Let \((R, m)\) be a local Noetherian ring. We show that if \(R\) is complete, then an \(R\)-module \(M\) satisfies local duality if and only if the Bass numbers \(\mu_i(G, M)\) are finite for all \(i\). The class of modules with finite Bass numbers includes all finitely generated, all Artinian, and all Matlis reflexive \(R\)-modules. If the ring \(R\) is not complete, we show by example that modules with finite Bass numbers need not satisfy local duality. We prove that Matlis reflexive modules satisfy local duality in general, where \(R\) is any local ring with a dualizing complex.

1. Introduction

In this paper all rings are commutative Noetherian rings with identities, and all modules are unitary. For a ring \(S\) and an \(S\)-module \(M, E_S(M)\) (or simply \(E(M)\) when the ring is understood) will denote the injective hull of \(M\).

Let \(R\) be a local ring with maximal ideal \(G\), and let \(E\) denote \(E_R(G)\). Recall that for an \(R\)-module \(M\),

\[
H^n_G(M) = \{ x \in M | m^n x = 0 \text{ for some natural number } n \},
\]

and that the right derived functors of \(H^n_G\) are the local cohomology functors \(H^i_G\) with support in the maximal ideal. For an \(R\)-module \(M\), \(Soc_M\) will denote the Matlis dual \(\text{Hom}_R(M, E)\) of the \(R\)-module \(M\). In this situation, a module \(M\) is called Matlis reflexive if the natural map \(M \to M^\vee \vee\) is an isomorphism. The term complete will always mean complete with respect to the \(m\)-adic topology.

The \(i\)th Bass number of an \(R\)-module \(M\) with respect to \(m\) will be denoted \(\mu_i(m, M)\), or simply \(\mu_i(M)\), and is the vector space dimension of \(\text{Ext}_R^i(R/m, M)\) over \(R/m\). Equivalently, if \(0 \to M \to I^\vee\) is a minimal injective resolution of \(M\), then \(\mu_i(M)\) is the number of times \(E\) appears as a direct summand in \(I^\vee\) (see \[1,\ Section 2\]).

A dualizing complex for a ring \(S\) of dimension \(n\) is a complex

\[
D^\cdot : 0 \longrightarrow D^n \longrightarrow \cdots \longrightarrow D^0 \longrightarrow 0
\]

de of injective \(S\)-modules such that the following two conditions are satisfied:

(i) \(D^i \cong \oplus E(S/p)\) for each \(i\), where the sum is taken over all \(p \in \text{Spec } S\) such that \(i = n - \dim R/p\);

(ii) \(H^i(D^\cdot)\) is finitely generated for each \(i\).

If such a complex exists, we say \(S\) has a dualizing complex \(D^\cdot\). If \(S\) is a Gorenstein ring, then the minimal \(S\)-injective resolution of \(S\) provides a dualizing complex for \(S\).
Suppose now that $R$ has a dualizing complex $D$ and $\dim R = n$. By Proposition (2.3) of [2], there is a natural transformation of functors (from the category of all $R$-modules to itself)

$$\phi^0 : H^n_m(-) \longrightarrow [H^n(\text{Hom}_R(-, D))]^\vee.$$ 

When $R$ is Gorenstein,

$$\phi^0 : H^n_m(-) \longrightarrow \text{Ext}^n(-, R)^\vee.$$ 

This $\phi^0$ can be incorporated into a uniquely determined homomorphism of (strongly) connected sequences of functors

$$\Phi = (\phi^i)_{i \geq 0} : (H^n_m(-))_{i \geq 0} \longrightarrow ([H^{n-i}(\text{Hom}_R(-, D))]^\vee)_{i \geq 0}.$$ 

For $R$ Gorenstein,

$$\Phi = (\phi^i)_{i \geq 0} : (H^n_m(-))_{i \geq 0} \longrightarrow (\text{Ext}^{n-i}(-, R)^\vee)_{i \geq 0}.$$ 

When $M^\prime$ is a finitely generated $R$-module, $\phi^i_{M^\prime}$ is an isomorphism for all $i \geq 0$.

