BOOK REVIEWS

DEFINABILITY AND COMPUTABILITY
(Siberian School of Algebra and Logic)

By Yuri L. Ershov: 264 pp., US$96.00, ISBN 0 306 11039 3

The development of recursion theory, the theory of recursive functions, as a precise counterpart to the intuitive notion of effectively computable functions was one of the major intellectual initiatives of the 1930s. Turing’s original analysis ([5], reprinted with other important contributions in [2]) remains well worth reading. Such a successful theory presents a temptation to generalise with a view to wider applications. We owe to Kreisel (see [4]) the observation that in the case of recursion theory one should seek analogues not only of the obvious key ideas (partial and total recursive function, semi-recursive and recursive set) but also of the less obvious crucial notion of finiteness; for one expects (semi-)recursive sets to be closed under finitary (positive) operations.

Generalised recursion theory emerged through the 1960s and 1970s in a variety of flavours, but the most widely applicable theory, the theory of admissible sets (Barwise [1]) couched in terms of definability, certainly supports Kreisel’s dictum. The analogy with a definability approach to ordinary recursion theory was clear to the originators of that theory, so it is strange that this is the first elementary treatment of the subject from the unified point of view. In Ershov’s treatment, one starts by considering an arbitrary (first-order) structure A equipped with a (not necessarily transitive) relation written <. One should think of the usual order on the natural numbers, and of the membership relation on some suitable collection of sets. The idea is that the analogue of a finite set is (the simple image of) one of the form \{b : b < a\} for a \in A. One augments the language with bounded quantification (\forall x < y and \exists x < y), and in the presence of suitable structure, the predicates definable using these and the propositional connectives, the \(\Delta^0\)-sets, suffice for recursion theoretic coding. The analogue of the recursively enumerable or (better) semi-recursive sets are the \(\Sigma^0\)-sets, that is, the projections of \(\Delta^0\)-sets. The first non-obvious effect of Kreisel’s observation is then that the \(\Sigma^0\)-sets are closed under bounded universal quantification.

This approach to recursion theory is based on explicit definability, but the major tool is the theory of inductive definitions first championed by Gandy (see [3]). The first two chapters of the book give an account of the basic theory leading to Gandy’s fundamental theorems on inductive definitions. This is a natural stopping point for the general mathematical reader, though one misses out on the more technical delights of the Barwise Compactness Theorem. The final chapter is distinctly odd. It contains accounts of a few further technical topics in the theory of admissible sets, together with some applications of the general theoretical apparatus of inductive definitions to dynamic logic and domain theory. The title of the chapter, ‘Selected topics’, is frank enough.

While the idea of an elementary integrated account of recursion theory from the perspective of definability is a good one, the achievement is rather disappointing. The
general exposition stops quite early, and thereafter there are separate treatments of recursion on the natural numbers and on admissible sets. So we hardly exploit the analogy, and there is painful redundancy. The truth definition for $\Delta^0$ formulae in arithmetic is written out in full on page 45, and then the corresponding definition in the context of admissible sets is given again in full on pages 113 and 114. Each of these definitions occupies two-thirds of a page!

There is certainly room for a readable book in this area, aimed at the general mathematical reader. Unfortunately, as such a reader will by now have guessed, Ershov’s style of presentation is very much in the Russian tradition. From one point of view, the book is completely honest: there is no shrinking from the gory details of the codings. Those who love syntactic detail will enjoy seeing it all laid out. However, others would probably trade that for clearer motivation and more explanation of ideas: it is not a book for the faint-hearted.

References


J. M. E. Hyland

SPECTRAL GRAPH THEORY

(CBMS Regional Conference Series in Mathematics 92)

(American Mathematical Society, 1997).

EIGENSPACES OF GRAPHS

(Encyclopedia of Mathematics and Its Applications 66)


A graph $G$ with $n$ vertices can be represented by an $n \times n$ symmetric matrix $A = (a_{ij})$, defined by the rule that $a_{ij} = 1$ if vertices $i$ and $j$ are joined by an edge, and $a_{ij} = 0$ otherwise. The matrix $A$ is known as the adjacency matrix of $G$. If $G'$ is isomorphic with $G$, then its adjacency matrix $A'$ is obtained from $A$ by a permutation of the rows and columns, and so it is similar to $A$. It follows that the similarity invariants of $A$ (which we shall refer to as algebraic invariants) are isomorphism invariants of $G$.

Two general questions arise naturally. What properties of $G$ can be deduced from a given set of algebraic invariants, and is it possible to find a complete set of algebraic invariants for $G$?

The most obvious algebraic invariants are the eigenvalues and their multiplicities. The first investigations into eigenvalues were carried out in 1939–45 by Collatz and
Sinogowitz, although their work was not published until 1957. It was quickly noticed that the eigenvalues do not form a complete set of invariants: that is, there are pairs of non-isomorphic graphs which have the same eigenvalues and multiplicities. But at the same time it became clear that a very great deal of useful information about a graph can be deduced from the eigenvalues. In the past forty years, our knowledge of this topic has advanced considerably, stimulated at first by applications to such fields as chemistry (molecular orbital theory) and electrical engineering (network theory). More recently, developments in other areas of mathematics have led to significant progress. The books under review illustrate the rapid advances which are currently taking place. Chung’s book mainly addresses the first question posed above, while the book of Cvetkovic, Rowlinson and Simic addresses the second.

Chung’s book is concerned with extracting as much information as possible from the eigenvalues. It incorporates a great deal of recent work, much of it due to the author herself, in which techniques from ‘continuous’ spectral theory are transferred to the discrete situation, and exploited. Motivation for much of this work has come from random walk theory, in particular the problem of estimating the rate of convergence to the stationary distribution. Results of this kind have important applications to statistical theory and the theory of algorithms.

Especially fruitful has been the analogy with differential geometry, for example with regard to the Laplacian operator. (The eigenvalues of the discrete Laplacian are not quite the same as those of the adjacency matrix, but the conceptual framework is identical.) For example, the Cheeger inequalities in spectral geometry have counterparts in graph theory, and the proof techniques are very similar. The inequalities tell us a great deal about the how the expansion properties of a graph depend upon the least non-zero eigenvalue of the Laplacian. This topic is developed in Chapters 2–6, leading up to explicit constructions of expander graphs, including the ones discovered by Lubotzky, Phillips and Sarnak. There is a mass of useful information here, much of which will surely be the basis for future research.

Chapter 7 deals with eigenvalues of symmetrical graphs, and contains elegant theoretical results, as well as practical applications. For example, the vibrational spectrum of the ‘buckyball’ (carbon-60) is computed. The rest of the book covers topics where the influence of ‘continuous’ methods is particularly strong. This includes eigenvalues of subgraphs with boundary conditions (Chapter 8), Harnack inequalities (Chapter 9), heat kernels (Chapter 10), and Sobolev inequalities (Chapter 11). The final chapter uses some of these techniques to study random walks on graphs.

The book by Cvetkovic, Rowlinson and Simic covers different ground: it goes beyond the eigenvalues in the search for finer algebraic invariants. Although, in general, there is no natural way to choose a canonical basis of eigenvectors, the eigenspaces themselves are invariant to within a permutation of the vertices, and one can build on this observation to construct invariants which are, in a sense, coordinate-free. For example, the angles between the coordinate axes and the eigenspaces can be ordered naturally, and so they meet this requirement. This leads to the important notion of a star partition, defined in terms of orthogonal projections of the axes onto the eigenspaces. Star partitions provide a natural (but not unique) bijection between eigenvalues and vertices, and a means of constructing natural bases, called star bases, of the ambient Euclidean space. There are only finitely many of these star bases, since there are only finitely many star partitions and only finitely many orderings of the vertices.

Unfortunately, the number of star bases increases exponentially with \( n \). Hence the
culmination of the book is the discussion in Chapter 8 of the complexity question for canonical star bases, and its relationship with the graph-isomorphism problem. The place of the graph-isomorphism problem in the hierarchy of complexity is a famously difficult question: roughly-speaking, all known approaches ‘just fail’ to show that the problem can be solved with a number of steps which is polynomial in \( n \). Here it is shown that the theory of star bases is powerful enough to provide an alternative proof of an earlier theorem of Babai, Grigoriev and Mount: isomorphism testing for graphs with bounded eigenvalue multiplicities can be done in polynomial time. Significantly perhaps, the machinery is shown to have theoretical links with other combinatorial problems, such as the maximal clique problem. Furthermore, it is effective when applied to the traditional benchmark problem in this field, that of distinguishing non-isomorphic strongly regular graphs with the same parameters.

Both books are clear, without being pedantic, and challenging, without being obscure. I enjoyed reading them, and I can recommend them without hesitation.

NORMAN BIGGS

LOW RANK REPRESENTATIONS AND GRAPHS
FOR SPORADIC GROUPS
(Australian Mathematical Society Lecture Series 8)

By CHERYL E. PRAEGER and LEONARD H. SOICHER: 141 pp., £24.95, isbn 0 521 56737 8 (Cambridge University Press, 1997).

