BOOK REVIEWS

REPRESENTATIONS OF INFINITE-DIMENSIONAL GROUPS
(Translations of Mathematical Monographs 152)

By R. S. ISMAGILOV: 197 pp., US$85.00, ISBN 0 8218 0418 9

CATEGORIES OF SYMMETRIES AND INFINITE-DIMENSIONAL GROUPS
(London Mathematical Society Monographs (N.S.) 16)

By Yu. A. NERETIN (translated by G. G. GOUJD): 417 pp., £65.00
(LMS Members’ price £48.75), ISBN 0 19 851186 8

Often in analysis, geometry and physics one encounters large groups such as the additive group of a Banach space, the group of all homeomorphisms or diffeomorphisms from $\mathbb{R}^n$ to itself, or the group of all smooth maps from a manifold into a Lie group (with pointwise multiplication). These have the structure of infinite-dimensional manifolds. Most such groups are not Type I groups, which means that their irreducible representations are not nicely parametrised, and that decompositions into irreducibles are of limited use.

Nonetheless, there are a number of infinite-dimensional groups for which particular classes of representations can be classified completely, and both these books by contributors to the subject discuss a number of these. Despite the similar theme and a considerable overlap in topics, the two books are very different in style. Neretin presents a vision of a systematic account into which the numerous apparently different examples may fit. The full structure and scope of that system is not yet clear, so that the account contains various loose ends. Ismagilov’s monograph is an elegant and tightly focussed account, concentrating on constructions useful for current and diffeomorphism groups (including Mickelsson–Rajeev extensions).

The flavour of some of the techniques can be illustrated by looking at the Araki construction of which Ismagilov, in particular, makes heavy use. One of the first groups for which large numbers of representations of an infinite-dimensional group were classified was the Heisenberg group of the canonical commutation relations, a central extension of the additive group of a Hilbert space, $H$. Viewed another way, one wants to find projective representations $W$ of the additive group of $H$ satisfying

$$W(u)W(v) = e^{i(u,v) - (v, u)/2} W(u + v).$$

One simple example of such a representation is realised on Fock space $\mathcal{F}$, which is the completion of the symmetric tensor algebra of $H$ with respect to a natural inner product inherited from $H$. For $v \in H$ we may define a vector $\text{Exp}(v)$ in $\mathcal{F}$ whose

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component in $\bigotimes_n^g H$ is $v^\otimes n/n!$, where $\otimes n$ denotes the tensor power, and then $\langle \text{Exp}(u), \text{Exp}(v) \rangle = \exp \langle u, v \rangle$. The linear span of these coherent state vectors $\text{Exp}(v)$ is dense in $\mathcal{H}$, so that a representation is defined by its action on them, and one has

$$W(u) \text{Exp}(v) = e^{\langle y + u/2, x \rangle} \text{Exp}(v + u).$$

More generally, for $A$ a bounded operator on $H$ and $a, x \in H$, we define $T(A, a, x)$ by

$$T(A, a, x) \text{Exp}(v) = e^{\langle x, v \rangle} \text{Exp}(Av + a),$$

which is bounded just when $A^* a + x$ is in the range of $(1-A^* A)^{1/2}$. The Araki construction works when there is a homomorphism taking each element $g$ of the group $G$ to a triple $(A(g), a(g), x(g))$. One then obtains a projective representation of $G$ by composing this homomorphism with $T$. In order that one has a homomorphism, the map $g \mapsto A(g)$ must be a representation, and $a(g)$ and $x(g^{-1})$ must be cocycles in the sense that

$$a(gh) = a(g) + A(g) a(h), \quad x((gh)^{-1}) = x(g^{-1}) + A(g^{-1}) x(h^{-1}).$$

If $A(g)$ is always unitary, then we may take $x(g^{-1})$ a multiple of $a(g)$.

Surprisingly often in cases of interest, this representation turns out to be irreducible, and this is particularly useful for groups of measurable maps from a measure space $(X, \mu)$ into a topological group $G$ and its semi-direct product with the group of invertible measure-preserving transformations of $X$. (One takes $H = L^2(X, \mu) \otimes H_o$ where $H_o$ carries a representation of $G$.) When $X$ is a manifold with a volume form, this group contains the group of volume-preserving diffeomorphisms. Ismagilov’s account also includes the construction of representations from Poisson measures. (There is a very useful introduction to this area by Graeme Segal in the LMS Lecture Notes 69 (Cambridge University Press, 1982) reprint of the basic papers of Vershik, Gel’fand and Graev from which much of this work sprang.) It culminates in a discussion of the projective representations of the group of diffeomorphisms of the torus which preserve the irrational angle foliation, which provides a subtle analogue of the Virasoro group of diffeomorphisms of the circle.

