ is a self-adjoint operator is a special case of Theorem 5 on page 77 of [10], with the $2n \times 2n$ matrices $A$ and $B$ being block diagonal matrices, $A$ having the $n \times n$ unit matrix as its top left-hand block and zeros elsewhere, while $B$ has the $n \times n$ unit matrix as its bottom right-hand block with zeros elsewhere.

To show that $M$ maps onto $L^2((0, 1)$, we want to show that given $f \in L^2((0, 1)$, we can find $g \in \text{dom}(M)$ such that $(-1)^ng^{(2n)} = f$. If such a $g$ exists, then we can integrate both sides, to obtain

$$(-1)^n \int_x^1 g^{(2n)}(u) \, du = \int_x^1 f(u_1) \, du_1$$

and so

$$(-1)^{n+1} g^{(2n-1)}(x) = \int_x^1 f(u_1) \, du_1.$$ 

Repeating this a further $n - 1$ times gives, using the boundary conditions at 1,

$$(-1)^{2n} g^{(n)}(x) = \int_x^1 du_n \int_{u_n}^1 du_{n-1} \ldots \int_{u_2}^1 f(u_1) \, du_1,$$

$$= \frac{1}{\Gamma(n)} \int_x^1 (u_1 - x)^{n-1} f(u_1) \, du_1.$$
Integrating this a further \( n \) times, but now between 0 and \( x \) and using the boundary conditions at 0, gives

\[
g(x) = \int_0^x du_2 \int_0^{u_2} du_2 \ldots \int_0^{u_{n+1}} (\int_0^{u_{n+1}} (u_1 - u_{n+1})^{n-1} f(u_1) du_1)
\]

\[
= \frac{1}{(\Gamma(n))^2} \int_0^x (x - u_n)^{n-1} \left( \int_0^1 (t - u)^{n-1} f(t) dt \right) du_n
\]

Thus, given \( f \in L^2(0,1) \), if we define \( g \) by

\[
g(x) = \frac{1}{(\Gamma(n))^2} \int_0^x (x - u)^{n-1} \left( \int_0^1 (t - u)^{n-1} f(t) dt \right) du,
\]

then it is not difficult to show that we can reverse the above argument to prove that \( g \in \text{dom}(M) \) and \( Mg = (-1)^n g^{(2n)} = f \). Hence \( M \) is onto.

(ii) Given \( f \in L^2(0,1) \), if we let \( g = Uf \), then (2.1) holds and so the argument at the end of (i) shows that \( MUf = Mg = f \).

(iii) Given \( g \in \text{dom}(M) \), if we let \( f = Mg \) and apply (ii) to this \( f \), then we obtain \( MU(Mg) = Mg \), and since \( M \) is injective, this gives \( UMG = g \).

**Corollary 1.** If \( f \in L^2(0,1) \) and \( g = Uf \), then \( g(x) = \int_0^1 a(x,t)f(t)dt \), where

\[
a(x,t) = \frac{1}{(\Gamma(n))^2} \int_0^{\min(x,t)} (x - u)^{n-1}(t - u)^{n-1} du
\]

\[
= \begin{cases} 
\sum_{k=0}^{n-1} \left( \frac{(-1)^{n-1-k}}{k!(2n-k-1)!} \right) x^k t^{2n-1-k} & \text{if } 0 \leq t \leq x \leq 1, \\
\sum_{k=0}^{n-1} \left( \frac{(-1)^{n-1-k}}{k!(2n-k-1)!} \right) x^{2n-1-k} t^k & \text{if } 0 \leq x \leq t \leq 1.
\end{cases}
\]

**Proof.** The representation of \( a(x,t) \) as an integral follows from (2.1), since we can interchange the order of integration by an application of Schwarz’s inequality and Fubini’s theorem. In order to evaluate the integral, we can assume that \( t < x \), and then

\[
\int_0^t (x - u)^{n-1}(t - u)^{n-1} du = \sum_{k=0}^{n-1} \left( \frac{n-1}{k} \right) (-1)^k x^{n-1-k} \int_0^t u^k(t - u)^{n-1} du
\]

\[
= \sum_{k=0}^{n-1} \left( \frac{n-1}{k} \right) (-1)^k x^{n-1-k} t^{n+k} \beta(k + 1, n)
\]

\[
= \sum_{k=0}^{n-1} \frac{((n-1)!)^2}{(n-1-k)! (n+k)!} (-1)^k x^{n-1-k} t^{n+k},
\]

which reduces to the expression above on putting \( k' = n - 1 - k \).