Groups are often best studied through their action on some combinatorial or geometric structure such as a block design, a graph or a geometry. Classical groups come equipped with a natural representation, but such actions for sporadic groups are isolated (and fascinating) ‘once-offs’. The rank of a transitive permutation action is the number of orbits of the stabilizer of a point. Such an action gives rise to a number of associated graphs which are more symmetrical and amenable if the rank is low. Several of the sporadic simple groups were discovered as automorphisms of graphs of low rank, and several others were first constructed in this way. Many such actions were determined some years ago, for example in Cambridge (in connection with the ATLAS project) and in Moscow at the VNIISI. However, the authors of this book have automated the process of finding permutation actions (see below), and have systematically obtained for the sporadic groups and their automorphism groups all permutation actions of rank less than or equal to 5 (primitive or imprimitive).

The method takes as its starting point a list of elementary facts about permutation representations which can be readily checked through the permutation character. (For instance, a power of a permutation must fix at least as many points as the original permutation.) A sum of irreducible characters which satisfies these elementary conditions is called a pseudo-permutation character. A pleasing short argument shows that a permutation character of rank 5 must be a sum of 5 distinct irreducible characters; a computer search is carried out to find all pseudo-permutation characters of rank up to 5. Various arguments are then employed to decide whether a given pseudo-permutation character is, in fact, a permutation character. At this point, more use could have been made of characters. The authors are often able to identify not only the isomorphism type but the conjugacy class of subgroups to which a possible
point stabilizer must belong. Fusion of elements between subgroup and group is thus known, and so induction of the trivial character would reveal whether the resulting permutation character has the required rank. Besides being more transparent, this would have furnished the reader with additional information. The authors clearly decided not to include diagrams of graphs, and it is certainly true that the experienced reader will have no difficulty in interpreting an adjacency matrix. However, just one diagram to show how the parameters $a_i$, $b_i$ and $c_i$ appear in a graph might have been helpful.

The introduction contains a concise survey of the required background, pitched at a level suitable for a reader with a knowledge of elementary group theory. Moreover, the book is a useful source of presentations of sporadic groups, and of generators for certain of their important subgroups. It should be kept on one’s bookshelves alongside the ATLAS of finite groups, to which it makes a valuable supplement.

ROBERT T. CURTIS

MATHEMATICS OF THE 19TH CENTURY:
GEOMETRY, ANALYTIC FUNCTION THEORY


This is a translation of two books first published jointly in Russian in 1980. Geometry is typical of its authors, B. L. Laptev and B. A. Rozenfel’d. An immense amount of material is covered clearly and accurately: differential geometry, projective geometry, algebraic geometry and geometric algebra (Hamilton and Grassmann), non-Euclidean geometry, higher-dimensional geometry, topology and geometric transformations. The book is at its most original when it describes Russian authors little known in the West: Somov, who used vectors in differential geometry, and Suvorov and Vasil’ev, non-Euclidean geometers after Lobachevskii. Lobachevskii’s work is described more thoroughly than in many an account, but the question of why Lobachevskii’s work exerted so little influence even in Kasan is not discussed. There is also a good account of the work of Kotel’nikov and Study, elucidating some obscure remarks of Riemann about how to express transformations of space. This relates to the geometry of dual numbers, and the account here is much easier to read than Rozenfel’d’s later and longer version [2]. The work of Riemann is analysed, and its implications for physical space drawn out very clearly. The authors look for mathematicians who attempted to construct a Riemannian geometry in spaces of dimension greater than 2; the story is nowhere told in the literature. They found some, but missed others, most notably Kronecker.

The style is generally to summarise a book or paper and then move on to the next book or paper which follows the earlier one thematically and chronologically. The result is clear but unexciting; the drama is lost. After 115 pages, three unexceptional conclusions are reached: that ordinary space was investigated with increasingly sophisticated methods, that the concept of space was enlarged, and that algebraic methods steadily penetrated geometry.

The second book, on analytic function theory, by A. I. Markushevich, who died in 1978, is much livelier. Markushevich is much more interested in how the new
mathematical ideas were disseminated, and what it could have meant to a mathematician at the time to hear the latest results.

The usual ingredient of a history of complex function theory is the triumvirate of Cauchy, Riemann and Weierstrass. They are all here, and the treatment of Riemann stands out for its range and depth. Markushevich rightly establishes that one of the most important sources of the new theory was the work of Abel and Jacobi on elliptic functions. As he points out, this had a formal character and did not call upon complex function theory when it was created; rather, it stimulated mathematicians to promote that theory in order to provide better formulations of the original work. The later development of Abelian functions had a different impact. Markushevich describes the work of Riemann and Weierstrass in this connection, which he knows very well, but the path from there to a function theory in several variables was to be a much more arduous one, opened up only in the 20th century.

Several other important topics are described: Picard’s Theorems and the stimulus this gave to French mathematicians like Hadamard and Borel, automorphic functions, and uniformisation. The treatment is necessarily brief, but the comments are to the point. Again, the treatment of Russian writers stands out, notably Sokhotskii’s contribution to the Sokhotskii–Casorati–Weierstrass Theorem, which is described at length. The result is a book that certainly surpasses his earlier, and interesting, account [1], and could become the standard account of the subject.

The translation, as one would expect, is fluent and a pleasure to read, although there are occasional disparities in titles and dates between the text and the Bibliography. One notes with regret that both the distinguished editors have also died since the Russian edition was published.

References
2. B. A. Rosenfel’d, A history of non-Euclidean geometry: evolution of the concept of a geometric space, translated by Abe Shenitzer, with the editorial assistance of Hardy Grant (Springer, Heidelberg, 1988).

Jeremy Gray

LECTURES ON ENTIRE FUNCTIONS
(Translations of Mathematical Monographs 150)


Entire functions

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]

or everywhere convergent power series constitute the simplest and most important class of analytic functions. The theory has made great strides in the last 100 years, starting with the fundamental work of Hadamard on order, genus and representation, and the work of Nevanlinna on meromorphic functions.
Among entire functions, those of exponential type have the most elegant theory and the most applications. These are functions for which

$$0 < \limsup \frac{\log M_f(r)}{r} < \infty,$$

where

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

Previous books in this general area include those by Paley and Wiener [4], Levinson [3], Boas [1] and Cartwright [2]. The first two of these are also published by the American Mathematical Society. Somewhat surprisingly, the authors refer to these two books but not to those by Boas and Cartwright, although the latter two books include some of the same basic material, such as the definitions of order and type, the Phragmén–Lindelöf Theorems and, for the functions of exponential type, Pólya’s indicator diagram and the indicator function $h(\theta)$.

However, much has happened in the last 40 years, and for this reason alone the present volume would be very welcome. It is superbly organised: the material is divided into 28 short chapters or lectures, thus the book would form an ideal basis for an advanced course. The student is not overloaded, but each lecture contains new and interesting material.

Each lecture contains problems to exercise the reader. There are many applications: to Banach algebras, signal processing, quasi-analytic classes, completeness and density theorems and the Titchmarsh Convolution Theorem, to mention but a few.

In the first lecture we find a characterisation of the order and type of a function in terms of the rate at which the coefficients tend to zero. This is an easy but useful result, and I am glad to have a reference to it here.

Among the highlights, I should like to mention a nice proof of Cartwright’s Theorem (p. 160), that a function of exponential type less than $\pi$, and bounded at the integers, is bounded on the real axis. There is also, in Lecture 15, a beautiful proof of Azarin’s form of the Phragmén–Lindelöf Theorem in a half plane. This is applied in the following lecture to functions of the Cartwright class $C$, that is, those functions of exponential type for which

$$\int_{-\infty}^{+\infty} \frac{\log^+ |f(x)|}{1 + x^2} dx < \infty.$$

Many results for $t(z) = \log |f(z)|$ extend to general subharmonic functions $u(z)$. This point of view is first developed in Lecture 7. The authors quote on p. 78 (unfortunately, without proof) the very useful theorem of Yulmukhametov, that if $u(z)$ is subharmonic of finite order in the plane, then there exists an entire function $f(z)$ such that

$$|u(z) - \log |f(z)|| = O(\log |z|), \quad \text{as } |z| \to \infty$$

outside a set of disks, the sum of whose radii is finite.

Levin’s book on the distribution of zeros of entire functions was the first book I read in the Russian language. It was and is a beautiful introduction to the fine work of this scholar and his school. The present book will keep Levin’s memory green and make him many further friends. I should like to recommend it not only as an essential reference book for libraries but also as a useful buy for anyone who wants to learn or teach entire functions.
SETS OF MULTIPLES
(Cambridge Tracts in Mathematics 118)


This volume is concerned with a single question. Given a sequence of positive integers, what can be said about the distribution of its positive integer multiples? The question was abstracted from studies of abundant numbers made by Behrend, Chowla, Davenport, Erdős, Schur and others during the thirties but abundant numbers played no part in further developments.

First ideas are presented in Chapter 0. The nature of the subject is indicated by two results: the multiples need not have an asymptotic density (Besicovitch, 1934); they always have (the weaker) logarithmic density (Davenport and Erdős, 1937).

In this volume a sequence of integers is Besicovitch if its set of multiples has asymptotic density. The sequence is Behrend if 1 does not belong to it and the multiples have density 1.

Straightforward application of the inclusion/exclusion principle shows that the distinct multiples of a finite sequence of integers \(a_j\) have an asymptotic density. A formula can be given for it in terms of the various least common multiples \([a_{j_1}, \ldots, a_{j_k}]\). Since no arithmetic properties of the \(a_j\) are assumed, computation of the individual common factors of the \(a_{ji}\) is not available. Even if we know it to exist, how might we express the asymptotic density of the multiples of an infinite sequence of \(a_j\) in an accessible form?