Neretin’s monograph is written in a colloquial style which sometimes reads more like the transcript of a good seminar, with numerous digressions and fascinating asides. His unconventional approach is apparent from the outset, when in the third sentence of the book the reader is advised that: ‘Anybody who is not acquainted with the topic could well skip this chapter’ . Nonetheless, with some reservations about the number of misprints (which afflict both books), the writing is clear. (Occasionally a little deconstruction is necessary to see what the author really meant, as in the Howe–Moore Theorem 1.4.6 in which the condition that the semi-simple group be non-compact is omitted.)

One of the first points stressed by Neretin is that representations of infinite-dimensional groups often carry a lot more structure than is at first apparent. For example, the weak closure of the image of a representation $U$ in the space of linear operators often includes an interesting semi-group of contractions, whose properties can also provide valuable information.

One of Neretin’s particular aims is to explore the representations of infinite-dimensional analogues of the classical semi-simple Lie groups, and these suggest two other special features. For example, the fundamental irreducible representations of the compact group $SU(n)$, Cartan’s $A_{n-1}$, are simply the exterior powers of the natural $n$-dimensional representation. This construction is universal in that it is independent of $n$, and can be described in terms of the covariant functors which take
vector spaces and linear maps into their exterior powers. Neretin argues that rather than looking at the groups one should concentrate on the categories and their representations. (A representation of a category \( \mathcal{C} \) is a functor from \( \mathcal{C} \) to the category of vector spaces and linear transformations. This approach, the theory of quivers, has been used in algebra since the early work of Gel'fand and Ponomarev on the Harish-Chandra modules of SL(2, \( \mathbb{C} \)) and the four-subspace problem, but curiously, apart from one oblique reference to the Kronecker problem, this is not mentioned by Neretin.) Neretin describes various categories closely related to the series of complex and real semi-simple Lie groups, and shows how various groups and semi-groups appear naturally as automorphisms and endomorphisms in the categories. The Riemann surface categories of conformal field theory which appear in the work of Graeme Segal and Kontsevich fit naturally into this framework.

Another lesson to be learned from the representation theory of finite-dimensional semi-simple Lie groups is that it is often fruitful to consider pairs consisting of the group and a distinguished subgroup (usually the maximal compact subgroup). This idea is exploited in the theory of Harish-Chandra modules for semi-simple groups (in which the irreducibles for the subgroup have finite multiplicities) and also in the theory of spherical representations, where one postulates that the subgroup fixes some vector in the representation space. Neretin discusses Ol'shanskii's theory of pairs of infinite-dimensional groups and their spherical representations, and shows how categories can be constructed from such pairs, thus linking in with earlier material.

Neretin's book also contains other more standard material, including chapters on the canonical commutation and anticommutation relations, and the associated spin and Weil representations. (Whilst applauding his attempt to do justice to the contributions of Friedrichs and Berezin to the investigation of Fock space, I felt uneasy that his potted history had also omitted a number of important papers, such as J. M. Cook's 'The mathematics of second quantization', *Trans. Amer. Math. Soc.* 74 (1953) 222–245, and Irving Segal's 'Quantization of non-linear systems', *J. Math. Phys.* 1 (1960) 468–488.)

Neretin postpones until his final chapter the main discussion of representations of various types of diffeomorphism group to which Ismagilov devotes the entire second half of his book. Nonetheless, the superficial differences are slightly misleading, because many of Ismagilov's examples are at least mentioned by Neretin, and conversely many of the structural features emphasised by Neretin also appear in Ismagilov's book, though often merely as technical tools where needed. One good example of the difference in style is provided by the discussion of semi-groups and the weak closure of representations. Having briefly explained the idea, Neretin provides a series of illuminating examples in the form of exercises, without revealing the general convexity theorem which underlies many of them. Ismagilov, on the other hand, provides the general theorem which is later used to provide an elegant proof of the irreducibility of various representations, but nowhere provides simple examples. This is one of many places where I found that the two books nicely complemented each other. The general pattern was not quite obvious from Neretin's examples, but they did illustrate the real significance of Ismagilov's theorem.

