COFINITENESS OF LOCAL COHOMOLOGY MODULES FOR PRINCIPAL IDEALS

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Abstract

In this note we show that if an ideal $I$ of a noetherian ring is principal, up to radical, then the local cohomology modules with support in $V(I)$ are $I$-cofinite.

1. Introduction

Throughout this note, we assume that all rings are commutative and noetherian with identity.

Definition 1. Let $R$ be a ring, $I$ an ideal of $R$, and $N$ an $R$-module. We say that $N$ is $I$-cofinite if the support of $N$ is contained in $V(I)$ and $\text{Ext}^i_R(R/I, N)$ is finitely generated for all $i \geq 0$ (compare [5, §2]).

There are many questions about local cohomology modules (compare [7]). In particular, Grothendieck (compare [4, Exposé XIII, 1.1]) proposed the following conjecture.

Conjecture 1. Let $M$ be a module of finite type over a ring $R$, and let $I$ be an ideal of $R$. Then the module $\text{Hom}_R(R/I, H^j_I(M))$ is of finite type for all $j \geq 0$.

Hartshorne later refined this conjecture (compare [5, §2]), and proposed the following.

Conjecture 2. Let $M$ be an $R$-module of finite type, and let $I$ be an ideal of $R$. Then $\text{Ext}_R^i(R/I, H^j_I(M))$ is of finite type for all $i \geq 0$ and $j \geq 0$.

Using the derived category, Hartshorne showed that if $M$ is a finitely generated $R$-module, where $R$ is a complete regular local ring, then $H^j_I(M)$ is $I$-cofinite in two cases:

(i) $I$ is a non-zero principal ideal [5, Corollary 6.3];
(ii) $I$ is a prime ideal with dimension 1 [5, Corollary 7.7].

Huneke, Koh, Delfino and Yoshida refined result (ii) to more general situations (compare [8, Theorem 4.1], [2, Theorem 3] and [14, Theorem (1.1)]).

Concerning result (i), Yassemi recently proved that the local cohomology module $H^j_I(M)$ is $I$-cofinite for a principal ideal $I$, by using generalized section functors if $M$ has finite projective dimension or $R$ is Gorenstein [13, Corollary 4.10].

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proved this independently for the local case [14, Proposition (4.1)]. We shall give a simple proof of this result without any restriction.

Hartshorne’s counterexample [5, §3] says that the above conjectures are not true for an ideal generated by 2 elements even if the ring \( R \) is regular.

2. The main theorem

In what follows, \( R \) is a ring, \( M \) is a finitely generated \( R \)-module, \( I \) is a proper ideal of \( R \), and \( X \) is the affine scheme \( \text{Spec} \, R \). We use \( D_I(-) \) to denote the functor \( \lim \to \text{Hom}_R(I^n, -) \). We note the Deligne formula:

\[
D_I(M) \simeq \Gamma(U, \mathcal{F}),
\]

where \( \mathcal{F} = M^\sim \) is the associated coherent \( \mathcal{O}_X \)-module of \( M \), and \( U = X \setminus V(I) \).

**Lemma 1.** Let \( p \) be a non-negative integer, and let \( N \) be an \( R \)-module. Then \( \text{Ext}^p_R(N, H^j_I(M)) \) is a finitely generated \( R \)-module if and only if \( \text{Ext}^p_R(N, DI(M)) \) is a finitely generated \( R \)-module.

**Proof.** By the standard exact sequence of [5, Proposition 2.2]

\[
0 \longrightarrow \Gamma_I(M) \longrightarrow M \longrightarrow \Gamma(U, M^\sim) \longrightarrow H^1_I(M) \longrightarrow 0,
\]

and the above formula, we have the short exact sequence

\[
0 \longrightarrow L \longrightarrow D_I(M) \longrightarrow H^1_I(M) \longrightarrow 0,
\]

where \( L \simeq M/\Gamma_I(M) \). Since \( L \) is finitely generated, the assertion follows from the long exact sequence obtained by application of \( \text{Ext}^i(N, -) \) to this sequence.