Lacking an arithmetic aesthetic, the subject of this volume employs implicitly or explicitly ideas from the study of large deviations in probability, together with (implicitly) the notion of truncation. These notions are translated into the statistics of the prime divisors of integers considered with respect to a natural frequency measure. The upshot is proofs that are an eclectic mix of Fourier analysis using positive reals (for the large deviation aspect), the study of Dirichlet series near to a singularity (for the density aspect), non-trivial inequalities established by induction (for the sieve aspect), and the rudimentary use of valuations.

Chapter 1 intensifies the results and methods of Chapter 0. A sequence is certainly Besicovitch if it is no denser than the primes (Erdős, 1948), or if it is multiplicatively like the primes (Erdős, Hall and Tenenbaum; Hall, 1994 onwards). Theorem 1.7 of the author (1990) gives a flavour. For a sequence \(a_1 < a_2 < \ldots\) to be Behrend, it is necessary that if \(0 < y \leq 1, \beta > y - 1 - \log y\), then the series \(\sum a_i^{-\beta y} y^{\Omega(a_i)}\) diverges. Here \(\Omega(n)\) denotes the total number of prime divisors of an integer \(n\).

Chapter 2 prepares a taxonomy of sequences by considering multiples of more than one integer at a time. Inequalities of Heilbronn and Rohrbach and of Behrend
play rôles. For multiples of mutually prime integers, a study of independent
Bernoulli trials suffices. Unrestricted sequences can be approached only in general
terms, in part using decompositions obtained via linear algebra.

The machinery carries over to Chapter 3 and the demonstration that sets of
multiples cannot be well distributed in every interval. Identities slide smoothly by, but
what calibrates the resulting inequalities? An example introduces a well-known
function of Buchstab. We glimpse, over the wall, that giant nettle-patch the theory
of sieves.

Chapter 4 combines probabilistic group theory and probabilistic number theory
to construct Behrend sequences (Erdős, 1965; Erdős and Hall, 1974 onwards).
There are nice applications of group characters. Helpful properties of \( \Omega \) and its
relatives range from central limit theorems to adumbrations of the law of the iterated
logarithm.

Divisor density, introduced by the author in 1978, is studied in Chapter 5, using
Tauberian theorems and classical, but substantial, Fourier inversion in the complex
plane. A sequence with positive divisor density is automatically Behrend.

Chapter 6 investigates functions that are almost uniformly distributed on the
divisors of almost all integers, combining argument from Chapter 5 with the Fourier
approach to uniform distribution initiated by H. Weyl.

In Chapter 7 the author estimates asymptotically the density of the multiples of
a short interval of integers, to settle a conjecture from the companion volume
*Divisors* (Hall and Tenenbaum, Cambridge Tracts in Mathematics 90). The argument
illustrates that the apparent simplicity of the problem is an illusion.

In these last two chapters the amount of hard analysis begins to climb, and
references range wider.

The author has gone to considerable pains to make the proofs elegant. When this
cannot be readily done, he just gives us the works, which is as it should be. A beginner
might follow many of the proofs in this book, but it is hardly a book for beginners.

One is struck by the large contributions of Erdős, of the author and of
Tenenbaum, individually and jointly. The new proof of Erdős’ criterion for a
sequence to be Besicovitch comes from a 1994 joint paper of Erdős, Hall and
Tenenbaum. In part, sixty years of Erdős’ interest in the question in this volume, and
of his taste and judgement in a particular intersection of number theory,
combinatorics, probability and group theory, are here memorialised.

As the author shows, the question is still lively.

P. D. T. A. Elliott

DOUBLE QUADRATICS
(Discrete Mathematics and its Applications)

By Richard A. Mollin: 387 pp., US$74.95, ISBN 0 8493 3983 9

The author’s purpose is to present a coherent account of recent research in the
theory of quadratic orders, an area which has seen considerable growth over the past
two decades. The topic is approached by using the continued fraction algorithm to
obtain the reduced ideals of a quadratic order. This approach, introduced by Dan
Shanks, brings in one of the major tools of modern computational number theory and leads to results in various areas of number theory.

After establishing a general background in algebraic number theory and the continued fraction algorithm, Chapter 3 shows how the divisibility properties for class numbers of quadratic fields have consequences for the solutions of certain Diophantine equations, and vice versa.

Chapter 4 is concerned with prime-producing quadratic polynomials. The celebrated example of Euler, $x^2 - x + 41$, is prime for the values $x = 0, 1, \ldots, 40$ and has discriminant $-163$. If the prime $k$-tuples conjecture is true, then there exist quadratic polynomials producing arbitrarily long sequences of primes. Prime-producing polynomials of both positive and negative discriminant are considered. Negative discriminants lead to connections with complex quadratic fields of class number one and two, whereas the continued fraction algorithm can be used when the discriminant is positive. Chapter 5 continues the connections with the class number one problem, concentrating on the problem for real quadratic orders.

Chapters 6 to 8 extend the results in diverse directions, including applications to cryptology, and the monograph concludes with several tables and an extensive bibliography containing over 400 references.

There is a great deal to be welcomed in this book. It collects together much information in one of the major areas of modern computational number theory. Numerous exercises and research problems make it a suitable foundation for various postgraduate courses.

ROGER COOK

THE ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF PARTIAL DIFFERENTIAL OPERATORS
(Translations of Mathematical Monographs 155)

By Yu. Safarov and D. Vassiliev: 354 pp., US$119.00, isbn 0 8218 4577 2

This book is an exposition of the work of the authors on the asymptotics of the eigenvalues of elliptic boundary value problems. The subject has a long history, which illustrates the development of the theory of partial differential equations over nearly a century, and I shall recall some of the main events before passing to a detailed discussion of the contents of the book under review.

In 1905, Rayleigh stated that if $N(\lambda)$ is the number of eigenvalues $\mu \leq \lambda^2$ of the Dirichlet problem for the Laplacian

$$-\Delta v = \mu v \text{ in } \Omega, \quad v = 0 \text{ on } \partial \Omega, \quad (1)$$

where $\Omega$ is an open set in $\mathbb{R}^d$, then

$$N(\lambda) = V \lambda^d / (6\pi^2 + o(\lambda^2)), \quad \text{as } \lambda \to \infty. \quad (2)$$

Here $V$ is the volume of $\Omega$, and the constant has been corrected according to an observation made at once by Jeans. The motivation given was that (2) is valid for a cube where the eigenvalues are explicitly known by separation of variables, and that $N(\lambda)$, for physical reasons, should asymptotically depend only on the volume. A proof was given by Weyl in 1912. He observed that decomposing the domain and introducing Dirichlet conditions on the new boundaries will lower $N(\lambda)$, whereas
introducing Neumann boundary conditions will increase $N(\lambda)$. The asymptotic formula (2) is also valid for cubes with Neumann boundary conditions, so by approximating $\Omega$ by a union of small cubes, a general proof of (2) can be obtained. The argument of Weyl was complicated by a detour over integral equations, and was later simplified by Courant to the very useful minimum–maximum principle for eigenvalues. By a careful discussion of the cubes intersecting the boundary, Courant proved that (2) can be refined to

$$N(\lambda) = V \lambda^3 / 6\pi^2 + O(\lambda^2 \log \lambda), \quad \lambda \to \infty,$$

as already observed by Weyl.

An entirely new method, and a new aspect of the asymptotics, was introduced by Carleman in 1934. With $v_j$ denoting orthonormal eigenfunctions of (1) with eigenvalues $\mu_j$, the orthogonal projection $E_j$ on the space spanned by the $v_j$ with $\mu_j \leq \lambda^3$ has the kernel

$$e(x, y, \lambda) = \sum_{\mu_j \leq \lambda} v_j(x) \overline{v_j(y)}, \quad N(\lambda) = \text{Tr} E_j = \int_{\Omega} e(x, x, \lambda) \, dx,$$

called the spectral function of (1). Carleman proved that when $\Omega \subset \mathbb{R}^2$,

$$e(x, x, \lambda) = \lambda^3 / 4\pi + o(\lambda^2), \quad \lambda \to \infty,$$

uniformly on compact subsets of $\Omega$. Combined with a fairly simple a priori bound allowing integration of (4) over $\Omega$, this implies the two-dimensional analogue of (2). To prove (4), Carleman studied the kernel of the resolvent $(A - z)^{-1}$ of the selfadjoint operator $A$ in $L^2(\Omega)$ defined by $-\Delta$ with Dirichlet boundary conditions, and deduced that the ‘zeta function’ $\sum |\phi_j(x)|^2 / \mu_j$ differs from $1/4\pi(s-1)$ by an entire analytic function. An application of Ikehara’s Tauberian theorem gave (4). The idea of studying the kernel of some function of the selfadjoint operator $A$, chosen so that it satisfies a differential equation, and then applying some Tauberian theorem, has been essential in almost all later work on spectral asymptotics. The most common functions were at first $(A - z)^{-k}, z \in \mathbb{C}, k \in \mathbb{Z}_+, \text{and } e^{-tA}, t > 0$. This led to the extension of (2) to general semibounded elliptic operators in the early 1950s (Browder, Gårding, and others). The development of pseudodifferential calculus in the 1960s allowed more explicit constructions of fundamental solutions. This made it possible for Greiner and Seeley to prove around 1970 that if $\mu_1 \leq \mu_2 \leq \cdots$ are the eigenvalues of an elliptic operator $A$ of order $2m$ on a manifold $\Omega$ of dimension $n$, with the domain of $A$ defined by elliptic boundary conditions on $\partial \Omega$, then there is an asymptotic expansion

$$\text{Tr } e^{-tA} = \sum_j e^{-\phi_j} \sim \sum_{k=0}^{\infty} \frac{h_k t^{(k-n)/2m}}, \quad \text{as } t \downarrow 0.$$

