The author describes this book as an introduction to the theory of automorphic forms of several variables and their Hecke theory. However, as often happens with an introduction to a highly technical subject by one of its leaders, there is plenty in this volume for both the beginner and the expert.

The reader familiar with only algebraic number theory will find in this volume a self-contained account of the main issues on the theory of automorphic forms on classical algebraic groups. There are eight useful appendices, which include four covering the necessary Fourier analysis, orbital integrals and convergence questions; I found these appendices particularly helpful. The remaining appendices cover parabolic and compact subgroups of the classical groups over the reals, theta functions of Hermitian forms, and the structure of central simple algebras over local fields. Also, throughout the book, one finds proofs of a number of ‘folk theorems’ concerning algebraic groups and their localisations.

Rather than trying to go into details and failing to do them justice for lack of space, a better way for me to present the importance of the results in Shimura’s book is to look back at the first types of automorphic forms, the modular forms of Siegel. A modular form of weight $k$ is an analytic function on the upper half-plane, growing nicely at infinity and transforming under $\text{SL}_2(\mathbb{Z})$ according to the rule

$$f \left( \begin{pmatrix} az + b \\ cz + d \end{pmatrix} \right) = (cz + d)^k f(z).$$

A Siegel modular form of weight $k$ is an analytic function on the space of complex $n \times n$ symmetric matrices, with positive definite real part and satisfying a similar transformation law when $a, b, c, d$ are replaced by a quartet of $n \times n$ integral matrices, $A, B, C, D$, comprising the corners of an integral $2n \times 2n$ symplectic matrix, $(cz + d)^k$ becoming $\det(CZ + D)^k$. Such a modular form, $f$, gives rise to a function, $\phi_f$, on the group, $\text{Sp}_{2n}(\mathbb{R})$, of real $2n \times 2n$ symplectic matrices defined by

$$\phi_f \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \det(Ci + D)^{-k}f(Ai + B)(Ci + D)^{-1}.$$

In fact, $\phi_f$ is left $\text{Sp}_{2n}(\mathbb{Z})$-invariant and, under mild conditions on $f$, the $\phi(f)$ may be characterised in terms of subrepresentations of the right-multiplication representation of $\text{Sp}_{2n}(\mathbb{R})$ on $L^2(\text{Sp}_{2n}(\mathbb{Z}) \backslash \text{Sp}_{2n}(\mathbb{R}))$. If $\mathcal{A}$ denotes the adele ring of the rationals, $\mathbb{Q}$, then $\mathcal{A}$ is the weak product of all the completions of the rationals. Functions on $\text{Sp}_{2n}(\mathbb{Z}) \backslash \text{Sp}_{2n}(\mathbb{R})$ may be regarded as functions on $\text{Sp}_{2n}(\mathbb{Q}) \cap \text{Sp}_{2n}(\mathcal{A}_0) \backslash \text{Sp}_{2n}(\mathcal{A}_0)$, where $\mathcal{A}_0 = \mathbb{R} \times \prod_v \mathbb{Z}_v$. The Hecke operators are integral operators defined on $L^2$ of this adelic quotient. They act on the functions on $\text{Sp}_{2n}(\mathbb{Z}) \backslash \text{Sp}_{2n}(\mathbb{R})$. In the simplest case,
where the group is $GL_1$, the decomposition of $L^2(GL_1(\mathbb{Q}) \backslash GL_1(\mathbb{A}))$ is accomplished by means of the continuous characters of $GL_1(\mathbb{Q}) \backslash GL_1(\mathbb{A})$, which therefore should be considered as the basic modular forms in this case. From these and the Hecke operators one can define an Euler product, which corresponds via Artin reciprocity to that of the Artin $L$-function of the character in question. Artin $L$-functions have analytic continuation to the whole complex plane, whereas Euler products may not.

In his James K. Whittemore Lecture given at Yale University in 1967, Langlands introduced his Eisenstein series to prove the analytic continuation of the Euler products associated by Hecke theory to automorphic forms on some reductive algebraic groups. At this time, he said: ‘the theory of modular forms (and Hecke operators) is far from complete. Indeed very little attempt has been made, as far as I can see, to understand what the goals should be’.

By now the main goals are clearer. In the particular case of this book, they include the determination of the local Euler factors on classical groups in an explicit rational form, the construction of Euler products and Eisenstein series on a unitary group of arbitrary signature, and the derivation of a class number formula (a mass formula in Siegel’s terminology) for a positive definite Hermitian form. All these new results are proved, and the author outlines, for the experts, related results which are to follow in subsequent papers. Finally, it should be noted that Shimura’s book is the first to treat the Hecke theory of general non-split classical groups.