**Corollary 2.** Let \( \lambda = \lambda_0(n) \) be the smallest positive eigenvalue of the differential equation

\[
(-1)^n f^{(2n)} = \lambda f
\]
subject to the boundary conditions

\[ f(0) = f'(0) = \ldots = f^{(n-1)}(0) = f^{(n)}(1) = f^{(n+1)}(1) = \ldots = f^{(2n-1)}(1) = 0. \]

Then

\[ \|V^n\| = \frac{1}{\sqrt{2\lambda_0(n)}}. \]

**Proof.** This follows from Theorem 1(ii) and the remarks made before Theorem 1.

**Remark 1.** The differential equation (2.2) (after a linear change of variable \( t' = 1 - t \)) was studied by Kupcov on page 119 of [6], in the context of finding an asymptotic expression for the best possible constant \( M_{n,k} \) as \( n \to \infty \) in Kolmogorov’s inequality

\[ \|f^{(k)}\| \leq M_{n,k} \|f\|^{(n-k)/n} \|f^{(n)}\|^{k/n}, \]

where \( 0 \leq k \leq n \) and \( f^{(k)} \in L^2(0,\infty) \). This work was taken up in [2], and on page 80 of [2] a list of the first 25 values of \( \lambda_0(n) \) is given, the cases \( n = 1, 2 \) and 3 confirming what D. Borwein and I had done by hand. Using this list, a table of norms of the first few powers of \( V \) is given at the end of this paper.

**Remark 2.** If we take \( a = 0, b = 1, U(z) \equiv 1, V(z) \equiv 1, p = q = 2 \) and \( k = n \) in Remark 2 of [9], then our inequality for \( \|V^n\| \) is a special case of the one considered in [9]. However, this gives the weight function \( v(x) = (1 + x)^n \), so that the hypothesis

\[ \int_0^{\infty} |v(x)|^{-2} \, dx = \infty \]

that is required for Theorem 3 of [9] is not satisfied, so our case is not covered in [9].

**Remark 3.** Although not given directly in [13], we can use Stepanov’s results on the Weyl fractional integral operator to give a lower bound for \( \|V^z\| \) for any \( z \geq 1 \). To see this, we first observe that \( f(t) \in L^2(0,1) \) if and only if \( g(x) = f(1/x)/x \in L^2(1,\infty) \), and so the inequality

\[ \left[ \int_0^1 \left( \frac{1}{\Gamma(z)} \int_0^x (x - t)^{z-1} f(t) \, dt \right)^2 \, dx \right]^{1/2} \leq M \left[ \int_0^1 (f(t))^2 \, dt \right]^{1/2} \]

is equivalent to

\[ \left[ \int_1^\infty \left( \frac{\Gamma(z)}{x} \int_x^\infty (t - x)^{z-1} \frac{f(1/t)}{t^{z+1}} \, dt \right)^2 \, dx \right]^{1/2} \leq M \left[ \int_1^\infty \left( \frac{f(1/t)}{t} \right)^2 \, dt \right]^{1/2}. \]

By writing \( h(t) = f(1/t)/t^{z+1} \), we see that the above are equivalent to

\[ \left[ \int_1^\infty \left( \frac{\Gamma(z)}{\Gamma(x)} \int_x^\infty (t - x)^{z-1} h(t) \, dt \right)^2 \, dx \right]^{1/2} \leq M \left[ \int_1^\infty (h(t)t^z)^2 \, dt \right]^{1/2}. \]

Thus, if \( z \geq 1 \), the least constant in the inequality for the norm of the Riemann–Liouville integral operator \( V^z : L^2(0,1) \to L^2(0,1) \) is the same as the least constant for the norm of the Weyl integral operator between the weighted spaces
$W^2 : L^2((1, \infty), x^{2s}dx) \rightarrow L^2((1, \infty), x^{-2s}dx)$, where the Weyl integral operator is given by

$$(W^2 h)(x) = \frac{1}{\Gamma(z)} \int_x^\infty (t-x)^{z-1} h(t) \, dt.$$ 

Stepanov shows in [12] and [13] that in this case there is a constant, $C(z)$ say, such that

$$\frac{1}{\Gamma(z)} \max\{B_{z,0}, B_{z,1}\} \leq \|V^z\| \leq C(z) \frac{1}{\Gamma(z)} \max\{B_{z,0}, B_{z,1}\},$$

where

$$B_{z,0} = \sup_{1 < t < \infty} \left( \int_1^t (t-x)^{2z-2}x^{-2} \, dx \right)^{1/2} \left( \int_t^\infty x^{-2} \, dx \right)^{1/2}$$

and

$$B_{z,1} = \sup_{1 < t < \infty} \left( \int_1^t x^{-2} \, dx \right)^{1/2} \left( \int_1^t (x-t)^{2z-2}x^{-2} \, dx \right)^{1/2}.$$  \hspace{1cm} (2.3)