**Definition 1.** Let $R$ be a local ring with a dualizing complex, and let the maps $\phi^i$ be given as above. For any $R$-module $M$, we shall say that $M$ satisfies local duality if and only if $\phi^i_M$ is an isomorphism for all $i \geq 0$.

Grothendieck’s Local Duality Theorem, as given in [4] or [2], states that finitely generated modules satisfy local duality. In Section 2 we shall show that Matlis reflexive modules satisfy local duality. In Section 3 we consider the special case in which $R$ is a complete local ring, and show that an $R$-module $M$ satisfies local duality if and only if the Bass numbers $\mu_i(M)$ are finite for all $i$. The class of modules with finite Bass numbers includes all finitely generated, all Artinian, and all Matlis reflexive modules. If the ring $R$ is not complete, we show by example that modules with finite Bass numbers need not satisfy local duality.

2. Matlis reflexive modules

In this section, we shall prove that all Matlis reflexive $R$-modules satisfy local duality, where $R$ is any local ring with a dualizing complex.

We begin by recalling some general facts about Matlis reflexive modules. First, any module of finite length is Matlis reflexive. An $R$-module $M$ is Matlis reflexive if and only if $M^\vee$ is Matlis reflexive. Also, submodules and quotients of Matlis reflexive modules are Matlis reflexive, and finite direct sums of Matlis reflexive modules are Matlis reflexive (see, for example, [7, Section 3.4]).

If $\hat{R}$ denotes the $m$-adic completion of $R$, then by Theorem 3.7 of [5],

$$\text{Hom}_R(E, E) \cong \hat{R}.$$ 

Hence the ring $R$ is a Matlis reflexive $R$-module if and only if $R$ is complete. In this case, finitely generated and Artinian $R$-modules are Matlis reflexive. Also, $M$ is an Artinian (Noetherian) $R$-module if and only if $M^\vee$ is Noetherian (Artinian); see [5, Corollary 4.3], for example. In particular, $R^\vee \cong E$ and, if $R$ is complete, $E^\vee \cong R$, and hence both are Matlis reflexive.
An example of a Matlis reflexive $R$-module that is neither finitely generated nor Artinian is given by $E(R)$ where $R$ is a complete Gorenstein ring of dimension one. To see this, observe that there is an exact sequence

$$0 \longrightarrow R \longrightarrow E(R) \longrightarrow E \longrightarrow 0.$$ 

Since $R$ and $E$ are Matlis reflexive, so is $E(R)$.

If $R$ is complete, then Matlis reflexive modules are characterized by the following proposition.

**Proposition 1.** If $R$ is complete, then $M$ is Matlis reflexive if and only if $M$ contains a finitely generated submodule $S$ such that $M/S$ is Artinian.

This proposition was proved by E. Enochs in [3, Proposition 1.3], and also by H. Zöschinger in [8] (see also [7, Theorem 3.4.13]). In fact, the proof of the ‘only if’ part, as given in [3] or [7], does not use the completeness assumption. So we shall actually use the following.

**Lemma 1.** Let $R$ be any local ring. If $M$ is a Matlis reflexive $R$-module, then there is a short exact sequence

$$0 \longrightarrow S \longrightarrow M \longrightarrow M/S \longrightarrow 0$$

with $S$ finitely generated and complete, and $M/S$ Artinian.

For the rest of this section we assume that $R$ is an $n$-dimensional local ring with maximal ideal $m$ which has a dualizing complex $D'$. Let $D'(-)$ denote the functor $H'(-) = (\text{Hom}_R(-, D'))$.