**Lemma 2.** Assume that \( H^j_I(M) = 0 \) for all \( j \neq 0, 1 \). Then \( \text{Ext}^i_R(N, H^j_I(M)) \) is finitely generated for each finitely generated \( R \)-module \( N \) with \( \text{Supp}(N) \subseteq V(I) \) and all \( i, j \geq 0 \). In particular, \( H^j_I(M) \) is \( I \)-cofinite for all \( j \geq 0 \).

**Proof.** We have only to prove the lemma for \( N = R/I \), by [8, Lemma 4.2]. Observe that, for an injective \( R \)-module \( E \), the standard exact sequence

\[
0 \longrightarrow H^0_I(E) \longrightarrow E \longrightarrow D_I(E) \longrightarrow H^1_I(E) \longrightarrow 0
\]

reduces to a split exact sequence

\[
0 \longrightarrow H^0_I(E) \longrightarrow E \longrightarrow D_I(E) \longrightarrow 0,
\]

so that \( D_I(E) \simeq E/H^0_I(E) \) is injective and \( I \)-torsion-free. In particular,

\[
\text{Hom}_R(R/I, D_I(E)) = 0.
\]

Now let \( E^* \) be an injective resolution of \( M \). Since the functor \( D_I \) is left exact, and \( \mathcal{R}/D_I(M) \simeq H^{j+1}_I(M) = 0 \) for all \( j > 0 \), it follows that \( D_I(E^*) \) is an injective resolution of \( D_I(M) \). Since \( \text{Hom}_R(R/I, D_I(E^*)) \) is a zero complex, it follows that \( \text{Ext}^i_R(R/I, D_I(M)) = 0 \) for all \( i \geq 0 \). The proof is completed by Lemma 1.
Theorem 1. Let \( f \) be an element of \( R \), and let \( I \) be an ideal generated by \( f \) up to radical. Then \( \text{Ext}^q_R(N,H^j_I(M)) \) is finitely generated for each finitely generated \( R \)-module \( N \) with support in \( V(I) \) and for all \( i,j \geq 0 \); that is, \( H^j_I(M) \) is \( 1 \)-cofinite for all \( j \geq 0 \).

Proof. Since \( \sqrt{I} \) and \( H^j_I(M) \) are equal to \( \sqrt{(Rf)} \) and \( H^j_{\sqrt{I}}(M) \), respectively, we have only to prove the theorem in the case \( I = Rf \) and \( N = R/I \). Then \( H^j_I(M) = 0 \) for all \( j > 1 \), by the Arithmetic Rank Vanishing Theorem, and so it is enough to show that \( \text{Ext}^q_R(N,D_i(M)) \) is finitely generated for all \( i \geq 0 \), by Lemma 1. However, since \( I = Rf \), we have \( D_i(M) \simeq M_f \). There exists \( t > 0 \) such that \( f^tN = 0 \). Therefore, since multiplication by \( f^t \) on \( \text{Ext}^q_R(N,M_f) \) is both zero and an automorphism, the latter module must be zero.

Corollary 1. If the dimension of \( M \) is no more than one, then \( \text{Ext}^q_R(N,H^j_I(M)) \) is finitely generated for each finitely generated \( R \)-module \( N \) with support in \( V(I) \) and for all \( i,j \geq 0 \). In particular, \( H^j_I(M) \) is \( 1 \)-cofinite for all \( j \geq 0 \).

Proof. We may assume \( N = R/I \). Since \( \dim M \leq 1 \), \( H^j_I(M) \) vanishes for \( q \neq 0,1 \), by [12, 6.1 Theorem]. The corollary follows from Lemma 2.

Corollary 2. Let \( Y \) be a closed subscheme of \( X \), defined by an ideal \( I \) of \( R \). If \( X \setminus Y \) is affine, then \( \text{Ext}^q_R(N,H^j_I(M)) \) is finitely generated for each finitely generated \( R \)-module \( N \) with support in \( Y \) and for all \( i,j \geq 0 \). In particular, \( H^j_I(M) \) is \( 1 \)-cofinite for all \( j \geq 0 \).

Proof. We may assume \( N = R/I \). Let \( \mathcal{F} = M^\sim \) be the associated coherent \( \mathcal{O}_X \)-module. We note that there is an isomorphism \( H^{q+1}_Y(\mathcal{F}) \simeq H^q(X \setminus Y,\mathcal{F}) \) for all \( q \geq 1 \), by [3, Proposition 2.2]. Since \( X \setminus Y \) is affine, we have \( H^q_Y(\mathcal{F}) = 0 \) for \( q \neq 0,1 \). The assertion follows from Lemma 2.