Similar essentially equivalent expansions were obtained for the resolvent at infinity. The coefficients $h_k$ are defined quite explicitly in terms of the symbols of the operators in the interior and on the boundary. A simple Tauberian argument shows that (5) implies that

$$N(\lambda) = c_0 \lambda^n + o(\lambda^n), \quad \lambda \to \infty, \quad c_0 \Gamma(1+n/2m) = h_0,$$

where $N(\lambda)$ is the number of eigenvalues $\mu_j \leq \lambda^{2m}$. If (6) could be refined to two-term asymptotics

$$N(\lambda) = c_0 \lambda^n + c_1 \lambda^{n-1} + o(\lambda^{n-1}), \quad \lambda \to \infty,$$
then it follows that we must have \( c_k \Gamma(1 + (n-k)/2m) = h_k \), \( k = 0, 1 \), so both coefficients are known. In the book under review, it is stated that Weyl conjectured in his 1913 paper that for the Dirichlet problem in a domain in \( \mathbb{R}^3 \) with volume \( V \) and surface area \( S \) of the boundary, it is possible to improve (2) to
\[
N(\lambda) = V/6 + S/16\pi + o(\lambda), \quad \lambda \to +\infty.
\]
(8)

However, the reviewer can find only (3) but not this conjecture in the paper of Weyl; in fact, Weyl wrote in a survey paper in *Bull. Amer. Math. Soc.* 56 (1950) (p. 130):

‘I have certain conjectures on what a complete analysis of their asymptotic behavior should aim at; but since for more than 35 years I have made no serious attempt to prove them, I think I had better keep them to myself.’ However, (8) is generally referred to as the Weyl conjecture, and we shall honour this tradition even if it is not quite accurate. The subject of the book under review is the work which has been done on verifying, modifying and extending the ‘Weyl conjecture’.

The first important step toward two-term asymptotics was taken by Levitan and Avakumović in the 1950s. For a positive second-order elliptic operator \( A \) on a manifold \( \Omega \) without boundary, consider the cosine transform of the spectral measure
\[
\cos(t\lambda) dE_{\lambda}.
\]
It is a fundamental solution of the wave operator \( \partial^2/\partial t^2 + A \), so the local singularities can be determined with any desired precision by a classical construction of Hadamard using corresponding geodesic coordinates. If \( \rho \in \mathcal{S}(\mathbb{R}) \) is a function with Fourier transform supported close to the origin, this gives at once an asymptotic expansion of a convolution \( \int \rho(\lambda - v) d\lambda(x, v) \) as \( \lambda \to \infty \), which implies (5), and by an elementary Tauberian argument using the monotonicity of \( e(x, x, v) \) proves that
\[
e(x, x, \lambda) - c_q(x) \lambda^n = O(\lambda^{n-1}).
\]
(9)

In 1968, the reviewer extended this to an arbitrary semibounded elliptic operator of order \( 2m \), on a manifold without boundary, by studying the Fourier transform
\[
\exp(-itA^{1/2m}) = \int \exp(-i\lambda) dE_{\lambda}.
\]
(10)

Here \( A^{1/2m} \) is a pseudodifferential operator, so this is a fundamental solution of the hyperbolic pseudodifferential operator \( \partial^2/\partial t - A^{1/2m} \). It could be analysed by means of Fourier integral operators which were developed for this purpose. The error estimate (9) is also valid on compact subsets of \( \Omega \) when \( \Omega \) is not compact and \( A \) is a semibounded selfadjoint extension of an elliptic operator, and it has applications to the convergence properties of expansions in eigenfunctions of \( A \). However, even for an elliptic boundary problem on a \( C^\infty \) manifold with boundary, an integration over \( \Omega \) leads only to an analogue of the Weyl–Courant estimate (3). The first error estimate,
\[
N(\lambda) = c_0 \lambda^n + O(\lambda^{n-1}), \quad \lambda \to \infty,
\]
(11)
was obtained by Seeley in 1978 for the Dirichlet problem for the Laplacian in \( \Omega \subset \mathbb{R}^3 \). The novelty was an improved estimate of the form (9) near the boundary, obtained by a variant of the Hadamard parametrix construction involving geodesics reflected once in the boundary. This opened the possibility of approaching two-term asymptotics such as (7) and (8). However, it was already observed by Avakumović that for the Laplacian on the sphere \( S^n \), the high multiplicities of the eigenvalues make it impossible to improve (11) to (7). In 1974, Duistermaat and Guillemin
extended the method of the reviewer to prove for manifolds without boundary that (7) is never true when the Hamilton flow of the principal symbol of $A^{1/2m}$ is periodic and the subprincipal symbol vanishes, because there are large clusters of eigenvalues. On the other hand, they proved that (11) can be improved to (7) with $c_1 = 0$ if the set of periodic orbits has measure zero. The proof required a study of (10) for arbitrarily large $t$. For the Dirichlet problem for the Laplacian in a Riemannian manifold with boundary, Ivrii finally proved the 'Weyl conjecture' (8) in 1980 under the assumption of zero measure for the set of closed broken geodesics, that is, curves which are geodesic in the interior and reflected at the boundary by the laws of geometrical optics.

The ambitious goal of the book under review is to prove results of this form for higher-order elliptic differential operators with fairly general differential elliptic boundary conditions, allowing also a 'nonclassical' second term which may not be a power of $\lambda$ as in (7). Theorem 1.2.1 states that (11) is always valid, of course with the classical value of $c_0$. Theorem 1.6.1 gives asymptotics of the form (7) under the hypothesis that the broken orbits of the Hamilton flow associated with the operator which are periodic or experience infinite reflections in a finite time form a set of measure 0. (Note that reflection is much more complicated than in the second-order case, since a ray impinging on the boundary may branch into several reflected rays.) In Theorem 1.7.14, certain periodic Hamilton flows are admitted, and the second term in (7) is then supplemented by a term taking the periodic orbits into account. Section 1.8 is devoted to two-term asymptotics for the spectral function improving (9).

The approach is again based on the study of (10). The operator $A^{1/2m}$ is not easily manageable, so the authors work instead with the differential operator $(i\partial / \partial t)^{2m} - A$. An essential point (Section 3.1) is that the characteristics of this operator which violate hyperbolicity can be handled fairly easily. Here is a short summary of the contents.

Chapter 1 presents without proofs the symplectic geometry and symbolic calculus required to state the main results.

Chapter 2 develops pseudodifferential operators and Fourier integral operators. The presentation starts from scratch, but the reviewer suspects that it would not be easy to follow for a reader without some previous experience of microlocal analysis. On the other hand, an expert in microlocal analysis may find it time-consuming to locate the points where something non-standard happens.

Chapters 3 and 4 give further technical preparations for the proof of the main results in Chapter 5. The study of the kernel of (10) is there divided into three zones. In a boundary zone where the boundary distance is $\ll \lambda^{-1}$ for a certain fixed small $\varepsilon$, the boundary operators play an essential role, and it is studied by means of resolvent estimates such as those which lead to (5). In the interior zone, at a fixed small distance from the boundary, a construction is made using Fourier integral operator techniques outside a set of small measure in the cotangent space where the Hamilton flow is not well behaved. The intermediate zone is divided into parts where the boundary distance does not vary by more than a fixed factor. The study of reflected Hamilton orbits is avoided entirely by separating orbits approaching or departing from the boundary, and noting that an estimate of the Fourier transform of a positive measure in one interval implies an estimate in the interval which is symmetric with respect to the origin.

Chapter 6 gives simple examples which illustrate the numerical importance of the second term, and also some more complicated examples which involve systems and really go beyond the theory developed in the book.
Finally, there are five appendices giving various technical parts of the proof. One was written by A. Holst and one by M. Levitin.

In the reviewer’s opinion, this book is indispensable for serious students of spectral asymptotics. It is not easy to read, but to a large extent this is no doubt unavoidable for a subject which is very demanding technically. The introductory chapter is very helpful in giving a general idea of the subject, but it might have eased the later presentation if a more geometric language had been used. Unfortunately, very few references are given in the text except to the publications of the authors, so the reader is to a large extent left to find out elsewhere what is new and what is standard knowledge in the field. This is the reason why I have tried to give at least a broader historical outline here.

LARS HÖRMANDER

ANALYTIC SEMIGROUPS AND SEMILINEAR INITIAL BOUNDARY VALUE PROBLEMS
(LMS Lecture Note Series 223)

By KAZUAKI TAIRA: 164 pp., £21.95, LMS Members’ price £16.45,
isbn 0 521 55603 1 (Cambridge University Press, 1995).