University of Southampton

SPECTRAL DECOMPOSITION AND EISENSTEIN SERIES

(Cambridge Tracts in Mathematics 113)


Let $G$ be the group of real points of a semisimple $\mathbb{Q}$-group $G$, and let $\Gamma$ be an arithmetic subgroup. A fundamental problem in the theory of automorphic forms is to decompose the action of $G$ on the space $L^2(\Gamma \backslash G)$ consisting of square integrable functions on $\Gamma \backslash G$. If $\Gamma \backslash G$ is compact, then one obtains a direct sum of irreducible square integrable subspaces under the action of $G$, each of which occurs with finite multiplicity in $L^2(\Gamma \backslash G)$; it is an important problem to determine these multiplicities. If $\Gamma \backslash G$ merely has finite volume, then the spectral decomposition of $L^2(\Gamma \backslash G)$ is more complicated to describe: it decomposes as a double direct sum of subspaces parametrised by pairs of families of parabolic subgroups, and these subspaces can be decomposed further via direct integrals. The theory of Eisenstein series is the tool that accomplishes this; roughly speaking, it starts by constructing automorphic forms on $\Gamma \backslash G$ from cusp forms on Levi components of parabolic subgroups, and involves proving that the new forms can be meromorphically continued along a complex analytic variety attached to the Levi component, and that they satisfy functional equations. In general, there are poles, whose residues provide non-cusp forms from which one constructs further Eisenstein series; this leads to a difficult induction.
The theory was begun by H. Maass, W. Roelke and A. Selberg (for \( \Gamma \) a discrete subgroup of a rank one group), and developed in general by R. P. Langlands \([8]\) (circa 1964) in a \textit{tour de force} which has had a striking impact on the theory of automorphic forms. Indeed, it is the first step in describing a trace formula for the action of \( G \) on the space \( L^2(\Gamma \backslash G) \); this has been carried out by J. G. Arthur in a fundamental series of papers (see \([5]\)). The insights gained from this analysis led Langlands to formulate, and provide evidence for, the Langlands’ program or ‘functoriality’; a fundamental tool here for comparison purposes is the above-mentioned trace formula. A particular aspect of this program is the hypothetical existence and meromorphic continuation of non-abelian \( L \)-functions associated to automorphic representations. One of the few ways known to do this is via the so-called ‘residual spectrum’ is a major problem.

For the purposes of Langlands’ program, one takes \( G \) to be a connected semi-simple group defined over a global field \( F \) with ring of adeles \( \mathbb{A}_F \). The group \( G(\mathbb{A}_F) \) is then naturally endowed with a topology for which it is locally compact and Hausdorff; further, it is unimodular and contains the discrete subgroup \( G(F) \). The problem is then to decompose the space \( L^2(G(F) \backslash G(\mathbb{A}_F)) \). This is essentially the framework of the book under review, which is likely to become a fundamental and valuable reference. First, it formulates and solves the problem in adelic form over arbitrary global fields suitable for applications to automorphic forms from the Langlands point of view. At the same time, it treats metaplectic coverings, and clears up some gaps in the literature (see, notably, Appendices 1 and 2). Finally, it explains and clarifies the elaborate inductive process alluded to above. On the other hand, the degree of generality will probably make it difficult for the browser or beginner to see the difficulties and subtleties; for this, the publications \([1, 3, 4, 6]\) might be helpful.

References


Clark University, Worcester, MA  Lawrence Morris
The Classification Theorem for Finite Simple Groups (traditionally abbreviated to CFSG) is one of the major achievements of 20th-century mathematics, and yet its proof remains to this day, nearly twenty years after its announcement, incomplete and in places less than thoroughly checked.

A significant part of the proof concerns the existence and uniqueness of the 26 sporadic simple groups. While these results themselves are not in any doubt, the published proofs (if any) often leave something to be desired.

Aschbacher has taken on the task of writing up proofs for all these results, attempting as far as possible to make them uniform, self-contained, and definitive. In his earlier book *Sporadic groups* [2], he does this for five of the 26 groups, while in the book under review he considers three more.

These are the three sporadic simple groups named after Bernd Fischer, who discovered them around 1969. Fischer’s proof of the existence and uniqueness of these groups was never published, but has been widely circulated in typescript. In essence, a 3-transposition group is a group generated by a set $D$ of elements of order 2, closed under conjugation, such that the product of any two elements of $D$ has order at most 3. Other conditions may be imposed for technical reasons. The most familiar examples are the symmetric groups, as well as certain classical groups defined over small fields. Fischer proved that (with suitable conditions) just three more groups arise.