Definition 2. Let \( L \) be an \( R \)-module and \( J \) an ideal of \( R \). We say that \( L \) is \( weakly \) \( J \)-cofinite if \( \text{Ext}^q_R(R/J,L) \) is finitely generated for all \( i \geq 0 \).

Proposition 1. Let \( I \subset J \) be ideals of a ring \( R \). If \( J \) is an ideal generated by an element \( f \) up to radical, then \( \text{Ext}^q_R(N,H^j_I(M)) \) is finitely generated for all finitely generated \( R \)-modules \( N \) with support in \( V(J) \) and for all \( i,j \geq 0 \).

Proof. We may assume that \( J \) and \( N \) are equal to \( Rf \) and \( R/J \), respectively, by [8, Lemma 4.2].

In this case, the standard long exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^{j+1}_I(Rf) & \longrightarrow & H^j_I(R) & \longrightarrow & H^j_I(Rf) \\
& & \longrightarrow & H^j_I(R) & \longrightarrow & H^j_I(M_f) \\
& & \longrightarrow & \cdots \\
& & \longrightarrow & H^j_I(R) & \longrightarrow & H^j_I(M_f) \\
& & \longrightarrow & \cdots \\
& & \longrightarrow & H^j_I(R) & \longrightarrow & H^j_I(Rf) \\
& & \longrightarrow & \cdots \\
\end{array}
\]

simplifies considerably, because \( I + Rf = Rf \) is principal, so that \( H^j_{Rf}(M_f) = 0 \) for all \( j > 1 \), by the Arithmetic Rank Vanishing Theorem.
Now, for all \( j \geq 0 \), the module \( H_j^1(M_1) \) is weakly \( Rf \)-cofinite, since for all \( i \geq 0 \), multiplication by \( f \) on \( \text{Ext}_R^i(R/Rf, H_j^1(M_1)) \) is both zero and an automorphism. Therefore, for \( j \geq 2 \), we have \( H_j^1(M) \cong H_j^1(M_1) \) is weakly \( Rf \)-cofinite. Also, \( H_j^0(M_1), H_j^1(M_1) \) and \( H_j^{p-Rf}(M) \) (which is zero), and the finitely generated \( R \)-modules \( H_j^0(M) \) and \( H_j^{p-Rf}(M) \), are weakly \( Rf \)-cofinite; furthermore, \( H_j^{p-Rf}(M) \cong H_j^2(M) \) is \( Rf \)-cofinite, by Theorem 1. It is therefore easy to split the first two (and a bit) rows of the above long exact sequence into short exact sequences to deduce that \( H_j^1(M) \) is weakly \( Rf \)-cofinite.

3. An example

Let \( R \) be a ring, \( M \) a finitely generated \( R \)-module, and \( X \) the affine scheme \( \text{Spec} \ R \).

**Lemma 3.** Let \( Y_1 \) and \( Y_2 \) be closed subschemes of \( X \) with defining ideals \( I_1 \) and \( I_2 \), respectively, and set \( I = I_1 + I_2 \). If \( X \setminus Y_i \) is affine for \( i = 1, 2 \), then \( \text{Hom}_R(R/I, H_j^1(M)) \) and \( \text{Ext}_R^i(R/I, H_j^1(M)) \) are finitely generated. Further, for an integer \( p \geq 0 \), \( \text{Ext}_R^{p+j}(R/I, H_j^1(M)) \) is finitely generated if and only if \( \text{Ext}_R^p(R/I, H_j^2(M)) \) is finitely generated.

**Proof.** First we note that Lemma 1 is the same as the following, by the Deligne formula: for an integer \( p \geq 0 \), a proper ideal \( J \) of \( R \) and an \( R \)-module \( N \), \( \text{Ext}_R^p(N, H_j^1(M)) \) is finitely generated if and only if \( \text{Ext}_R^p(N, I(U, M^-)) \) is finitely generated, where \( U = X \setminus \text{V}(J) \).