These lecture notes deal with the existence and uniqueness of solutions of elliptic equations. Let \( \Omega \) be a bounded and smooth domain in \( \mathbb{R}^n \), and let \( \Gamma \) denote its boundary. The main interest is in the study of the boundary condition

\[
Bu = a \frac{\partial u}{\partial v} + bu = 0 \quad \text{on } \Gamma,
\]

where \( a \) and \( b \) are real \( C^\infty \) functions satisfying

\[
(H_a) \quad a(x) \geq 0, \quad b(x) \geq 0, \quad x \in \Gamma,
\]

\[
(H_b) \quad b(x) > 0 \quad \text{on } \Gamma_a = \{ x \in \Gamma : a(x) = 0 \}.
\]

Taira introduces a scale of Besov type spaces adapted to these boundary conditions: \( B_{m-1/p}^{m-1-1/p} (\Gamma) \), where \( m \geq 2 \) is a natural number. One of his main results reads as follows.

Let \( f \in H^m_\tau (\Omega) \) and \( \phi \in B_{m-1-1/p}^{m-1-1/p} (\Gamma) \) be given functions. Then there exists a unique solution \( u \in H^\infty (\Omega) \) of the boundary value problem

\[
\begin{align*}
Au(x) &= f(x), \quad x \in \Omega, \\
Bu(x) &= \phi(x), \quad x \in \Gamma.
\end{align*}
\]

Here \( A \) is a second-order elliptic differential operator with real \( C^\infty \) coefficients.

Next, Taira studies existence, uniqueness and regularity of the solutions of the semilinear parabolic differential equation

\[
\left( \frac{\partial}{\partial t} - A \right) u(x, t) = f(x, t, u, \text{grad } u) \quad \text{in } \Omega \times (0, T),
\]

\[
Bu(x, t) = a(x) \frac{\partial}{\partial v} u(x, t) + b(x) u(x, t) = 0 \quad \text{on } \Gamma \times [0, T),
\]

\[
u(x, 0) = u_0(x) \quad \text{in } \Omega.
\]
In proving existence, regularity and uniqueness, Taira makes use of the theory of analytic semigroups combined with the boundedness of pseudo-differential operators on Besov spaces.

The book contains an introduction and seven chapters. In the introductory part, problems (1) and (2) are formulated and the main results are stated. Chapter 1 is devoted to a review of standard topics from the theory of analytic semigroups. Here also, complete proofs of local existence and uniqueness theorems for the abstract linear and semilinear Cauchy problems are given. In Chapter 2 the function spaces are introduced, and a detailed proof of generalized Gagliardo–Nirenberg estimates is presented. The approach used here is more or less elementary but of some length. It is based on the classical derivation of Gagliardo, and does not apply modern tools from the theory of function spaces. Chapter 3 contains a very rough introduction to the $L_p$ theory of pseudo-differential operators. In Chapters 4 and 5, a proof of the existence and uniqueness of the solution of problem (1) is given. Finally, Chapters 6 and 7 are used to establish the same in the case of problem (2).

The book is essentially self-contained, with two exceptions: the theory of pseudo-differential operators and, in particular, the boundedness of these operators in Besov spaces; and the theory of Sobolev and Besov spaces, and, in particular, the trace theorem which connects them. Otherwise, the book contains detailed and readable proofs.

There are other books in this field, investigating semilinear parabolic boundary value problems in connection with analytic semigroups: compare, for example, the recent book of Lunardi [1] and the references given there. In comparison with the existing literature, one of the advantages of this book is the concentration on the two problems mentioned above. So the strategy for tackling problems (1) and (2) becomes more obvious.

Reference

1. A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems (Birkhäuser, Basel, 1995).

W. Sickel
quantum mechanics as it had been formulated by Dirac and others. Self-adjoint operator algebras which are closed in the weak operator topology were first studied by Murray and von Neumann in the 1930s under the name *rings of operators* (they are now known as von Neumann algebras). Slightly later, Gelfand and Naimark investigated the structure of the more general class of norm-closed self-adjoint operator algebras, or C*-algebras as they are known.

Until the end of the 1960s, there was no more than a handful of textbooks on the subject, two volumes by Dixmier occupying a central position. The situation has changed since the early 1970s. There have been spectacular developments both in the theory of von Neumann algebras, and in the study of C*-algebras, where new techniques such as those of K-theory have revolutionised the subject. The last twenty years have seen the appearance of a plethora of books, both introductory and more specialized, on C*- and von Neumann algebras. At the same time the subject has expanded vigorously and formed quite unexpected links with other parts of mathematics, such as differential geometry and knot theory. Whereas a student starting research in 1970 might reasonably have expected to encompass the whole area, its scope has now become so vast, its topics so disparate, that there are few even among the most experienced who are master of more than one corner of the subject. It has become unreasonable to expect any single book to give a comprehensive account of more than a few of its aspects.

One of the difficulties for anyone currently embarking upon research in operator algebras is to gain a complete overview of the subject. In particular, much of the current research on von Neumann algebras, on subfactors for example, has relatively little contact with other parts of operator algebras, though there are significant connections with other parts of mathematics and physics. There is a clear need for works which survey and connect major parts of the subject, especially those parts that have not hitherto received adequate textbook exposure, without necessarily treating every topic exhaustively. It is also important that the area should be made more accessible to mathematicians in other disciplines. The two books under review, though very different in scope, break new ground and go some way to meeting these needs. Both authors are based in Canada, and the books, each based on a graduate course, were written during a special year on operator algebras held at the Fields Institute in Waterloo, Ontario, in 1994–95.

As Davidson’s title indicates, his emphasis is on the properties of particular classes of C*-algebras. One of the most fruitful lines of research in C*-algebras in recent decades has been the investigation of such specific classes as the AF (approximately finite-dimensional) algebras, irrational rotation algebras and Cuntz algebras, using K-theory and other methods. Davidson’s book is, in the main, a fairly full account of the structure theories of these algebras.

The book is reasonably self-contained. After a concise introduction to the basic theory of C*- and von Neumann algebras, there is a useful chapter on spectral theory and what have come to be known as Weyl–von Neumann results, the most celebrated of which is Voiculescu’s theorem. The core of the book is a presentation of the theories of AF algebras, various algebras of isometries, irrational rotation algebras and group C*-algebras. The two chapters devoted to AF algebras give one of the most complete expositions of this class to be found, and include the classification by dimension groups and some nice applications. The other classes of algebras are perhaps less amply explored, but there is much useful material and the ideas are well explained. The final chapter, which contains an account of the Brown–Douglas–
Fillmore theory of extensions for abelian C*-algebras, has rather little connection with what has gone before. Its inclusion is perhaps motivated primarily by the author's own research interests.

It is most useful to have all this material collected under one cover. What one misses, though, is more of the general picture. One of the as yet unattained goals in C*-algebra theory is a general classification of C*-algebras. Important steps towards this were taken in the 1960s, by Glimm with his results on type I C*-algebras, and by Takesaki, and later others, with the formulation of the concept of a nuclear C*-algebra. More recently, Kirchberg has obtained profound results on exact C*-algebras, a class which includes the nuclear algebras. An extra chapter giving at least a brief survey of this work would have been most welcome, given that the classes of C*-algebras discussed in the text happen to be the principal examples of nuclear and exact C*-algebras. I was also a little disappointed to find few pointers to current research. After all, Elliott has, in the last few years, instigated a vigorous programme to classify separable nuclear C*-algebras. Nevertheless, I consider this book very useful as a compendium of the common classes of nuclear C*-algebras, to which it provides an excellent introduction. A good collection of exercises is provided.

Fillmore has, by contrast, attempted the daunting task of surveying almost the entire field of self-adjoint operator algebras. In a book of little over two hundred pages it is an achievement just to set the scene and state the major results, but he achieves this and more with remarkable deftness. There is almost no part of operator algebra theory that is not touched upon, though many topics are sketched only lightly. Topics covered include von Neumann algebras, the structure of factors and Jones' theory of subfactors; the structure of type I C*-algebras; injective von Neumann algebras and nuclear C*-algebras; crossed products of C*- and von Neumann algebras; and Hilbert C*-modules, K-theory and Kasparov’s KK-theory. Many of the important examples of C*- and von Neumann algebras are introduced.

The approach has an attractive simplicity, resulting partly from the omission of much detail and of the more involved proofs. A felicitous choice of notation makes the printed text unusually easy on the eye. The order in which the material is presented is well thought out, and the space given to different topics judiciously apportioned. Moreover, the treatment is topical, with many indications of the direction in which current research is going. It was pleasing to find concise accounts of Jones' fundamental results on subfactors and their connection with knot theory, and of the E-theory of Connes and Higson.

This book provides an ideal way for someone approaching operator algebras for the first time to survey the subject. Almost every major area of current research, with the exception of cyclic cohomology, is represented. Although the treatment of any given topic is inevitably limited, in many cases just a taster, there are ample references to other, fuller, accounts (including Davidson's book).

To whom should one commend these two volumes? To research students, certainly, but experienced operator algebraists will also find Davidson's book a valuable reference work to add to their library. On the other hand, Fillmore's book is one of the most accessible introductions to operator algebras for non-specialists. Of course, most topics receive only a thumbnail sketch, yet I am sure that even experienced workers in the area will enjoy browsing through this book. I certainly did. However, I am left with the feeling that its price is on the high side considering
its length, and the fact that it is a survey rather than a reference work. Perhaps a paperback edition selling for about £25 would have been more appropriate.