Aschbacher’s aim in the present book is threefold. First, he presents a re-working of Fischer’s original classification, incorporating some more recent improvements. For this part of the book the groups are defined as 3-transposition groups. Secondly, he proves uniqueness of the groups, as a contribution to the full proof of CFSG. Here the groups are defined in terms of the centralizers of involutions. Finally, he develops the basic structure of local subgroups of the Fischer groups, some of which is needed to complete the proofs in the earlier parts of the book. His specific aim is to determine the conjugacy classes of elements, and the normalizers of the cyclic subgroups of prime order.

This strategy allows the author to reduce the total length of the proof by carefully switching definitions several times in the middle, at the cost of causing some confusion, and engendering an uncomfortable feeling (albeit unjustified) that the argument may be circular.

Roughly speaking, he proves that (a) the 3-transposition groups are unique, and quotes (b) simple groups with the given involution centralizers exist, and then proves (c) the latter are (essentially) 3-transposition groups. It follows that (d) they are also unique, and (e) the 3-transposition groups exist.

The one obvious gap in the treatment is the existence question. This is dealt with merely by referring to the existence of the Monster—clearly, the author’s definition of ‘self-contained’ is weaker than the reviewer’s. There are also numerous references to the author’s previous two books [1, 2], which the reader will need to have read, understood and inwardly digested.

The author’s attempt to introduce a (relatively) uniform definition for all the
sporadic groups is certainly helpful, if not completely convincing, and is a good starting point for a ‘uniform’ proof of uniqueness (but not existence). However, no amount of general notation will hide the fact that it is still necessary to prove 52 separate theorems.

This book serves a very useful function, both by making Fischer’s work available in print for the first time, and by forming part of the second-generation proof of CFSG. It is not, however, an easy book to read, and it demands total commitment from the reader. It is designed as a single edifice, and a typical proof of one of the main theorems contains many references to lemmas and propositions (and even exercises) scattered throughout the book. It is also very light on external references and bibliographical material. As a result, this is not a book that one can dip into and select choice morsels—it must be eaten whole (and thoroughly chewed), or not at all. That said, its place as an important reference work is assured, and it should be in every serious finite group theorist’s library.

References


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ROBERT WILSON

INTRODUCTION TO SUBFACTORS
(LMS Lecture Note Series 234)

By V. Jones and V. S. Sunder: 162 pp., £22.95 (US$37.95, LMS Members’ price £17.20), isbn 0 521 58420 5 (Cambridge University Press, 1997).

A von Neumann factor on a complex Hilbert space is a self-adjoint algebra of bounded operators which is closed in the weak operator topology, with centre just the scalar multiples of the identity operator. In the 1930s Murray and von Neumann classified factors into three types. Those of type I are simply of form \( B(H) \), the bounded linear operators on a finite or infinite dimensional Hilbert space. The type II factors can be divided into those of type II\(_a\), the infinite dimensional factors possessing a non-zero trace functional, and those of type II\(_\infty\), the factors which are the tensor product of a type II\(_1\) with a type I factor. The type III factors are the remainder.

Until about 1980, research on factors was concerned primarily with classification up to isomorphism. More recently, the thrust has shifted to classifying the possible inclusions \( N \subseteq M \) for a given pair of factors \( M \) and \( N \). In a seminal paper of 1983, Vaughan Jones, one of the authors of the book under review, defined a numerical invariant, the index, of such an inclusion. He showed that the index for II\(_1\) factors always lies in the set \( \{ 4\cos^2(\pi/n): n = 1, 2, \ldots \} \cup [4, \infty) \), and that any of these values can occur. An important constituent of the proof is a family of polynomials, now known as the Jones polynomials. Quite fortuitously, these polynomials were observed to give rise to new invariants in knot theory, leading to the solution of some old problems in that subject. Much of the subsequent work on subfactors, by Jones and
others, has been devoted to finding constructions of II₁ factor inclusions, in particular when both factors in the inclusion are isomorphic to the hyperfinite II₁ factor (the unique II₁ factor which contains a dense subalgebra isomorphic to an increasing union of self-adjoint finite dimensional algebras).