Put \( U_i = X \setminus Y_i \) for \( i = 1, 2 \), and consider the Mayer–Vietoris exact sequence

\[
0 \longrightarrow \Gamma(U_1 \cup U_2, M^-) \longrightarrow \Gamma(U_1, M^-) \oplus \Gamma(U_2, M^-) \longrightarrow \Gamma(U_1 \cap U_2, M^-) \longrightarrow H^1(U_1 \cup U_2, M^-) \longrightarrow H^1(U_1, M^-) \oplus H^1(U_2, M^-) \longrightarrow \cdots.
\]

We then have an exact sequence

\[
0 \longrightarrow \Gamma(U_1 \cup U_2, M^-) \longrightarrow \Gamma(U_1, M^-) \oplus \Gamma(U_2, M^-) \\
\longrightarrow \Gamma(U_1 \cap U_2, M^-) \longrightarrow H^1(U_1 \cup U_2, M^-) \longrightarrow 0,
\]

since \( U_1 \) and \( U_2 \) are affine. By this sequence, we have two short exact sequences

\[
0 \longrightarrow \Gamma(U_1 \cup U_2, M^-) \longrightarrow \Gamma(U_1, M^-) \oplus \Gamma(U_2, M^-) \longrightarrow C \longrightarrow 0, \quad (x)
\]

\[
0 \longrightarrow C \longrightarrow \Gamma(U_1 \cap U_2, M^-) \longrightarrow H_j^2(M) \longrightarrow 0, \quad (\beta)
\]

where \( C \) is the image of \( \Gamma(U_1, M^-) \oplus \Gamma(U_2, M^-) \to \Gamma(U_1 \cap U_2, M^-) \). Since an affine scheme is separated, \( U_1 \cap U_2 \) is again affine. Then \( \text{Ext}_R^i(R/I, \Gamma(U_1 \cap U_2, M^-)) \) is finitely generated for all \( q \geq 0 \), by Corollary 2 and Lemma 1, since \( \text{Supp}(R/I) \subseteq V(I_1I_2) \).

For an integer \( p \geq 0 \), it follows from (\( \beta \)) that \( \text{Hom}_R(R/I, C) \) is finitely generated, and \( \text{Ext}_R^{p+1}(R/I, C) \) is finitely generated if and only if \( \text{Ext}_R^p(R/I, H_j^2(M)) \) is finitely generated. Since \( U_j \) is affine, \( H_j^2(M) \) is \( I_j \)-cofinite for \( i = 1, 2 \), by Corollary 2 again.

Hence \( \text{Ext}_R^p(R/I, \Gamma(U_1, M^-)) \) is finitely generated, by Lemma 1, for \( i = 1, 2 \) and all \( q \geq 0 \). It follows that \( \text{Ext}_R^p(R/I, \Gamma(U_i, M^-)) \) is finitely generated for \( i = 1, 2 \) and all \( q \geq 0 \), since \( I \) contains \( I_j \) [8, Lemma 4.2]. For an integer \( p \geq 0 \), we have that \( \text{Hom}_R(R/I, \Gamma(U_1 \cup U_2, M^-)) \) is finitely generated, and \( \text{Ext}_R^{p+1}(R/I, \Gamma(U_1 \cup U_2, M^-)) \) is finitely generated if and only if \( \text{Ext}_R^p(R/I, C) \) is finitely generated, by (\( x \)). Therefore \( \text{Hom}_R(R/I, \Gamma(U_1 \cup U_2, M^-)) \) and \( \text{Ext}_R^p(R/I, \Gamma(U_1 \cup U_2, M^-)) \) are finitely generated,
Theorem 1. Hence the Bass numbers of Ext₁κ numbers of following.