**Lemma 2.** Let $M$ be an Artinian $R$-module. Then $D'(M) = 0$ for all $i \neq n$, and $D'(M) = M$. Furthermore, the map $\phi^l_M: H^l(M) \to D'(M)$ is the natural map $M \to M$. Proof. The first part follows directly from the definition of a dualizing complex and the fact that $\text{Hom}_R(M, E(R/p)) = 0$ if $p$ is a non-maximal prime ideal. Recall that if $M$ is Artinian, then $H^i_M = 0$ for $i > 0$. The second part then follows from the definition of $\phi^l_M$.

**Corollary 1.** Let $M$ be an $R$-module which contains a submodule $S$ such that $M/S$ is an Artinian module. If $j: S \to M$ is the inclusion map, then the induced maps

$$H^i_m(j): H^i_m(S) \longrightarrow H^i_m(M) \quad \text{and} \quad D^{* -(j)}: D^{* -(j)} \longrightarrow D^{* -(M)}$$

are isomorphisms for $i > 1$.

Proof. For any $i > 1$ we have the following commutative diagram.
Since $M/S$ is Artinian, $H_{m}^{i}(M/S)$ and $H_{m}^{i}(M/S)$ are zero, and by Lemma 2, $D^{n+1}(M/S)$ and $D^{n+1}(M/S)$ are zero. The conclusion then follows.

**Theorem 1.** Let $M$ be an $R$-module which contains a finitely generated submodule $S$ such that $M/S$ is an Artinian, reflexive $R$-module. Then the maps

$$
\phi_{i}^{j}: H_{m}^{i}(M) \rightarrow D^{n-j}(M)
$$

are isomorphisms for all $i \geq 0$. In particular, if $M$ is a Matlis reflexive module, then $M$ satisfies local duality.

**Proof.** By Corollary 1, $H_{m}^{i}(S) \cong H_{m}^{i}(M)$ and $D^{n-j}(S) \cong D^{n-j}(M)$ for $i > 1$. Since $S$ is finitely generated, the maps $\phi_{S}^{i}: H_{m}^{i}(S) \rightarrow D^{n-j}(S)$ are isomorphisms for all $i$. Thus $\phi_{M}^{i}: H_{m}^{i}(M) \rightarrow D^{n-j}(M)$ are isomorphisms for $i > 1$.

For the other two isomorphisms, the exact sequence $0 \rightarrow S \rightarrow M \rightarrow M/S \rightarrow 0$ gives rise to the following commutative diagram.

$$
\begin{array}{cccccc}
0 & \rightarrow & H_{m}^{0}(S) & \rightarrow & H_{m}^{0}(M) & \rightarrow & H_{m}^{0}(M/S) & \rightarrow & H_{m}^{1}(S) & \rightarrow & H_{m}^{1}(M) & \rightarrow & 0 \\
\downarrow \phi_{S}^{0} & & \downarrow \phi_{M}^{0} & & \downarrow \phi_{M/S}^{0} & & \downarrow \phi_{S}^{1} & & \downarrow \phi_{M}^{1} & \\
0 & \rightarrow & D^{n}(S) & \rightarrow & D^{n}(M) & \rightarrow & D^{n}(M/S) & \rightarrow & D^{n+1}(S) & \rightarrow & D^{n+1}(M) & \rightarrow & 0
\end{array}
$$

Since $M/S$ is Artinian, by Lemma 2 the map

$$
\phi_{M/S}^{0}: H_{m}^{0}(M/S) \rightarrow D^{n}(M/S)
$$

is the natural map

$$
M/S \rightarrow (M/S)^{\vee}.
$$

Now $M/S$ is assumed to be reflexive, so the map $\phi_{M/S}^{0}$ is an isomorphism. Since $S$ is finitely generated, $\phi_{S}^{0}$ and $\phi_{S}^{1}$ are isomorphisms. Thus $\phi_{M}^{0}$ and $\phi_{M}^{1}$ are also isomorphisms.