TeX and its various dialects have offered authors and translators unrivalled opportunities to adorn their books with a rich variety of typographical errors, and these two are no exceptions. Dirac's only comment on *Crime and punishment* is said to have been the observation that in one chapter Dostoevsky had made the mistake
of describing the sun as rising twice on the same day. It may be equally pedantic to note that Ismagilov’s book contains two theorems numbered 9.3 only four pages apart, or that Neretin’s related equation (1.4.2) has become rather garbled, but these errors do make life more difficult for the reader. Neretin’s book contains an adequate index and useful index of notation, whereas Ismagilov includes neither. This makes it much harder to dip into the latter, though my particular interests are such that Ismagilov’s is probably the book to which I shall more often want to refer. Despite such carping, I learned a lot from each of these books, and can recommend both to those who wish to learn about the subject.

K. C. Hannabuss

**SHAFAREVICH MAPS AND AUTOMORPHIC FORMS**  
(M. B. Porter Lectures)  

By János Kollár: 199 pp., £29.50, ISBN 0 691 04381 7  

The relationship between function theory and topology has been studied since Riemann and Poincaré. In his monograph [3], S. Lefschetz added considerable depth to the study of the topology of complex algebraic varieties, and since then mathematicians have sought to understand more about this relationship. Lefschetz proved two crucial theorems and introduced methods still being studied and adapted. His hyperplane theorem relating the topology of a variety and that of its intersection with a generic hyperplane has been generalised in many directions, and new versions are discussed in the present book. The ‘hard’ Lefschetz theorem has found its ultimate home in Kähler geometry, but such methods do not impinge very strongly on the work presented here. The main theme is the influence of topology on function theory, and the study involves many fascinating methods and ideas from algebraic geometry, analysis and topology.

The book under review, which is based on the author’s M. B. Porter lectures at Rice University, gives new perspectives on the relationship between topological properties of smooth algebraic varieties and their function theory. It provides modern accounts of several well established methods as well as recent results—many of these are due to the author, and he is not afraid to speculate about future developments. The author therefore provides a perspective on mathematics, past, present and future.

There have been a number of attempts to understand which groups can be the fundamental groups of compact varieties; much of this work is conveniently brought together in [1], but the emphasis in the present book is on the influence of the fundamental group on the function theory of the variety. In his text [4], I. R. Shafarevich conjectured that if $X$ is a smooth projective variety which has a ‘big’ fundamental group, then its universal covering is holomorphically convex. In particular, he defined a normal, proper variety $X$ to have large fundamental group if the image of $\pi_1(Y) \to \pi_1(X)$ is infinite whenever $Y \to X$ is a non-constant morphism. He conjectured that this condition is equivalent to the universal cover, $\tilde{X}$, of $X$ being Stein, that is, there are enough holomorphic functions to separate points. The difficulty is in deducing the function theoretic information from the topological condition. The majority of this book is dedicated to exploring this conjecture and its background; a great deal of other interesting material is discussed en route.
To explain the title: if $X$ is normal and proper, a map $sh: X \to \text{Sh}(X)$ is called a Shafarevich morphism of $X$ if it has connected fibres and, for $Z \subseteq X$, $sh(Z)$ is a point unless the image of $\pi_1 Z$ in $\pi_1 X$ is infinite. If $\text{Sh}(X)$ exists, it is unique and its existence is closely related to the Shafarevich conjecture. Automorphic forms arise naturally through the function theory of the universal cover.

The book consists of an extended introduction and five parts. The first part consists of a discussion of Shafarevich maps and various variants, as well as the relationships between rival definitions of what a large fundamental group should be. The second part is more analytic, and discusses the classical theory of automorphic forms, including Poincaré’s method, Atiyah’s method using the $L^2$ index theorem, and Earle’s theorem on Poincaré series. Chapter 8 is devoted to illustrating the use of these methods on the unit ball in $\mathbb{C}^n$. Parts three and four respectively discuss vanishing and non-vanishing theorems, and use them to construct automorphic forms. The classical theorems of Kodaira are reviewed, and then a number of variants are discussed, including Gromov’s novel topological approach [2]. The final part discusses applications and speculations. Varieties that admit a generically finite morphism to an abelian variety are the simplest examples for the problems considered here. The methods of the book are applied in some detail to these examples.

There is a wealth of open problems and directions involving different ideas from a number of areas. Although many excellent mathematicians (amongst whom Kollár is very prominent) are already involved, there is tremendous scope to find problems which are both challenging and significant.