The publication of this book, seemingly the first devoted purely to subfactors, is opportune. Though not an exhaustive exposition of the state of the art, it provides an ideal introduction to the subject. Little or no prior knowledge of von Neumann algebras is assumed; what is required is presented with admirable deftness in the first chapter. The central concept of the subject, the index, receives a particularly accessible treatment using bimodules in Chapter 2, and the Jones basic construction, one of the subject’s most fundamental tools, is described in Chapter 3. The remaining three chapters present various techniques for constructing inclusions of factors, the main unifying element being the idea of a commuting square of inclusions of finite dimensional algebras. There are useful appendices which give further background on von Neumann algebras and also the connection between subfactors and knot theory.

This book is a model of clear exposition and keen pricing. Those working in operator algebras will find it self-recommending, and research students and others outside the area wishing to learn the subject need look no further. It is a most valuable addition to the literature.

University of Glasgow

Simon Wassermann

CARLESON CURVES, MUCKENHOUPT WEIGHTS, AND TOEPLITZ OPERATORS
(Progress in Mathematics 154)

By Albrecht Böttcher and Yuri I. Karlovich: 397 pp., SFr.98.00, isbn 3 7643 5796 7 (0 8176 5796 7) (Birkhäuser, 1997).

The book under review is devoted to the spectral theory of Toeplitz operators with piecewise continuous symbols and of singular integral operators with piecewise continuous coefficients on Carleson curves with Muckenhoupt weights. A locally rectifiable curve \( \Gamma \) is a Carleson curve if \( e^{-\varepsilon} |\Gamma(t, \varepsilon)| \) is bounded independently of \( t \in \Gamma \) and \( \varepsilon > 0 \), where \( \Gamma(t, \varepsilon) = \{ \tau \in \Gamma; |\tau - t| < \varepsilon \} \). It has been known since the 1970s that the essential spectrum of the Cauchy singular integral operator \( S \) consists of two circular arcs between \(-1\) and \(1\) when \( \Gamma \) is a bounded and simple Lyapunov curve (roughly, a graph of a function whose derivative satisfies the Hölder condition) and \( w \) is a power weight with exponent in a certain appropriate range. In 1990, Spitkovsky proved a quite surprising fact that non-power Muckenhoupt weights on Lyapunov curves might force those circular arcs to turn into horns. An important point here is that \( S \) is not only bounded for Lyapunov curves with power weights, but in fact it is bounded on \( L^p(\Gamma, w) \) for a locally rectifiable curve \( \Gamma \), endowed with a weight \( w \), if and only if \( 1 < p < \infty \), \( \Gamma \) is a Carleson curve, and \( w \) belongs to the Muckenhoupt \( A_p \) class. This is an extremely deep result which was proved in the 1980s through efforts of many leading scientists in the field. This result, whose proof takes up a non-trivial portion of the book, is a certain ‘point of departure’ towards the authors’ highlighted topic: the spectral theory of Toeplitz operators.
The authors’ central discovery is that, in addition to the above-mentioned Spitkovsky’s result, Carleson curves and Muckenhoupt weights bring qualitatively new phenomena into spectral theory. In particular, certain interference between the curve and the weight leads to the appearance of (logarithmic) double spirals, (spiralic) horns and, beyond a certain critical point, ‘leaves’. Thus the resulting spectral theory is surprisingly rich, and the book is aimed at showing all its beauty to a broad mathematical public.

The authors present an excellent exposition, leading the reader step by step through a formidable theory, overcoming considerable technical difficulties in a most elegant manner. As an example, let us mention the ingenious solution to the authors’ dilemma about whether or not to include the proof of the boundedness of $S$ (an assertion nicknamed at one stage as ‘Theorem 1 of the book’). The whole of Chapter 5 is devoted to the detailed proof, but the preceding chapter, containing the necessity part and an outline of the proof with historical survey, enables Chapter 5 to be skipped should a reader wish to omit it.

The first three chapters contain much more than just an introduction to Carleson curves and Muckenhoupt weights. A concept of the indicator function and the indicator set is developed here, extracting all the useful information contained by the curve and the weight. Further, submultiplicative functions and their indices are used, in an innovating way, to supply non-trivial Muckenhoupt weights.

In Chapter 6, the Toeplitz operators take over the leading role. The problem of finding the spectrum of $S$ is reduced to the characterization of the spectrum of Toeplitz operators with piecewise continuous symbols on weighted Hardy spaces over Jordan curves. The localization principle and the Wiener–Hopf factorization are applied to determine whether a Toeplitz operator is Fredholm.

The main results of the book are contained in Chapter 7, where a complete description of the essential spectrum and of the spectrum of Toeplitz operators is given. This part is almost exclusively based on very recent results of both authors.