3. Modules with finite Bass numbers

**Lemma 3.** Let $R$ be a local ring, and let $M$ be an $R$-module such that $\mu_{i}(M) < \infty$ for all $i$. Then $H_{m}^{i}(M)$ is Artinian for all $i$.

**Proof.** Suppose that $\mu_{i} := \mu_{i}(M)$ is finite for all $i$, and let $M \rightarrow I$ be the minimal injective resolution of $M$. Using the fact that $H_{m}^{i}(E) = E$ and $H_{m}^{i}(E(R/p)) = 0$ if $p$ is a non-maximal prime ideal of $R$, it follows that $H_{m}^{i}(I) = E^{\mu_{i}}$, where $E^{\mu_{i}}$ denotes the direct sum of $\mu_{i}$ copies of $E$. Then $H_{m}^{i}(I)$ is the complex

$$
0 \rightarrow E^{\mu_{0}} \rightarrow \cdots \rightarrow E^{\mu_{i-1}} \rightarrow E^{\mu_{i}} \rightarrow \cdots
$$

Every module in this complex is Artinian. Hence $\ker(d_{i})$, $\text{im}(d_{i-1})$ and their quotient $H_{m}^{i}(M)$ are all Artinian.

Now let $R$ be a complete local Gorenstein ring with dim $R = n$, and let $M$ be an $R$-module. Since $R$ is complete, there is a natural isomorphism

$$
\text{Tor}_{m}^{n}(M, E)^{\vee} \cong \text{Ext}_{R}^{n+1}(M, R).
$$
Since \( R \) is Gorenstein, \( H^n_m(R) = E \) and \( H^n_m(M) \cong \text{Tor}^m_n(M, E) \) for all \( i \), functorially in \( M \) (see 9.1.4 and 10.1.10 of [7], for example). Therefore, for any \( i \geq 0 \), the map
\[
\phi^i_M : H^i_m(M) \rightarrow \text{Ext}^i_{R^e}(M, R)\]
is an isomorphism if and only if the natural map
\[
\text{Tor}^i_{w(G)}(M, E) \rightarrow \text{Tor}^i_{w(G)}(M, E)\]
is an isomorphism. So \( M \) satisfies local duality if and only if \( H^i_m(M) \) is Matlis reflexive.

**Proposition 2.** For a complete local Gorenstein ring \( R \), an \( R \)-module \( M \) satisfies local duality if and only if \( \mu_i(M) < \infty \) for all \( i \).

**Proof.** Assume that \( M \) satisfies local duality. Then each \( H^i_m(M) \) is Matlis reflexive by the remarks above. Since a Matlis reflexive module cannot contain an infinite direct sum (see [7], the paragraph just before 3.4.3 on p. 41), \( \dim_k(\text{Soc} H^i_m(M)) \) is finite, where \( k = R/m \). But \( \text{Soc} H^i_m(M) = \text{Soc} M \cong \text{Hom}_n(k, M) \), and so
\[
\dim_k(\text{Soc} H^i_m(M)) = \mu_i(M)
\]
is finite. To see that \( \mu_i(M) < \infty \) for \( i \geq 1 \) consider the short exact sequence
\[
0 \rightarrow M \rightarrow I \rightarrow C \rightarrow 0,
\]
where \( I = E(M) \) and \( C = I/M \). Then \( \mu_i(M) = \mu_{i+1}(C) \) for \( i \geq 1 \). Since \( H^i_m(M) \) and \( H^i_m(I) \) are Matlis reflexive (see the proof of Lemma 3), it follows from the long exact sequence of local cohomology modules that \( H^i_m(C) \) and \( H^i_m(C) \cong H^{i+1}_m(M) \) for \( i \geq 1 \) are Matlis reflexive. Therefore \( \mu_i(M) = \mu_i(C) \) is finite. Repeating the argument with \( C \) in place of \( M \) shows that \( \mu_i(M) < \infty \) for all \( i \).