References


E. G. Rees

FINITE-DIMENSIONAL DIVISION ALGEBRAS OVER FIELDS


Progress in division algebras has been in sharp bursts of activity, separated by long periods of quiescence. The foundations were laid by Frobenius, Molien, Dickson and Wedderburn around the turn of the century; after several quiet decades, this was followed by a spate of papers in the 1930s, resulting in the classification of all division algebras over algebraic number fields, as well as A. A. Albert’s classification of Riemann matrices by means of involutorial division algebras. There followed another quiet period, broken by Amitsur’s construction of non-crossed products in 1972, since when there has been a good deal of activity, much of it concerned with valuations. An adequate exposition of all this work would require a pretty hefty tome, ranging over many diverse areas of mathematics. This is hardly to be expected, and certainly not intended in the volume under review. Professor Jacobson’s aim was to present a few
basic topics: involutorial algebras, \( p \)-algebras and generic splitting fields, in a coherent development. He has been able to do this in relatively few pages by taking for granted most of the standard theory as described in Chapter 4 of the author’s Basic algebra II \([1]\) or Chapter 7 of the reviewer’s Algebra 3 \([2]\).

The book begins by constructing cyclic algebras as residue-class rings of skew polynomial rings, as well as the generalization obtained by replacing the maximal subfield by a division algebra. Likewise, differential polynomial rings are used to build \( p \)-algebras, and generic trace and norm are introduced to derive splitting criteria. Even though the standard theory is assumed known, a construction of crossed products is given, including a concise description of Brauer and Noether factor sets. This is again done in a more general setting, where the maximal subfield is replaced by a (commutative) Frobenius algebra. Albert’s criterion for the cyclicity of a division algebra of prime degree is proved, and Brauer’s construction of division algebras of prescribed degree and exponent is given, as well as Albert’s example of a non-cyclic algebra. In all, these two chapters, of less than a hundred pages, provide a lot of detail that is not present in more basic expositions, and much of it can be read locally, without having to wade through pages of general theory.

The remaining three chapters are devoted to the topics mentioned earlier. For a central simple algebra, its splitting fields are important; they can all be obtained as specializations of the generic splitting field, which forms the topic of Chapter 3. There are three ways of constructing a generic splitting field for a central simple algebra \( A \): (i) as a field of rational functions of the Brauer–Severi variety associated with \( A \), (ii) as a function field of the norm hypersurface, and (iii) as a function field of the variety of rank one elements of \( A \). Moreover, with every crossed product algebra, there is associated a field obtained from the cocycle defining \( A \) by Galois descent, the Brauer field, and this also turns out to be a generic splitting field. The relations between different algebras related by restriction, inflation or corestriction are described in terms of their generic splitting fields. All this material, until now available only in the papers of Chatelet, Amitsur, Roquette and Heuser, finds a concise but complete exposition here, with a lightning tour (omitting proofs) of the necessary algebraic geometry background.

In characteristic \( p \), the simple algebras of \( p \)-power exponent, called \( p \)-algebras, play a special role; they always have a splitting field that is purely inseparable, as well as one that is an Artin–Schreier extension of the ground field. Using a notion of generic abelian crossed product, Amitsur and Saltman have given a simple construction of non-cyclic \( p \)-algebras, and this, as well as the problem of the existence of \( p \)-algebras for a given field extension, is described in Chapter 4.

The final chapter, the longest in the book, is devoted to simple algebras with involution. Involutions, that is, anti-automorphisms whose square is the identity, come in two kinds: leaving the centre fixed or not. By splitting the algebra, an involution of the first kind reduces either to transposition or to transposition plus conjugation by a skew-symmetric matrix, and accordingly it is said to be of orthogonal or of symplectic type. A basic theorem of Albert states that a central simple algebra has an involution if and only if its exponent is 1 or 2. An anti-automorphism \( \alpha \) of a central simple algebra can often be replaced by an involution, namely when \( \alpha^2 \) is inner, induced by a norm element. There are many detailed results on \( 2 \)-algebras of low degree, giving conditions for representability as a tensor product of quaternion algebras, some going back to Albert, but most of more recent origin (Rowen, Tignol and others). This is followed by counter-examples, in the main the construction by
Amitsur, Rowen and Tignol of an involutorial division algebra of degree eight which is not a tensor product of quaternion algebras. For a further study, one defines $H(A)$, the set of symmetric elements of $A$; this is a special Jordan algebra under the multiplication $a \cdot b = \frac{1}{2}(ab + ba)$, and it may be used to study involutorial division algebras of low dimensions. In characteristic two this has to be replaced by the operation $U_a(b) = aba$ and its linearization. There is an extensive study of the Jordan algebra of symmetric elements, its norm and trace and its special universal envelope, at some points using results from the theory of division algebras, but there are no significant applications (yet!) going the other way.