Simon Wassermann

FUNCTION SPACES, ENTROPY NUMBERS AND DIFFERENTIAL OPERATORS
(Cambridge Tracts in Mathematics 120)

By David E. Edmunds and Hans Triebel: 252 pp., £40.00, ISBN 0 521 56036 5

In the modern study of differential equations, a boundary-value problem is viewed as an equation for a (differential) operator defined on a subspace of some underlying function space which has a natural connection with the problem. If one is concerned with the Lebesgue spaces $L_s(R^n)$, then the domain of the operator is usually given in terms of a Sobolev space $W_1^s(R^n)$ of functions having derivatives (in the weak or distributional sense) up to order $k$ in $L_s(R^n)$, where $k$ is the order of the differential equation. Situations often arise in which the standard Sobolev spaces are inadequate for the problem under consideration, and in such a case finer scales of function spaces are introduced. Properties of these function spaces, like the existence and form of Hölder-type inequalities, embedding theorems and the nature of the embedding maps, provide invaluable information. For instance, regularity theory uses embedding theorems to determine the degree of smoothness of solutions of a differential equation which are initially given merely as members of the abstract operator domain in the function space; the compactness or otherwise of the embedding map has implications for the spectral properties of the differential operator, in particular, whether or not its spectrum consists only of eigenvalues.

The scales of function spaces of main concern in this book are those of type $B_{p,q}^s$ and $F_{p,q}^s$ for $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. In the full range of indices considered, they are quasi-Banach spaces, and include many of the well-known classical spaces: for $p \in (1, \infty)$, $F_{p,q}^s(R^n) = L_s(R^n)$ when $s = 0$, and $F_{p,q}^s(R^n) = W_1^s(R^n)$ when $s \in \mathbb{N}$; for $p \in (0, \infty)$ and $s \in \mathbb{R}$, $F_{p,q}^s(R^n)$ is the fractional Sobolev space $H_1^s(R^n)$, and $F_{p,q}^s(R^n) = h_s(R^n)$, the inhomogeneous Hardy space; $B_{p,q}^s(R^n)$, $s \in \mathbb{R}$, are the Hölder–Zygmund spaces. Embedding theorems for these spaces $B_{p,q}^s$, $F_{p,q}^s$ are of special interest when the ‘differential dimension’ $s - n/p = 0$. As a typical example, if $\Omega$ is a bounded domain in $R^n$, then $H_1^{a/p}(\Omega) = F_{1,1}(\Omega)$ is continuously embedded in the Zygmund space $L_s((\log L)_s(\Omega))$ when $1 < p < \infty$ if and only if $a \leq (1/p) - 1$, and the embedding is compact if and only if $a < (1/p) - 1$; when $0 < p \leq 1$, $H_1^{a/p}(\Omega)$ is continuously embedded in $L_s(\Omega)$ and $\text{bmo}(\Omega)$, but neither embedding is compact.

The book is essentially in three parts, corresponding to Chapter 2, Chapters 3 and 4, and Chapter 5, and is largely a coherent account of the impressive body of work produced by the authors and their co-workers in recent years. The first part is a rich store of up-to-the-minute results on function spaces and the powerful techniques developed to handle the intricate analysis. In the second part, upper and lower bounds are derived initially (Chapter 3) for the entropy and approximation numbers of embeddings between function spaces in the scales $B_{p,q}^s(\Omega)$ and $F_{p,q}^s(\Omega)$ when $\Omega$ is a bounded domain in $R^n$, the results for entropy numbers being optimal for an $\Omega$ with
a $C^\infty$ boundary; in Chapter 4 the estimates are obtained for weighted spaces with $\Omega = \mathbb{R}^n$. In the final (and climactic) chapter, the authors apply the earlier results, and a variant of the Birman–Schwinger Principle given in terms of entropy numbers, to the determination of bounds on the eigenvalues of operators of the form $B = b_2 C^{-1} b_1$, where $b_1, b_2$ lie in some function space $H^s$, and $C$ is a regular differential operator, a fractional power of it or a pseudo-differential operator. These estimates are then used to obtain bounds for the number of negative eigenvalues of self-adjoint operators in $L^2$ of the form $A - \beta b$ for some positive definite $A$.

The authors’ mastery of the subject is obvious, and they make every effort to guide the reader through the difficult analysis. One has to work hard to assimilate the material, but the effort is worthwhile. The book is recommended to anyone with an interest in function spaces and differential equations.

W. D. Evans

METRICS, CONNECTIONS AND GLUING THEOREMS
(CBMS Regional Conference Series in Mathematics 89)

By Clifford Henry Taubes: 90 pp., USS15.00, ISBN 0 8218 0323 9

For many years, topologists interested in manifolds found smooth 4-dimensional manifolds the most difficult to understand and classify. In 1983, Simon Donaldson used gauge theory (then a branch of physics) to discover new information about 4-manifolds that was, and is still, inaccessible using conventional topological techniques. His method was to study a nonlinear, elliptic partial differential equation called the anti-self-dual or instanton equation on the 4-manifold. The topology of the set of solutions to this equation, called the moduli space, carries valuable information about the 4-manifold. To extract this information, and understand the properties of the moduli space, requires a lot of difficult analysis.

The subject of Taubes’ book is the analysis of the anti-self-dual equations. Two versions of the equation are studied. The first, anti-self-dual connections on a vector bundle over a fixed Riemannian 4-manifold, is the central plank of Donaldson theory. The second version, anti-self-dual conformal metrics on a 4-manifold, has deep connections to twistor theory and complex geometry.

The author first reviews the anti-self-dual equations and elementary properties of the moduli spaces, and gives some motivation from Donaldson theory. The central portion of the book proves the existence of solutions to the anti-self-dual equations, first for connections and then for metrics. The construction uses an idea called grafting. A known solution to the equations on $\mathbb{R}^4$ is used to make an approximate solution on the 4-manifold, which is then deformed to a nearby exact solution. The deformation process is explained in detail, with careful attention to the analytic details. The book finishes with a list of open problems.

This book is easy to read, and is well written in a pleasant, informal style, with occasional humour. It should be accessible to graduate students in differential geometry, and others; a grounding in the differential and Riemannian geometry of manifolds, with some understanding of vector bundles and connections, is about all that is required.

Dominic Joyce
INDEX THEORY, COARSE GEOMETRY, AND TOPOLOGY OF
MANIFOLDS
(CBMS Regional Conference Series in Mathematics 90)

By John Roe: 100 pp., US$17.00, ISBN 0 8218 0413 8

Having chosen a set of generators, the word length function makes a discrete group $\Gamma$ into a metric space $[\Gamma]$. Two different generating sets will generally produce two different metrics, but if both sets are finite then the large-scale features of the two metric spaces $[\Gamma]$ will be the same. If $M$ is a compact Riemannian manifold with fundamental group $\Gamma$ then the large-scale geometry of $[\Gamma]$ is the same as the large-scale geometry of the universal covering space $\hat{M}$. Coarse geometry is this ‘large-scale geometry’ of metric spaces, and as the above examples might suggest, its most important theorems lie in the region connecting group theory to Riemannian geometry. The subject has blossomed in recent years, thanks largely to remarkable work by Mikhael Gromov, especially his notion of word-hyperbolic group which incorporates the notion of negative curvature into combinatorial group theory.

In the present context index theory is the study of the Fredholm indices (dimension of the kernel minus dimension of the cokernel) of linear elliptic operators on manifolds. In the first instance the manifolds are compact, so that the kernels and cokernels of linear elliptic operators are finite dimensional, and the Fredholm indices are integers: the celebrated Atiyah–Singer index theorem is, of course, the central result. There are a number of possible extensions of index theory to the realm of noncompact manifolds. Most can be viewed as attaching to an elliptic operator not an integer Fredholm index but an index in the $K$-theory group of some operator algebra. An important problem—which often must precede the formulation of an index theorem—is to calculate the $K$-theory groups which arise in this way.

John Roe begins his attractive CBMS memoir by linking coarse geometry to the index theory of elliptic operators on complete Riemannian manifolds. A Dirac-type operator $D$ on $X$ is invertible modulo the algebra $S(X)$ of those smoothing operators whose Schwartz kernels are supported within a uniform neighbourhood of the diagonal in $X \times X$; it follows that $D$ has an index in the $K$-theory group of $S(X)$. Now it turns out that the algebra $S(X)$ and its $K$-theory depend more or less only on the coarse geometric structure of $X$, and so index invariants of elliptic operators on $X$ are coarse invariants of $X$. This begins a useful interplay of ideas between index theory and coarse geometry, building on the many links between the Atiyah–Singer theorem and differential geometry. But the main theme of Roe’s lectures is the unexpectedly close relationship (first noted by Shmuel Weinberger) between this index theory–coarse geometry mixture and surgery theory, particularly the fairly recently developed ‘bounded’ surgery theory. It is impossible to summarize the connection quickly, except to say that it is formed around the signature operator on $X$, and that a key role is played by $C^*$-algebras, particularly the $C^*$-algebra completion $C^*(X)$ of the smoothing operator algebra $S(X)$. Roe has managed to organize all these parts into an attractive whole which includes: a conjectural formula for the $K$-theory of $C^*(X)$, analogous to the Baum–Connes conjecture in $C^*$-algebra $K$-theory; a common framework for the Novikov higher-signature conjecture and the Gromov–Lawson–Rosenberg positive scalar curvature conjecture; and a ‘descent principle’ which makes these conjectures accessible by coarse geometric techniques. One very nice
consequence is a coarse proof of the Novikov conjecture for word-hyperbolic groups. An outstanding open problem is to properly understand the relation of this coarse proof to the original analytic proof of Alain Connes and Henri Moscovici.