Conversely, if \( \mu_i(M) \) is finite for all \( i \), then \( H^i_m(M) \) is Artinian by Lemma 3. Hence \( H^i_m(M) \) is Matlis reflexive, and so \( M \) satisfies local duality.

Recall that if \( S \) is a Gorenstein ring, then the minimal \( S \)-injective resolution \( I' \) of \( S \) is a dualizing complex for \( S \). If \( T \) is a homomorphic image of \( S \), then \( \text{Hom}_n(T, I') \) is a dualizing complex for \( T \). So any homomorphic image of a Gorenstein ring has a dualizing complex (see, for example, [6]).

Now let \( S \) be a complete local ring. By the Cohen structure theorem, \( S = R/I \) where \( R \) is a regular local ring and \( I \) is an ideal of \( R \). Since a regular local ring is Gorenstein, a complete local ring has a dualizing complex, namely \( D' = \text{Hom}_n(S, I') \), where \( I' \) is the minimal \( R \)-injective resolution of the Gorenstein ring \( R \).

**Theorem 2.** Let \( S \) be a complete local ring, and let \( M \) be an \( S \)-module. Then \( M \) satisfies local duality if and only if \( \mu_i(M) < \infty \) for all \( i \).

**Proof.** We shall assume that \( S = R/I \), and let the notation be as in the paragraph just before the statement of the theorem.

By a standard reduction (see, for example, various proofs of the local Lichtenbaum–Hartshorne theorem) we can assume that \( S \) and \( R \) have the same Krull dimension. Now let \( M \) be an \( S \)-module. We have
\[
\text{Hom}_n(M, D') = \text{Hom}_n(M, \text{Hom}_n(S, I')) \cong \text{Hom}_n(M, I').
\]
Thus for any $i$ we have $H^{*+i}(\text{Hom}_s(M, D^*)) = \Ext^n_{R_i}(M, R)$ and
$$[H^{*+i}(\text{Hom}_s(M, D^*))]^\vee = \Ext^n_{R_i}(M, R)^\vee.$$ Also, $H^i_R(\mathord{-})$ and $H^i_{ms}(\mathord{-})$ are naturally isomorphic functors from $S$-modules to $R$-modules (see Proposition 4.1.4 of [7]).

The local rings $(R, m)$ and $(S, mS)$ have the same residue field $k$, so $\mu_i(M) = \dim_k \Ext^i(k, M)$ does not depend on whether we regard $M$ as an $S$-module or as an $R$-module. By the previous proposition, the $R$-linear maps
$$H^i_R(M) \xrightarrow{\phi^i_M} \Ext^{*+i}(M, R)^\vee$$
are isomorphisms for all $i \geq 0$ if and only if $\mu_i(M) < \infty$ for all $i$. Therefore the $S$-linear maps
$$H^i_{ms}(M) \xrightarrow{\phi^i_M} [H^{*+i}(\text{Hom}_s(M, D^*))]^\vee$$
are isomorphisms for all $i \geq 0$ if and only if $\mu_i(M) < \infty$ for all $i$.

A natural question to ask is whether one can drop the completeness assumption in Theorem 2. The next proposition answers the question and gives us the following example.

**Example.** The Artinian module $E$ does not satisfy local duality if $R$ is a Gorenstein ring that is not complete.

**Proposition 3.** Let $R$ be a local Gorenstein ring. The Artinian module $E$ satisfies local duality if and only if $R$ is complete.

**Proof.** If $R$ is complete, then $E$ satisfies local duality by Proposition 2.

For the converse, we shall show that if $E$ satisfies local duality, then $R$ is complete. Since $E$ is Artinian, we have $H^i_R(E) = 0$ for $i > 0$, and $H^0_R(E) = E$. By the remarks before Proposition 2, $E$ satisfies local duality if and only if the natural map $E \rightarrow E^\vee$ is an isomorphism. But $E = R^\vee$, and if $R^\vee$ is reflexive then $R$ is reflexive. Hence $R$ is complete.

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**References**


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