Example. Let A be a regular ring k[x,y][[u,v]] over a field k which is complete with the J = (u,v)-adic topology. Let M be A/(xu + yv). Hartshorne shows that

\[ \text{Hom}_A(A/J, H^2_1(M)) \]

is not finitely generated as A-module [5, Example 1]. That is, the second local cohomology module is not J-cofinite in the sense of Grothendieck (compare [4, Exposé XIII, 1.1]), even if the base ring is regular. One can see that the first local cohomology module \( H^1_1(M) \) is J-cofinite in the sense of Grothendieck, by Lemma 1 together with the fact that \( \text{Hom}_R(R/J, D_1(M)) = 0 \). Unfortunately, this is not J-cofinite in the sense of Hartshorne, as in Definition 1. Indeed, since the standard open subsets \( D(u) \) and \( D(v) \) of SpecA are affine, \( \text{Hom}_A(A/J, H^1_1(M)) \) and \( \text{Ext}^1_A(A/J, H^1_1(M)) \) are finitely generated, and \( \text{Ext}^2_A(A/J, H^1_1(M)) \) is finitely generated if and only if \( \text{Ext}^2_A(A/J, H^2_1(M)) \) is finitely generated by Lemma 3. It follows from (γ) that \( \text{Ext}^2_A(A/J, H^2_1(M)) \) is not finitely generated.

4. Bass numbers

There are some results on the finiteness of the Bass numbers of local cohomology modules (compare [9], [10] and [11]). Let \( R \) be a ring. For a prime ideal \( P \) of \( R \) and an \( R \)-module \( T \), the \( i \)-th Bass number \( \mu_i(P, T) \) is defined to be \( \dim_{R/P} \text{Ext}^i_{R/P}( \kappa(P), T_P) \), where \( \kappa(P) = R_P/P \cdot R_P \) (compare [1, 2.7 Lemma]). If \( T \) is \( I \)-cofinite, then the Bass numbers of \( T \) are finite [10, Remark 2]. The results in Sections 2 and 3 yield the following.

Corollary 3. Let \( M \) be a finitely generated \( R \)-module, and let \( I \) be an ideal of \( R \). Suppose that \( I \) is a principal ideal of \( R \), or a subideal of a principal ideal generated by a non-unit element \( f \) of \( R \). If the prime ideal \( P \) contains \( f \), then the Bass numbers \( \mu_i(P, H^1_1(M)) \) of the local cohomology module \( H^1_1(M) \) are finite for all \( i, j \geq 0 \).

Proof. If \( I \) is a principal ideal, then \( H^1_1(M) \) is \( I \)-cofinite for all \( j \geq 0 \), by Theorem 1. Hence the Bass numbers of \( H^1_1(M) \) are finite for all \( j \geq 0 \).

Let \( f \) be an element of \( R \) such that the ideal \( Rf \) contains \( I \). In this case, \( \text{Ext}^i_A(R/Rf, H^1_1(M)) \) is finitely generated for all \( i, j \geq 0 \), by Proposition 1. Let \( P \) be a prime ideal of \( R \) which contains \( f \). Then \( \text{Ext}^i_A(R/P, H^1_1(M)) \) is finitely generated for all \( i, j \geq 0 \). So its localization \( \text{Ext}^i_A(R/P, H^1_1(M))_P \) is a finitely generated \( R_P \)-module for all \( i, j \geq 0 \). Therefore \( \mu_i(P, H^1_1(M)) \) is finite for all \( i, j \geq 0 \).

Remark. If \( P \) does not contain \( f \), then the Bass numbers \( \mu_i(P, H^1_1(M)) \) are not necessarily finite. Let \( J \) be an ideal, and consider the ideal \( I \) of the form \( Rf \cap J \). Since \( P \) does not contain \( f \), it follows from the Mayer–Vietoris exact sequence that \( H^1_1(M)_P \) is isomorphic to \( H^1_1(M)_P \) for all \( j \geq 0 \). Then \( \mu_i(P, H^1_1(M)) \) is finite if and only if \( \mu_i(P, H^1_1(M)) \) is finite for each \( i, j \geq 0 \). This implies that there is an example in which \( \mu_i(P, H^1_1(M)) \) is not finite for some \( i, j \geq 0 \). Indeed, Hartshorne shows that \( \mu^0(P, H^2_1(M)) \) is not finite [5, §3].
Corollary 4. Let \((R, \mathfrak{m})\) be local, and let \(M\) and \(I\) be as above. Then the socle \(\text{Soc}(H^j_I(M))\) is a finitely generated \(R\)-module for all \(j \geq 0\). Further, if the local cohomology module \(H^j_I(M)\) has support only at \(\mathfrak{m}\) for some \(j\), then it is Artinian.

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References