This book is a concise but clear account of a number of important aspects of division algebras, just the kind of work we have come to expect from the author, who has enriched the mathematical literature with his books for over 50 years. Inevitably, much that one would like to have seen included has been left out: for example, there is no discussion of the Merkuryev–Suslin theorem, the Tannaka–Artin problem, or Amitsur’s example of a non-crossed product. Instead, the author takes us on a tour of division algebras, pointing out the salient facts, often with little-known proofs, but never going on so long as to bore the reader. This makes the book a pleasure to read, a pleasure only slightly dimmed by the absence of an index or list of notations, and only a partial list of references.

References


P. M. Cohn

PROBABILITY MEASURES ON SEMIGROUPS: CONVOLUTION PRODUCTS, RANDOM WALKS, AND RANDOM MATRICES
(The University Series in Mathematics)

By Göran Högnäs and Arunava Mukherjea: 388 pp., US$89.50
(US$107.40 outside US and Canada), ISBN 0 306 44964 1
(Plenum Press, 1995).

The authors’ aim in writing this book was to provide up-to-date information on the theory of weak convergence of convolution products of probability measures on semigroups, the theory of random walks on semigroups, and applications of these theories to products of random matrices. The book is intended to be suitable for a two-semester graduate student course. The result is a valuable addition to the literature that should encourage more mathematicians to become involved in this fascinating field, which combines the attractions of the interplay between probability measures and algebraic and topological structure with the potential for wide applicability. The ideas in the book are well illustrated by examples, and open problems are indicated. The presentation is biased towards results which are new, interesting and useful in the context of semigroups; some results are presented in groups rather than semigroups when the semigroup situation is so far only partially resolved; some parts of the subject, such as the theory of infinitely divisible distributions, are excluded.
The first chapter of the book provides background material on locally compact semigroups, which is needed for the subsequent probability theory. Of particular importance for this are completely simple semigroups with compact group factor, for which a detailed structure theorem is given.

The second chapter is the core of the book. It introduces the basic facts about probability measures on locally compact semigroups and describes what is presently known about the limiting behaviour under weak convergence of convolutions of probability measures. For convolution powers of a single probability measure on a compact or discrete semigroup, the problem is almost completely solved, whereas some questions still remain unanswered in the non-compact non-discrete case. For convolutions of non-identical probability measures, the problem is largely solved in discrete groups and in abelian semigroups that are compact or discrete, but is still far from solution in other cases. An important rôle in these results is played by idempotent probability measures. The support of such a measure is always a closed completely simple subsemigroup with compact group factor.

The third chapter provides a detailed study of random walks on locally compact semigroups. Left, right, bilateral and mixed random walks are considered, and correspond to whether the increments multiply on the right, on the left, on both sides simultaneously, or on one side that is randomly chosen at each step. The increments are usually assumed to be independent and identically distributed. The usual topics for random walks are dealt with, including transience and recurrence of the walks and stationary distributions. The results demonstrate the strong interplay between the probabilistic properties of a random walk and the topological and algebraic structure of the semigroup that supports it.

In the final chapter the previous results are applied to random matrices. To include only results that are reasonably complete and avoid duplicating work that is readily accessible elsewhere, attention is restricted to problems of recurrence, tightness, invariant measures and laws of large numbers for products of random matrices with nonnegative entries. There are connections here with attractors and fractals.

Michael S. Bingham

Potential Theory on Infinite-Dimensional Abelian Groups
(de Gruyter Studies in Mathematics 21)

(Walter de Gruyter, 1995).

The main object of study of this book is an infinite-dimensional elliptic second-order differential operator with constant coefficients on the infinite-dimensional torus.