The last three chapters contain various applications, extensions and further results, including the $A_p$-projection theorem, a symbol calculus of Banach algebras of singular integrals over Jordan curves, the Fredholm criteria for operators in certain algebras, and many more.

The main scientific asset of the book lies in the spectral theory, but the book has something to say to a much broader mathematical public, in particular to those interested in functional analysis or harmonic analysis. To name just one example, the book brings a new rich supply of highly non-trivial $A_p$ weights; that might be useful in many parts of analysis!

Academy of Sciences of the Czech Republic

Luboš Pick

**FLAVORS OF GEOMETRY**


This book comprises four essays, each based on a lecture series delivered as part of an MSRI programme in geometry during 1995 and 1996. Each essay is intended
as an introduction to a particular geometric topic, and hence to give a ‘flavor’ of some aspect of current research.

The first essay, by Keith Ball, is an introduction to euclidean convex geometry. It covers an array of elegant results, both old and new, including some of the author’s own. Much of this is already interesting in low dimensions, though the main objective is to study the asymptotic structure of convex bodies in higher dimensions. In the course of this, we learn how higher-dimensional convex sets often behave in surprising and counterintuitive ways. One of the central results is Fritz John’s theorem, which gives a characterisation of the unique ellipsoid of maximal volume contained in a given convex body. Other topics include the existence of almost spherical sections of convex bodies, inequalities related to the Brunn–Minkowski theory, and the isoperimetric and reverse isoperimetric problems. We are introduced to a diverse array of techniques, with outline proofs of the main results. The essay is very readable, and the network of interrelationships is clearly described. It is a particularly nice introduction to the subject.

The second essay, by J. W. Cannon, W. J. Floyd, R. Kenyon and W. R. Parry, is an introduction to hyperbolic geometry. It begins with a potted history of the subject, and goes on to describe in some detail the various standard models of hyperbolic space. The main qualitative differences with euclidean geometry are also highlighted. The account is clear, and well illustrated with diagrams. About three-quarters of the way through, the pace quickens dramatically, and we are given a fleeting glance of Thurston’s geometrisation programme for 3-manifolds. The account eventually focuses on the authors’ own contribution, involving combinatorial moduli, and tilings produced by recursive subdivision rules—a subject of much beauty in itself. The discussion of recent development in hyperbolic geometry is far from comprehensive, and there are now many good introductory texts available. However, this one has its own distinctive flavour, which I hope may serve to whet more appetites.

The third essay, by John Smillie and Gregery T. Buzzard, lies on the fringe of what many would classify as geometry. It describes the dynamics of the complex Henon map. This is a polynomial diffeomorphism of \( \mathbb{C}^2 \) (or a 2-complex-parameter family thereof). As the simplest non-trivial example of an invertible polynomial map, it has come under much scrutiny by dynamicists. In it, one can see examples of axiom A systems, such as horseshoes and solenoids, as well as kinds of behaviour more familiar from the iteration of rational maps. The authors give an account of several general topics, including horseshoes, stable and unstable manifolds, potential theory, currents, entropy, and the distribution of periodic points. An ambitious attempt has been made to convey in a small space the intricacies of a complex subject. However, the result is somewhat lacking in cohesion, making it difficult to carry away more than an impressionistic view of many of the main ideas. For those wishing to gain insight into the main general principles of dynamics, the complex Henon map is perhaps not the optimal place to start.

The final essay, by Béla Bollobás, concerns algorithms for estimating volumes of convex bodies. The aim is to find a polynomial-time algorithm which, when confronted with an oracle which describes some convex body, will compute upper and lower bounds for its volume whose ratio has good asymptotic dependence on the dimension. In the first section, which ties in with Ball’s essay, the author explains why one cannot expect a good deterministic algorithm. In the second and third sections, there is an account of rapid mixing of random walks on graphs. And in the final
section, there is an outline of the probabilistic algorithm of Dyer, Frieze and Kannan which uses these ideas. The essay is generally well written. Even if the subject is rather specialised, it is an intriguing advertisement for Monte Carlo algorithms.

Overall, the book should be viewed as a set of distinct contributions rather than a coherent whole. The choice of topics is somewhat arbitrary. There is no mention of differential geometry, for example, and some readers may be disappointed that the many exciting recent applications of geometry to topology and group theory do not feature more prominently (or, indeed, at all). However, I expect that the individual contributions will serve as useful introductory references to their respective domains.