Roe's memoir is a clear introduction to a subject which is obviously quite complicated, drawing as it does on topics as far apart as surgery theory and $C^*$-algebras, to mention just two. Several concise accounts of background topics make the book accessible to quite a wide audience, and in fact these accounts should be of interest to a variety of readers who are not necessarily intent on studying the whole of the book: for the topologists there are sketches of $C^*$-algebra theory, $C^*$-algebra $K$-theory and Kasparov theory; for the analysts there is a marvellous lecture devoted to surgery theory and the positive scalar curvature problem. But for those interested in acquainting themselves with the $C^*$-algebraic index theory and its applications, the entire book should be required reading.

NIGEL HIGSON

DYNAMICAL SYSTEMS: STABILITY, SYMBOLIC DYNAMICS AND CHAOS

By CLARK ROBINSON: 468 pp., US$65.95, ISBN 0 8493 8493 1
(CRC Press, 1995).

The influential survey article [4] by Stephen Smale set out a framework for a global qualitative theory of dynamical systems which has underpinned much of the progress in understanding nonlinear dynamics over the subsequent thirty years. It drew attention to the important role played in the dynamics of a flow or iterated map by the nonwandering set $\Omega$ (the set of points all of whose neighbourhoods eventually return to themselves), and to the significance of uniform hyperbolic structure of exponential attraction/repulsion on $\Omega$. This structure not only provides a way into coding the dynamics on $\Omega$ in terms of symbol sequences via Markov partitions, but is crucial to the proof of structural stability or robustness of the system as a whole. Structurally stable systems are those whose dynamical properties remain qualitatively unchanged when the system is subjected to small perturbations (a lot of machinery from analysis and topology is needed to make precise mathematics out of this, of course), and common sense seems to dictate that these should be the systems of most relevance to applications in the real world—with certain exceptions, such as in celestial mechanics, for example, where energy is conserved and the rules of the game are rather special.

Whether or not this philosophy is convincing, it is the case that structural stability has provided an important handle on understanding and classifying many interesting types of dynamical behaviour. Moreover, if a given system is not structurally stable, then it seems reasonable that a small nudge should convert it into one which is. Reasonable or not, however, this is in fact false, as Smale also showed [3]: structurally stable systems are not dense in the (suitable defined) space of all systems on a given manifold. This means that a typical or generic system cannot safely be assumed to be stable. So what then is generic behaviour? What are the dynamical properties we can expect to see in ‘almost all’ systems? Various plans were set out (and even divinations attempted using the I Ching [1]) for extensions of the Smale program to address this problem. Meanwhile, however, in another part of the forest ...
The story has often been told of how Ed Lorenz stumbled across chaotic dynamics in the system he was studying in the early 1960s to model atmospheric flow [2], now well known as the Lorenz equations. Other systems with strange-looking attracting sets studied by physicists (M. Hénon), engineers (Y. Ueda) and others began to emerge from the applied sciences in increasing numbers in the 1970s. The stage appeared set for a triumphant meeting of theory with experiment—but, alas, the bride would not come to the ball. Experiments refused to conform to the existing mathematical theory. Evidently the experimental systems had some kind of robustness properties in order for them to be repeatedly observed, but uniform hyperbolicity did not get beyond first base. Not for the first time in applied mathematics, there was clear water between the order and discipline of the theory and the untidiness of the observed phenomena. Many of the more recent developments in the theory of dynamical systems have been motivated by attempts to bridge this gap.

Of the multitude of books on nonlinear dynamics and chaos to appear in the last few years, this volume has a particular niche as a straight mathematical text, comprehensive in its treatment of the Smale theory and many of its natural developments, and yet accessible to beginning graduate students. It does not include complex dynamics (so no pictures of Julia or Mandelbrot sets), nor does it discuss Hamiltonian systems or much ergodic theory. Knowledge of calculus and analysis is assumed up to the Implicit Function Theorem, linear algebra up to Jordan canonical form, and some point set topology, but not including differential topology or theory of manifolds (most of which here are, in any case, Euclidean space, tori or graphs of functions). However, as the author emphasises, it is one thing to know the proof of a key result, but quite another to understand how it is used and why it is important. Thus, throughout the book, theorems and lemmas are typically preceded and followed by comments on why a result is needed and what are the main points of the argument, and sometimes also a comparison of alternative proofs. It is a comfortable book to read.

The main text begins with 1-dimensional dynamics of iteration, introducing limit sets, the nonwandering set $\Omega$, and the chain recurrent set $\mathcal{R}$ for a continuous map on a metric space. There is detailed study of the dynamics on the uniformly hyperbolic invariant Cantor set possessed by the system $F_\mu(x) = \mu x(1-x)$ with $\mu > 4$, and its representation in terms of a shift operation on symbol sequences—a paradigm for the general theory to come. Sharkovskii’s Theorem is proved in full (some details left as routine exercises), leading to subshifts of finite type and so another glimpse of the wider theory.

Proofs of fundamental local results in $n$ dimensions, such as the Hartman–Grobman Theorem and the Stable Manifold Theorem, are included, emphasising the important role of the Contraction Mapping Theorem (also proved). Standard uniformly hyperbolic examples such as horseshoes and hyperbolic toral automorphisms are thoroughly discussed, as are the celebrated nonuniform examples of the Hénon map and the Lorenz attractor as far as their structure is known. The global theory of systems with uniform hyperbolic structure on $\mathcal{R}$ (often assumed merely on the closure of the set of periodic points or the closure of the union of limit sets) is given full treatment. Some of this theory does not need hyperbolicity, and can be seen as a particular case of Charles Conley’s fundamental theorem on the decomposition of an arbitrary system into chain-recurrent pieces with gradient-like flow between them: this theorem is proved, and important related results on recurrence and attracting sets are explored. Hyperbolicity, however, puts a firm grip on the general
structure and allows deeper analysis, including the construction of Markov partitions and the \( \Omega \)-Stability Theorem; all this is proved here in detail. There are comments on the Structural Stability Theorem, the pinnacle of the global theory which the author was among the first to scale, and key results on generic properties of systems and on necessary conditions for stability. Of more current interest to applications are the techniques for measurement of chaos such as Liapunov exponents, fractal dimension and topological entropy, all discussed here in a rigorous yet highly accessible way (no mean feat for the latter, as far as this reviewer is concerned).

This book is a pleasure to work from, both as a reference work and as a vehicle for a lecture course. The core sections of the text are indicated, and there are abundant exercises that help to take care of some of the spadework. It is nicely typeset in AMS-\TeX, with few misprints, and reasonably priced: strongly recommended as a source for the mathematics of dynamical systems.

References


DAVID CHILLINGWORTH

VERTEX ALGEBRAS FOR BEGINNERS

(University Lecture Series 10)

\textit{By Victor Kac}: 141 pp., US$25.00, ISBN 0 8218 0643 2


I shall start by giving a vague idea of what a vertex algebra is, and why anyone should be interested in them. A vertex algebra is very roughly a sort of commutative ring with derivation, except that the ring multiplication has ‘singularities’. The relation between vertex algebras and commutative rings is roughly like the relation between rational functions and regular functions, as one can informally think of the vertex algebra as having a ring multiplication which is not defined everywhere because of some sort of poles. In particular, any commutative ring with derivation can be considered as a vertex algebra; unfortunately for beginners, these are the only examples that are easy to construct (for example, any finite-dimensional vertex algebra over the reals is really just a commutative ring with derivation). However, if one thinks of vertex algebras as some sort of generalization of commutative rings, this does at least give a good idea of many of their properties: for example, one can define modules over them, homomorphisms between them, and ideals, and one can (sometimes) take tensor products of modules or of two vertex algebras.

The reason vertex algebras are useful is that many infinite-dimensional spaces in areas of mathematics related to physics unexpectedly turn out to have a vertex algebra structure. Some typical examples are given by the highest weight representations of Kac–Moody algebras, the ‘moonshine’ module for the monster
finite simple group, spaces of physical states of chiral strings on a torus, representations of the Virasoro algebra of vector fields on a circle, and modules over (a central extension of) the Lie algebra of polynomial differential operators on a circle. Vertex algebras also turn up implicitly under several other names: vertex operator algebras, \( \mathcal{W} \) algebras, chiral algebras in conformal field theory, and meromorphic conformal field theories are all more or less vertex algebras with a small amount of extra structure.

Unfortunately, the field of vertex algebras is notoriously difficult to enter, with most people taking only one glance at the pages of gruesome formulas infesting the subject before deciding to do something easier. Kac’s book goes a long way towards remediying this problem. The past few years have produced a lot of cleaning up of the original definitions and constructions, so that the subject has become almost civilized. In particular, Goddard found a more conceptual definition of vertex algebras, and several people noticed that almost any ‘formally commuting’ set of vertex operators generates a vertex algebra in the same way that a set of commuting operators generates a commutative ring, which greatly simplifies many constructions. Because of these improvements, Kac’s book is much easier to read than most earlier expositions, and is the first book I would recommend to anyone entering the subject. (For some more advanced books and papers, see anything in Kac’s bibliography with the word ‘vertex’ in the title, in particular the book *Vertex operator algebras and the monster* by Frenkel, Lepowsky and Meurman.)