By noticing that

$$(t-x)^{2z-2}x^{-2} = \frac{1}{x^2} (t/x - 1)^{2z-2} = \frac{d}{dx} \left( \frac{-(t/x - 1)^{2z-1}}{(2z-1)t} \right),$$

(2.3) gives

$$B_{z,0} = \frac{1}{2z-1} \sup_{1 < t < \infty} (t-1)^{2z-1/2}t^{-2z},$$

and similarly (2.4) gives

$$B_{z,1} = \frac{1}{2z-1} \sup_{1 < t < \infty} (1-t)^{1-2z}t^{1/2}.$$

Elementary calculations show that the maximum in (2.3) occurs at $t = 2x$ and gives $B_{z,0} = (2x-1)^{(2z-3)/2}(2x)^{-z}$. The maximum in (2.4) occurs at $t = (2x)^{1/(2z-1)}$ and gives $B_{z,1} = (2x-1)^{-1/2}(2x)^{-z/(2z-1)}$. Using these, we see that $B_{z,0} < B_{z,1}$ if and only if $(2x - 1)^{1-1} < (2x)^{3(1-1)/(2z-1)}$, and this holds since $(2x - 1)^{2z} < (2x)^{2z}$. Thus Stepanov’s estimate for $\|V^z\|$ becomes

$$\frac{1}{\Gamma(z) \sqrt{2x - 1} \Gamma(2z-1)^{1/(2z-1)}} \leq \|V^z\| \leq \frac{C(z)}{\Gamma(z) \sqrt{2x - 1} \Gamma(2z)^{1/(2z-1)}}.$$  \hspace{1cm} (2.4)

The Hilbert–Schmidt norm of $V^z$, and hence an upper bound for $\|V^z\|$, is

$$\frac{1}{\Gamma(z)} \left[ \int_0^1 \int_0^x (x-t)^{2z-2} \, dt \, dx \right]^{1/2} = \frac{1}{\Gamma(z)} \sqrt{\frac{1}{2x(2x-1)}},$$

so putting this together with the lower bound for $\|V^z\|$ from above, we obtain, as $z \to \infty$,

$$\|V^z\| \sim \frac{1}{\sqrt{2x(2x-1)}},$$

since $(2x)^{1/2z} \to 1$ as $z \to \infty$.

The table below compares the operator norm obtained from Corollary 2 and [2] with the Hilbert–Schmidt norm of $V^z$ for a few values of $n$. 

<table>
<thead>
<tr>
<th>n</th>
<th>$|V^z|$ from Corollary 2</th>
<th>$|V^z|$ from Hilbert–Schmidt</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(</td>
<td>\mathcal{L}^{1/2}(1, \infty)|</td>
</tr>
</tbody>
</table>
Table 1. The operator norm of $V^n: L^2(0,1) \rightarrow L^2(0,1)$ compared to the Hilbert–Schmidt norm

<table>
<thead>
<tr>
<th>$V^n$</th>
<th>Operator norm</th>
<th>Hilbert–Schmidt norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$</td>
<td>0.6366</td>
<td>0.7071</td>
</tr>
<tr>
<td>$V^2$</td>
<td>0.2844</td>
<td>0.2887</td>
</tr>
<tr>
<td>$V^3$</td>
<td>0.9081e - 01</td>
<td>0.9129e - 01</td>
</tr>
<tr>
<td>$V^5$</td>
<td>0.4385e - 02</td>
<td>0.4385e - 02</td>
</tr>
<tr>
<td>$V^{10}$</td>
<td>0.1413e - 06</td>
<td>0.1414e - 06</td>
</tr>
<tr>
<td>$V^{15}$</td>
<td>0.3888e - 12</td>
<td>0.3889e - 12</td>
</tr>
<tr>
<td>$V^{20}$</td>
<td>0.2081e - 18</td>
<td>0.2081e - 18</td>
</tr>
</tbody>
</table>

Acknowledgement. I should like to thank the referee for his helpful comments.

Note added in revision. (1) After having submitted this paper, I noticed [7], in which $\|V^n\|$ is calculated for $n = 1, 2, \ldots, 10$, and the conjecture is made that the operator norm of $V^n$ is asymptotic to its Hilbert–Schmidt norm. Subsequently, two papers [5, 8] have been accepted for publication in which this conjecture has been proved and so anticipate the result in Remark 3. These proofs use methods different to that described here, [8] being particularly succinct.

(2) The referee has pointed out that Stepanov’s result is valid for an arbitrary interval (see page 447 of [9]) and so could be applied directly to $V^n: L^2(0,1) \rightarrow L^2(0,1)$ instead of introducing $W^n$.

References


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