University of Southampton

B. H. Bowditch

AN INTRODUCTION TO INFINITE ERGODIC THEORY
(Mathematical Surveys and Monographs 50)

By Jon Aaronson: 284 pp., US$79.00, ISBN 0 8218 0494 4
(American Mathematical Society, 1997).

Given a measure space \((X, \mathcal{B}, m)\), a measurable transformation \(T: X \to X\) is said to be non-singular if the image of the measure \(m\) under \(T\) is equivalent to \(m\). A particular case is that of a \(T\)-invariant probability measure. Most books on ergodic theory concentrate on this case. The present monograph addresses the general situation.

It starts with a systematic introduction to general non-singular ergodic theory. After a brief introduction, in particular to the basics of standard measure spaces, conservativity and recurrence are discussed. Then the Frobenius–Perron operator and induced transformations are introduced and applied to the question of existence and uniqueness of absolutely continuous invariant measures. The first chapter concludes with a brief discussion of ergodicity for flows and more general group actions.

The second chapter deals with what can be considered the core of ergodic theory. It discusses mean and pointwise ergodic theorems, ergodic decomposition, and converses to Birkhoff’s theorem. The organisation of the individual ergodic theorems starts from Hurewicz’s ratio ergodic theorem (for \(T\) conservative non-singular with respect to a \(\sigma\)-finite measure \(m\)), presenting Hopf’s theorem (\(T\) preserving a \(\sigma\)-finite \(m\)), the stochastic ergodic theorem of Krengel (\(T\) non-singular with respect to a probability measure) and Birkhoff’s ergodic theorem (\(T\) with an invariant probability) as corollaries. Then it is derived that for a conservative ergodic transformation with an infinite invariant measure, \(1/a_n \sum f \circ T^n\) does not converge to the integral of \(f\) a.e. for all integrable \(f\), whichever sequence \(a_n \to 0\) one takes under consideration. The chapter concludes with a discussion of mixing properties as well as other aspects of spectral theory. These first two chapters make up about one third of the volume.

The third chapter concentrates on infinite invariant measures. The notions of factors, extensions and isomorphism need more sophistication than in the case of invariant probabilities. For instance, taking the product of two transformations, these transformations usually are no longer factors of the product. Denoting by a law of large numbers (LLN) for a conservative, ergodic measure preserving transformation a map \(L: [0,1] \to [0, \infty]\) such that \(L(1_B(x), 1_B(Tx), \ldots) = m(B)\) for all \(B \in \mathcal{B}\) a.e.,
relations with rational ergodicity and genericity of LLN are investigated. Then the Darling–Kac theorem, pointwise dual ergodicity and wandering rates are addressed. This chapter contains a large amount of material: it can be considered the core of infinite ergodic theory.

The second half of the book, comprising Chapters 4–8, takes up several more or less disjoint topics, following the lines of the author’s research. The topics included are (null-recurrent) Markov maps and Markov shifts, in particular recurrent events and similarity in this context (Chapters 4 and 5), inner functions (Chapter 6) and hyperbolic geodesic flows (Chapter 7). Cocycles and skew products are the topic of the final chapter.

The author claims that prerequisites for reading this monograph are only metric topology and measure theoretic probability. This is true so far as the methods employed are concerned. In order to understand the book, however, some knowledge of finite ergodic theory might be helpful. This might further motivate and help the reader to become more aware of those situations where infinite measures give rise to new problems, some of them quite subtle, which do not arise for finite measures.

The book is a research monograph and contains an impressive amount of material. The presentation is tight. Some chapters start with ‘Definition: ...’; the reader is expected to bring sufficient motivation. The style is close to that of a blackboard course: in particular, the symbols ∀, ∃ are used frequently, even in running text. Still, the presentation is careful, well organised, and reliable. The text is typeset with \TeX; the number of misprints and minor errors is a little above average. A list of corrections is available from www.math.tau.ac.il/~aaro/book/book.html

This monograph is definitely a valuable complement to the ergodic theory literature. According to my knowledge, this volume is the first systematic treatment of non-singular and infinite ergodic theory appearing in book form. It will be useful for graduate students and researchers in ergodic theory and related fields. Furthermore, it is of interest for everybody interested in the great variety of ergodic behaviours which appear when not restricting attention to invariant probability measures, providing ‘the whole picture’ in the field of ergodic theory.