Classical potential theory is the study of the Laplace operator \( \Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \) and the Newtonian potentials \( \frac{d\mu(y)}{|x-y|} \) on the Euclidean space \( \mathbb{R}^3 \). The Laplace operator and the Newtonian kernel being translation-invariant, it is natural to consider similar objects on a group. Twenty years ago, a book entitled *Potential theory on locally
compact abelian groups was written by Berg and Forst [1]. It explained potential theory on groups like \( \mathbb{R}^n, \mathbb{T}^m \) or \( \mathbb{R}^m \times \mathbb{T}^n \), where \( \mathbb{T} = \mathbb{R}/2\pi \mathbb{Z} \) is the one-dimensional torus. Bendikov’s book explains what happens when the dimension becomes infinite, mainly on the infinite-dimensional torus. This research was initiated in the seventies by Berg, Bliedtner, Forst and others, but the present book rests on its author’s own work.

The starting point of Bendikov is axiomatic potential theory as developed by Brelot and Bauer around 1960, and its probabilistic interpretation given by Doob, Hunt, Meyer and others at the same time. In the axiomatic view, the basic object of potential theory is not a Newton or Riesz kernel, but a ‘sheaf’ of ‘harmonic’ functions on a locally compact space. This sheaf is subject to various conditions satisfied by the classical harmonic functions. The motive of this theory is to study, from a unified point of view, solutions of elliptic or parabolic second-order differential equations on \( \mathbb{R}^n \) or on differentiable manifolds. Around 1940 it was discovered that there is a deep connection between classical potential theory and the theory of Brownian motion rigorously defined by Wiener. The Laplace operator is the infinitesimal generator of the Brownian transition probability semigroup; the Newtonian kernel is the integral with respect to time of the Brownian semigroup. A similar connection was discovered between axiomatic potential theory and the general theory of Markov processes. In 1963 Meyer showed that the cone of non-negative superharmonic functions on a ‘harmonic Brelot’ space coincides with the cone of excessive functions for a good continuous Markov process. Bendikov uses twenty-five pages to recall briefly this story. Then he embarks upon the description of the translation-invariant harmonic sheaves on \( \mathbb{T}^\ast \) and their connections with Gaussian diffusions generated by infinite-dimensional elliptic operators

\[
\mathcal{L} = \sum_{i,j=1}^{\infty} a_{ij} \partial_i \partial_j + \sum_{i=1}^{\infty} b_i \partial_i.
\]

To obtain an idea of the precise content of the book, let us consider the following objects:

- \( \mathbb{T}^\ast \) is the \( \ast \)-dimensional torus;
- \( \Delta^\ast = \sum_{k=1}^{\infty} \partial_k^2 \) is the Laplace operator on \( \mathbb{T}^\ast \);
- more generally, given a positive sequence \((a_k)_{k=1}^{\infty}\), \( \mathcal{L} = \sum_{k=1}^{\infty} a_k \partial_k^2 \) is an elliptic differential operator on \( \mathbb{T}^\ast \);
- \( \pi_{\alpha \beta} \) is the natural projection, when \( \alpha \leq \beta \), of \( \mathbb{T}^\beta \) onto \( \mathbb{T}^\alpha \);
- \( \mathcal{H}_\alpha \) is the harmonic sheaf on \( \mathbb{T}^\alpha \) of the local solutions of \( \mathcal{L}_\alpha u = 0 \).

It is obvious that for every \( \alpha \), \( (\mathbb{T}^\ast, \mathcal{H}_\alpha) \) is a translation-invariant Brelot harmonic space. The inclusions \( \mathcal{H}_\alpha; \pi_{\alpha \beta} = \{ u : \pi_{\alpha \beta} u \in \mathcal{H}_\beta \} \subset \mathcal{H}_\beta \) are satisfied. Thus we obtain an increasing sequence of harmonic spaces. Is it possible to close, in some sense, this sequence? This question leads to the exploration of the potential-theoretic properties defined on the compact infinite-dimensional abelian group \( \mathbb{T}^\ast \) by the sheaf \( \mathcal{H} \), which is the projective limit of the sequence \((\mathcal{H}_\alpha)\). It turns out that the answer depends strongly on the speed of growth of the sequence \((a_k)\). To attack this problem, it is necessary to consider the diffusions generated by the operators \( \mathcal{L}_\alpha \) on \( \mathbb{T}^\ast \). Their transition probability semigroups are the Gaussian semigroups of measures

\[
\mu_\alpha^t = \bigotimes_{k=1}^{\alpha} n_{n,t},
\]

where

\[
n_n(x) = 2\pi \sum_{m=-\infty}^{\infty} \frac{1}{\sqrt{(4\pi t)}} \exp \{ -(x-2\pi nm)^2 / 4t \}.
\]
is the density on $\mathbb{T}$ which is the image of the Gaussian density by the natural homomorphism $\mathbb{R} \to \mathbb{R}/2\pi \mathbb{Z}$. Passing to the projective limit, one obtains the process $X$ on $\mathbb{T}^{\infty}$ associated to the semigroup $\mu_t = \bigotimes_{j=1}^{\infty} n_{a_j t}$. Then the answer to the previous question is the equivalence of the two properties:

1. $(\mathbb{T}^{\infty}, \mathcal{H})$ is a translation-invariant Brelot harmonic space;
2. the growth of $a_k$ to infinity is fast enough to yield

$$\lim_{\lambda \to \infty} \frac{1}{\lambda} \text{card} \{ k \geq 1; a_k \leq \lambda \} = 0.$$ 

In this case, $\mathcal{H}$ is exactly the sheaf of the local weak solutions of $\mathcal{L}u = 0$ with $\mathcal{L} = \sum_{k=1}^{\infty} a_k \partial^2_k$. The proof relies on good duality properties for the process $X$; these properties demand the absolute continuity on $\mathbb{T}^{\infty}$ of the measure $\mu_t$; at this point, Kakutani's criterion for the equivalence of infinite product measures is essential. Moreover the growth condition (2) is the exact condition which ensures the validity of Weyl's lemma, that is the hypoellipticity, for the operators $\mathcal{L} - \lambda$.

This example gives only a glimpse of the book. The author treats fully the representation of translation-invariant harmonic sheaves by constant-coefficient elliptic differential operators, on groups $\mathbb{R}^m \times \mathbb{T}^{\infty}$. He proves the equivalence of the three kinds of objects elliptic harmonic group, elliptic differential operator, and space homogeneous Markov process; this equivalence relies on an infinite-dimensional version of a classical theorem of Bony. By the way, he shows that among the abelian groups which are not Lie groups, only groups of the type $\mathbb{R}^m \times \mathbb{T}^{\infty}$ admit a good potential theory. In the last chapter he deals with the infinite-dimensional Sobolev inequality, using a notion of 'mixed norm' on a space of martingales.

The comparative discussion of the different notions of axiomatic potential theory does not make very exciting reading. The repetition of words like harmonic or projective limit might be tedious. But this book is not difficult to read. It is even pleasant when the author keeps his own style, giving a view on the essential. The apparatus of symbols and formulae is light and transparent: it needed no virtuosity with the computer-aided composition and does not need any effort to be deciphered.

The book is short. This is the result of a choice of the author: he uses systematic references for the classical results he needs. Elements of Fourier analysis, Markov theory and potential theory are used but not repeated. The beginner student might find that hard: it is advisable to have at hand the book by Berg and Forst. The dominant spirit of the book under review is analytical. Markov theory appears as it was before Ito calculus imperialism. Several important specific notions of potential theory are not considered: capacity is not mentioned; barriers are defined at the beginning but disappear at once. The book closes with an attractive short chapter called 'Some thoughts on probability and analysis on locally compact groups'; the non-abelian case is evoked with the ingenious affirmation that without doubt, the structure of a group and properties of Gaussian measures and harmonic functions on it are closely connected! But there is not a single word on all the work on these problems over the last thirty years, showing the big differences between nilpotent, solvable or semi-simple Lie groups! Our second reference [2] is given just to recall a landmark paper in this story. Bendikov's book is not a treatise, it is a personal essay.

It is the usual style in a review to pick out a few, more or less serious, mistakes. Let us offer that Beppo Levi's property is attributed to Lévy (p. 143). Other required elements of a review are the following questions. Who are the potential readers? What
are the applications? If they are not irrelevant, these are really hard questions. Applications, which nowadays are fiercely demanded, here receive one line, evocating problems in statistical mechanics.

This elegant book, sometimes reminiscent of the good old Bourbaki time, takes its reader on a trip in a well-defined problem of mathematical analysis.