As the title suggests, this is mainly an introductory book, giving the definition and basic properties of vertex algebras, and constructing the main examples of them from Lie algebras and lattices, but stopping short of most applications. However, even most experts will find something new: the first two sections rapidly discuss the relation with quantum field theory (and should probably be skipped by most readers!), Chapter 4 mentions a non-commutative generalization of vertex algebras (which seems closely related to the quantum algebras defined by Lian and Zuckerman), and Chapters 2 and 5 discuss ‘conformal algebras’, which generate vertex algebras and have the advantage that they are finite dimensional and can often be classified (and which seem to be roughly what physicists mean by chiral algebras).

To summarize, I would rate this book as essential reading for anyone trying to learn about vertex algebras, and as being well worth buying for experts.

R. E. Borcherds

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**EQUIVARIANT HOMOTOPY AND COHOMOLOGY THEORY**

*(CBMS Regional Conference Series in Mathematics 91)*


The theory of topological transformation groups is a substantial part of algebraic and geometric topology. But more is true: consideration of geometrical objects from the viewpoint of symmetry has been an important method since the days of Felix Klein’s ‘Erlanger Programm’ in 1872. It is a useful principle to consider groups which
appear in algebraic topology not just as ‘invariants’ but rather as structural assertions about symmetry.

A typical first example is the fundamental group. It is best treated as the symmetry group of the universal covering. The advantage of this symmetry viewpoint lies in the fact that it immediately leads to the consideration of non-free group actions, and this in turn is essential for the use of functorial and representation-theoretic methods. Since fundamental groups are in general not compact Lie groups, this example is only occasionally the topic of the book, although some of the methods can and should be extended to the more general situation since discrete group actions have ever-increasing importance in geometric topology.

A typical second example is the codification of geometric properties in the language of bundle theory. This brings universal bundles and classifying spaces into the foreground, and non-free actions appear in the guise of the now ubiquitous classifying spaces for families of isotropy groups (introduced in 1972 by the reviewer, based on work of Bredon).

The importance of the book under review for algebraic topology in general emerges from the usefulness of this symmetry viewpoint. This viewpoint has to be elaborated in two ways. First, general experience shows that to each result or notion in ordinary topology, there exists an equivariant counterpart. Sometimes this is automatic. But often it is good for surprises or additional difficulties: for example, the lack of equivariant transversality leaves open, still, the basic question (asked by the reviewer nearly 30 years ago) about the relation between geometrical and homotopical equivariant bordism. Secondly, equivariant machinery is used to understand better or to prove non-equivariant results. A typical context is the use of non-free actions, the appearance of several groups, the use of methods from group and representation theory; for example, filtration by the set of subgroups (splitting theorems) or induction theorems (reduction to particular classes of groups like \( p \)-groups). It may seem unfortunate, but is unavoidable, that the equivariant world requires a lot of categorical book-keeping. In this book you will find important examples for both lines of thinking.

The book is not a textbook; in particular, it contains few proofs. Here are several of its aims.

1. A coherent treatment and presentation of a whole area of mathematics. Basic notions are explained clearly. Motivation and ideas are developed, and the notions are put into context.

2. Overview and guide to the literature (223 items in the Bibliography). The purpose is to inform the reader without assuming too much technical background.

3. To advertise certain fundamental technical tools; in particular, categories of spectra with good formal properties such that constructions of homological algebra, say, can immediately be imitated in topology. (A method, here called ‘brave new algebra’, which is the study of point-set level topological algebra in stable homotopy theory.) Example: topological Hochschild homology.

4. Description of recent progress in the area, based on the technical tools.

5. Introduction to some of the work of the authors. These parts tend to be more special themes, but they are well related to the mainstream.

The overall strategy is directed more to the explanation of the theory in general than to the presentation of specific calculations.

Since a detailed presentation of the material covered in this book requires thousands of pages (of not always easily accessible text), it seems absolutely necessary
to have this guide-book on the desk. This remark applies to the advanced student as well as to the educated scientist. The presentation is clear, reliable, informative and motivating.

What can you expect to find in the book? Among other things, the following.

1. The basics of equivariant (for compact Lie groups) topology reviewed: CW-complexes, bundle theory and classifying spaces, equivariant cohomology graded over the representation ring, stable homotopy.

2. Statements, generalizations, and the role in equivariant topology of by now famous results like: the Segal conjecture, the Sullivan conjecture, and Lannes theory.


5. A large portion devoted to categories of equivariant spectra and their basic properties, especially with respect to smash product. Non-technical explanation of the setting and basic properties.

6. Fundamental new results of the authors which show the usefulness of the techniques. For example, their proof of the localization and completion theorems in complex cobordism using local and (generalized) Tate cohomology and norm maps is the first basic progress in cobordism theory for many years.

Please read the table of contents (5 pages) and the Introduction (10 pages, dated 29 February and so pointing at something exceptional).

There is no comparable recent book in algebraic topology. One of its influences may be that it gives you ideas of what to teach in basic courses, namely prerequisites for this book. It also almost certainly guides further research. This remark applies especially to the fundamental part about ‘brave new algebra’. In this respect it may well serve as a reference book, also with respect to terminology and notation.

It should be pointed out, however, that the use of ideas from transformation group theory has larger scope than even this text suggests: there is no geometric topology, no discrete groups in general, little use of representation theory. Thus the reviewer cannot resist the joke: the work does not describe a universe but certainly a galaxy.

Tammo Tom Dieck

LÉVY PROCESSES
(Cambridge Tracts in Mathematics 121)

By Jean Bertoin: 265 pp., £35.00, ISBN 0 521 56243 0

It is at least fifteen years since I started to ask potential authors if they would write a connected account organising the results for Lévy processes in a self-contained monograph. Actually, the delay has been helpful, as we have seen during this period substantial progress in both the theory and applications of Lévy processes. The author of this book has himself made many important contributions to the developing theory, and he has had to arrange the material into an organised graduate
course for presentation at the Laboratoire de Probabilités in Paris V, so he is well equipped to write this account.

Any stochastic process in continuous time with stationary independent increments must have increments whose distribution is infinitely divisible. The class of such distributions is described by the Lévy–Khintchine formula, and the resulting process is called a Lévy process. A Markov process can be described as one for which the future and the past are independent given the present: the class of Lévy processes is the same as that of those Markov processes which are homogeneous in both space and time. Mathematical Brownian motion is the most studied as well as the most important example of a Lévy process, for it is the basic tool needed for stochastic analysis.

There has been a great deal of detailed research into the sample path properties of particular classes of Lévy processes. (For a particular process, a sample path property is one which is valid for almost all trajectories of the process.) Before this book appeared, it was frustrating to those looking for an answer to some easily formulated question about the sample path of a specific process that there was no reference book where one could discover the different pieces of machinery which are available and could be relevant to the question. This book will now be the first place to look, for it ranges widely in examining the concepts and techniques relevant to the study of Lévy processes.

After a preliminary chapter, the book summarises the effect of the Markov property and the use of probabilistic potential theory. In many ways, the use of this more general theory for Lévy processes is illuminating while being easier than the general case. An increasing Lévy process $X_t$ is called a subordinator. Its value at time $t$ is the sum of its jumps up to time $t$. The detailed properties of a general subordinator are developed in Chapter 3; they will be much used in later chapters. There is a detailed description of the excursion process and its use for understanding the structure of local time for a Markov process. In applying this to a Lévy process, the recent results of Barlow and Hawkes on the existence of a version of local time which is jointly continuous in time and space are described, and the necessary theory is developed.

Fluctuation theory in discrete time was developed by Spitzer and others; it relied heavily on quite deep analytic methods. If $X_t$ is a real-valued Lévy process and $S_t$ is its supremum up to time $t$, then $(S_t - X_t)$ is Markov and its local time at zero, $L(t)$, has an inverse which is called the ladder time process: this inverse of $L(t)$ is always a subordinator. The sample path properties of subordinators now give useful ‘fluctuation identities’, which lead to interesting results including the surprising arcsine law valid for any Lévy process for which $P(X > 0)$ is the same for all $t$.

Chapter 7 explores the properties of Lévy processes with no positive jumps. This forces the tail of certain distributions to be much smaller, leading to precise asymptotic behaviour at 0 or $\infty$. Many sample path properties in the literature were first obtained for Brownian motion and then extended to the class of strictly stable processes. Many arguments become easier because strictly stable processes satisfy a scaling property so that a simultaneous change of time and space variables produces a new version of the same process. The final Chapter 8 deals with these stable processes, and develops a theory of bridges and excursions which is surprisingly simple.

The book is largely self-contained, with extensive references. Each chapter has a section of exercises in which the reader is asked to use the theory of the chapter to
obtain deep results which had often been first found by different methods. When appropriate, the exercise contains a hint, or even the outline of an argument. The last section of each chapter is headed ‘Comments’: this provides a detailed bibliography, and a discussion of different approaches to the material in the chapter, as well as further extensions. This book would form an excellent text for a graduate course intended for probabilists. But it also provides a quick way of accessing and understanding fifty years of the work of many researchers. I strongly approve of both the choice of material and the pleasant manner in which it has been assembled.

S. J. Taylor