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HANS CRAUEL

CONFOLIATIONS
(University Lecture Series 13)


Let ξ be a two-dimensional plane field on the compact, oriented 3-manifold M, and suppose that ξ is defined at least locally by the Pfaffian equation \( \alpha = 0 \). The 1-form \( \alpha \) is defined up to multiplication by a non-vanishing function, and defines a positive confoliation on M if \( \alpha, d\alpha \) is non-negative everywhere. Thus a confoliation is intermediate between a foliation (equal to zero everywhere) and a contact structure
(strictly positive everywhere). The purpose of this monograph is to introduce the reader to confoliations, and to use their properties to show how the 1-form defining a codimension-one foliation on $M$ can be continuously deformed to a 1-form defining a contact structure. A revealing example is provided by manifolds modelled on $\text{SL}_3(\mathbb{R})$, if we start with the foliation described by left cosets of the subgroup of matrices $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, with $ac = 1$. The Lie algebra of left-invariant vector fields tangent to this foliation corresponds to a subalgebra of the Lie algebra generated by the matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = X$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = Y$, while the normal bundle corresponds to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = Z$. Let $X^*, Y^*, Z^*$ be the dual basis of 1-forms. Easy matrix manipulation then shows that $X^*$ satisfies the contact condition (originally this was used to show that our foliation has non-vanishing Godbillon–Vey invariant), and that the same holds for the line segment joining $X^*$ to $Z^*$, except for $Z^*$ itself. Thus $Z^*$ is an example of a foliation which can be linearly perturbed to a contact form. That this is not always the case is shown by the 3-torus, but here non-linear perturbation is possible. Taken together, these two examples illustrate one of the main theorems proved by Eliashberg and Thurston. Note, in passing, that the geometric duality between the two limiting structures is illustrated by the fact that it is easy to construct a contact form on $S^3$ (fibre over $\mathbb{C}P^2$), but harder to construct a codimension-one foliation (Reeb’s construction on two solid tori). The other way round, its non-trivial fundamental group allows us to foliate $T^3$ by copies of $T^2$, but a little more work is needed to find a contact form. Hence it is all the more striking that if a $C^2$-foliation $\xi$ on the oriented manifold $M$ is different from the standard foliation on $S^2 \times S^1$, then $\xi$ can be $C^\infty$-approximated by positive contact structures. The $\text{SL}_3(\mathbb{R})$ and $T^3$ examples above give the clue to the proof; a delicate geometric argument combined with real analysis is first used to perturb $\xi$ to $\xi'$ satisfying a transitivity condition. This states that any two points in $M$ can be connected by paths tangent to $\xi'$. One can then use short-time existence results for a linearised equation of heat type to deform $\xi$ further to a contact structure. In the text, Eliashberg and Thurston replace the appeal to the theory of PDEs by an explicit geometric argument.

The importance of this result is shown by the following considerations. Having proved the result above, the authors proceed to show that if the original foliation is taut (technically, contains no Reeb component), then the approximating contact structure is tight. This implies that it is (at worst) a connected component of the boundary of a symplectic 4-manifold, and hence amenable to analysis by Seiberg–Witten theory. This has enabled P. Kronheimer and T. Mrowka to prove, for example, that the number of homotopy classes of 2-plane fields on $M$ which are tangent to smooth, taut foliations is finite. More recently still, the same authors have applied the same method to examine one well-known suggested method for constructing a counterexample to the Poincaré conjecture—namely, Dehn surgery on the complement of a knot. As this reviewer understands it, the argument starts by constructing a taut foliation on the modified complement of one of a wide class of
knots, replaces this by a tight contact structure, and uses Seiberg–Witten theory to show that the fundamental group has to be non-trivial. This is very impressive, and confirms the reviewer’s long-held belief that contact geometric methods have been under-exploited in the study of 3-manifolds.

Go out and buy this book before the pound weakens; the American Mathematical Society will sell it to you for at most sixteen dollars.

University of Cambridge

C. B. THOMAS

EVOLUTION EQUATIONS AND LAGRANGIAN COORDINATES
(de Gruyter Expositions in Mathematics 24)


It is well known that in continuum mechanics, the Lagrangian approach consists in expressing all physical quantities in terms of coordinates labelling each material particle (usually the coordinates in the reference configuration). In comparison with the Eulerian approach (based on the use of ordinary Cartesian coordinates), the Lagrangian approach is complementary and sometimes more convenient. This basic concept is also applicable to large classes of problems governed by evolution equations. In dealing with free boundary problems in which the free boundary is a material surface, easily identifiable by means of Lagrangian coordinates, obvious advantages are expected from the adoption of the Lagrangian point of view. This idea is also successful for abstract problems that can be cast into a form resembling a set of conservation laws of the form encountered in continuum mechanics.