References


Yves Derriennic

BORDISM, STABLE HOMOTOPIY AND ADAMS SPECTRAL SEQUENCES
(Fields Institute Monographs 7)

By S. O. Kochman: 272 pp., US$49.00, ISBN 0 8218 0600 9

Classical algebraic topology no longer epitomises the cool and fashionable, as it did during the 1950s, 1960s and 1970s. Throughout that era, graduate students around the world would jostle to learn K-theory, cobordism and homotopy theory, excited by thoughts of joining a powerful mathematical movement. Since then, and like all others, the empire has declined. With hindsight, this was partly due to its own success in attracting some of the greatest mathematical talent to its service; major problems were solved, and Fields medals were won in quick succession by Serre, Thom, Milnor, Atiyah, Novikov and Quillen. When such leaders progressed to pastures new, the pickings for those who remained grew more austere. The gains of the 1980s and 1990s, inspired particularly by Morava, and by Hopkins and his collaborators, are therefore less familiar to the wider mathematical community, although time will surely rank them with the best.

The book under review is pitched specifically at those graduate students who remain captivated by the problem of computing the stable homotopy groups of spheres (or stable stems), and are willing to explore the two demanding routes which have been established to pursue this holy grail. The first is quantitative, and develops traditional methods to the point where spectral sequences are the tool of choice, and themselves require a further spectral sequence to get the computation started. The second route is qualitative, leading to recent results which have identified nilpotence and periodicity as structural features of the stable category, yet still require the Adams spectral sequence and other algebraic and geometric machinery.

A third pathway has been worn by external applications and innovations, equally formidable to students and outsiders. Examples include Adams’s confirmation [1] that the complex numbers, quaternions and Cayley numbers are the only real division algebras of dimension > 1 (in the slipstream of his work on Hopf invariants), and Stolz’s verification [5] of the Gromov–Lawson conjecture in Riemannian geometry; both of these again appeal to the Adams spectral sequence. Meanwhile, algebraic topologists have stimulated the growth of novel forms of K-theory, and are currently
posed tantalisingly on the number-theoretic brink of the age of elliptic cohomology. Those of us who love the subject believe that such developments will become commonplace, as its techniques percolate through to the mathematical masses.

The contents of Kochman’s book look promising to the would-be student, with five well-balanced chapters augmented by sections on further reading, and successively covering bordism, characteristic classes, stable category, complex bordism and computing stable stems. In the last of these, he calculates the 2-torsion subgroup of the stable stems in dimensions 1 to 30, although his avoidance of odd primes may leave students unaware of the in-built advantages of using Novikov’s version of the Adams spectral sequence. His choice of topics is based on a remix of appropriate portions of Adams’s classic text \[2\], laced with material from unpublished courses by Lück and May of similar 1960s vintage. Kochman’s own imprint is unmistakable, with smatterings of the prodigious computations which have made him famous. I would plead, however, for clearer guidance on the origins of certain material. For example, the first chapter’s coordinate-free approach to classifying spaces is novel, and most welcome, yet remains curiously unflagged under ‘further reading’; given its demanding nature, references to alternative treatments would surely assist the student.

The book is clearly self-contained for those of quantitative bent, and gives an uncompromising feel for the joys and sorrows ahead. Qualitatively speaking, Kochman seeks to prepare his readers for Ravenel’s exposition \[4\] of the nilpotence and periodicity results of Hopkins, Devinatz and Smith. He succeeds with honour, but I would like to have been aware of his intentions long before the penultimate page, because students might realistically consider reading both texts simultaneously (or even approach Ravenel first). I would also have encouraged the author to devote a few pages to an updated sprinkling of his favourite items from pathway three, since the future credibility of the subject depends on the continuing ability of its practitioners to apply and innovate.

The book is carefully and clearly typeset, and seems well proof-read, although certain notation and nomenclature surprises. For example, the first reference I found to formal groups was on page 187, where the unsuspecting student could be confused by their relationship with the formal products so prominently featured elsewhere.

Unfortunate lapses surface in the text. During the author’s introduction to Hopf algebroids \((U, H)\), essential for a proper understanding of the coactions underlying the Adams–Novikov spectral sequence, he asserts in 4.5.2 (f) that the product map on \(H\) factors through \(H \otimes \mu, H\). This is generally false with the module structures he describes, and is illustrated with an incorrect equation. Similarly, he states on page 178 that ‘…we have no formula or algorithm for computing the composite of two Quillen operations’. For better or worse, such formulae were provided by Li \[3\] (to which I was alerted by my Mancunian colleague Reg Wood), and readers should be told.

The book is beautifully produced, with diagrams recalling the more expansive 1960s, and students who attend Stan’s lectures will surely find it an invaluable and encyclopaedic accompaniment. For others, it might more appropriately be added to the shelves of their supervisors; I myself can already testify to its usefulness in that rôle.
References


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