This book reviews both the general theory underlying transformations to Lagrangian coordinates and its most striking applications. Very appropriately, it opens with a chapter dedicated to the so-called Verigin’s problem (displacement of two compressible immiscible fluids in a porous medium separated by a sharp interface). Indeed, the transition to Lagrangian coordinates appears in a quite natural way, the new frame of reference being chosen so that the material interface is at rest, and it is possible to fully appreciate the power of this technique. There follows a very elegant presentation of the theory applied to quasilinear parabolic equations (possibly degenerate), with a number of interesting applications, including the Stefan problem and the porous media equation. Questions such as group properties, hidden symmetry and linearization are treated in a concise but effective way. A large amount of space is devoted to nonlinear parabolic equations with degeneracy both in one dimension and with several space variables, analysing regularity, asymptotic properties, structure of the level curves and various other questions. In conclusion, the book is at the same time a well-written review and a valuable research tool.

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Suppose that I have a noncompact manifold \( W \). What would it take to find that it is, in fact, the interior of a manifold with boundary?

Over 30 years ago, this problem was solved for high-dimensional manifolds—dimension at least six will be a blanket hypothesis in what follows—although it is not so easy to describe the answer. If \( W \) is ‘simply connected at infinity’ (that is, for each compact set \( K \) there is yet a larger one \( K' \), such that loops in the complement of \( K' \) always can be filled in by disks in the complement of \( K \)), then, according to [1], the beautiful answer is: if and only if the homology of \( W \) is finitely generated. In general (see [4]), there is a ‘fundamental group at infinity’ whose algebraic properties enter in the complete analysis; for instance, if this fundamental group is cyclic of prime order \( p \), then the above theorem remains true only if the ring of integers in the cyclotomic field of \( p \)th roots of unity is a unique factorization domain (that is, \( p \) has class number one).

Already there are at least three points that need explaining. The first is: how does one see an unpublished document such as [4]? The second is more mathematical: what is meant by ‘fundamental group at infinity’? It should be the fundamental group of the boundary, if a boundary exists, but since we do not yet have such an object, we seem to be in trouble. Thus we should, in any case, want pointers about what \( W \) really does look like, in case we do not succeed in finding such a good compactification, for example, as a manifold with boundary. And, finally: what does adding a boundary have to do with cyclotomic fields?

This last point is actually at the beginning of some very beautiful connections between algebraic \( K \)-theory and topology, and is also intimately connected with the nonsimply connected generalizations of the key part of Smale’s proof of the high-dimensional Poincaré conjecture. An introduction to the compact aspects of this can already be found in [2]. The cyclotomic fields are quite close to the group ring of the fundamental groups involved, and the projective modules and matrices over this ring encode a great deal of the relevant topology. After the early and mid-1960s saw geometric applications of this sort of algebra, as in these compactification theorems and the \( h \)-cobordism theorem, the late 1960s and early 1970s saw topologists returning the favour, and proving theorems about algebraic \( K \)-groups of twisted polynomial extensions and the like by either direct geometric considerations or algebraic analogues of such. However, just as in the unpublished [4], in this fast-paced field there were a number of other ideas ‘well known to experts’ and, indeed, exploited throughout the literature, yet not themselves a part of that literature.

The book under review goes a good way towards remedying the situation, and thus dealing with the three issues mentioned above. There is a good and careful discussion of that Cheshire Cat-like ‘space at infinity’ (= ‘end of the manifold’, explaining, in part, the title of the book) that is not quite a space yet has a homotopy type, and, therefore, a fundamental group and all sorts of homology theories. It also sketches the proofs of these connections between \( K \)-theory and topology, and even makes some interesting contributions in this direction—unfortunately, but perhaps
of necessity, without complete detail. The material that has been covered in detail seems to be precisely the material that has proven itself to be useful in more recent work, and is necessary for good, lively, intelligible flow, and might have been slighted in the literature. Thus a good student armed with this book and the theorems genuinely proven within should be able to write down a proof of Siebenmann’s result after spending an afternoon in the library.

Most importantly, while a book like this could easily have had a desiccated feel, since so much of its raison d’etre is embedded in mathematics of the past, this is well avoided because behind the scenes the authors have not-so-secretly intended their work to apply to an entirely different set of problems (see pages xix–xxi) with roots in the theory of group actions, embeddings and stratified spaces. The detailed picture of ends that this book espouses, in an extremely classical setting, does shed light on these contemporary issues. Probably, I should just refer to [3].

References

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