Supporting Material of

Video-Based Tracking of Single Molecules Exhibiting Directed In-frame Motion

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Derivatives of the log-likelihood function

In this note, we derive the first and second order derivatives of the log-likelihood function (Eq. 7 in the main article), with respect to Θ, which are used both in the Cramer-Rao Lower Bound computation and the iterative Maximum Likelihood algorithm.

First Order Derivatives:

A general expression for the first order partial derivative of the log-likelihood function with respect to the components of Θ can be written as:

$$\frac{\partial L(\Theta)}{\partial \Theta_i} = \sum_{k=1}^{N} \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \frac{\partial \Lambda_k(T)}{\partial \Theta_i}. \quad (1)$$

• Partial derivative with respect to \( x_0 \):

$$\frac{\partial L(\Theta)}{\partial x_0} = \sum_{k=1}^{N} \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \frac{\partial \Lambda_k(T)}{\partial x_0}, \quad (2)$$

where

$$\frac{\partial \Lambda_k(T)}{\partial x_0} = \lambda_0 \int_{0}^{T} \frac{\partial p_k(\tau)}{\partial x_0} d\tau. \quad (3)$$

We can write \( p_k(\tau) \) as

$$p_k(\tau) = \int_{y_{k1}}^{y_{k2}} \int_{x_{k1}}^{x_{k2}} g(x - x_0 - v_x \tau, y - y_0 - v_y \tau) dx dy$$

$$= \int_{y_{k1}-y_0-v_y \tau}^{y_{k2}-y_0-v_y \tau} \int_{x_{k1}-x_0-v_x \tau}^{x_{k2}-x_0-v_x \tau} g(x, y) dx dy.$$

1
Therefore, the partial derivative of $p_k(\tau)$ with respect to $x_0$ can be written as
\[
\frac{\partial p_k(\tau)}{\partial x_0} = \int_{y_{k1}-y_0-v_y\tau}^{y_{k2}-y_0-v_y\tau} \int_{x_{k1}-x_0-v_x\tau}^{x_{k2}-x_0-v_x\tau} \frac{\partial}{\partial x_0} g(x, y) \, dx \, dy
\]
\[
= \int_{y_{k1}-y_0-v_y\tau}^{y_{k2}-y_0-v_y\tau} \left( g(x_{k1} - x_0 - v_x\tau, y) - g(x_{k2} - x_0 - v_x\tau, y) \right) \, dy.
\]

If the point spread function is Gaussian, with parameter $\sigma$, i.e.,
\[
g(x, y) = \frac{1}{2\pi \sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} = \frac{1}{\sigma^2} G\left(\frac{x}{\sigma}\right)G\left(\frac{y}{\sigma}\right),
\]
where $G(a) = \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}}$, we can write the probability $p_k(\tau)$ as
\[
p_k(\tau) = Q_{kx}(\tau)Q_{ky}(\tau),
\]
where
\[
Q_{kx}(\tau) = \int_{x_{k1}-x_0-v_x\tau}^{x_{k2}-x_0-v_x\tau} \frac{e^{-x^2}}{\sqrt{2\pi}} \, dx
\]
\[
= \frac{1}{2} \left( \text{erf}\left(\frac{x_{k2} - x_0 - v_x\tau}{\sqrt{2}\sigma}\right) - \text{erf}\left(\frac{x_{k1} - x_0 - v_x\tau}{\sqrt{2}\sigma}\right) \right)
\]
as
\[
\text{erf}(a) = \frac{2}{\sqrt{\pi}} \int_0^a e^{-t^2} \, dt.
\]

Similarly
\[
Q_{ky}(\tau) = \int_{y_{k1}-y_0-v_y\tau}^{y_{k2}-y_0-v_y\tau} \frac{e^{-y^2}}{\sqrt{2\pi}} \, dx
\]
\[
= \frac{1}{2} \left( \text{erf}\left(\frac{y_{k2} - y_0 - v_y\tau}{\sqrt{2}\sigma}\right) - \text{erf}\left(\frac{y_{k1} - y_0 - v_y\tau}{\sqrt{2}\sigma}\right) \right).
\]

Thus we obtain
\[
\frac{\partial p_k(\tau)}{\partial x_0} = \frac{\partial Q_{kx}(\tau)}{\partial x_0} Q_{ky},
\]
where
\[
\frac{\partial Q_{kx}}{\partial x_0} = \frac{1}{\sqrt{2\pi} \sigma} \left( e^{-\frac{(x_{k1} - x_0 - v_x\tau)^2}{2\sigma^2}} - e^{-\frac{(x_{k2} - x_0 - v_x\tau)^2}{2\sigma^2}} \right)
\]
\[
= -\frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x_{k1} - x_0 - v_x\tau)^2}{2\sigma^2}} \left( e^{\frac{2\Delta_x(x_0 + v_x\tau - x_{k1}) - \Delta_x^2}{2\sigma^2}} - 1 \right).
\]
• Partial derivative with respect to \( y_0 \) can be similarly obtained as:

\[
\frac{\partial L(\Theta)}{\partial y_0} = \sum_{k=1}^{N} \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \frac{\partial \Lambda_k(T)}{\partial y_0},
\]

where

\[
\frac{\partial \Lambda_k(T)}{\partial y_0} = \lambda_0 \int_{0}^{T} \frac{\partial p_k(\tau)}{\partial y_0} d\tau,
\]

and

\[
\frac{\partial p_k(\tau)}{\partial y_0} = \int_{x_{k1}-x_0-v_x \tau}^{x_{k2}-x_0-v_x \tau} (g(x, y_{k1} - y_0 - y \tau) - g(x, y_{k2} - y_0 - y \tau)) dx.
\]

If the point spread function is Gaussian, with parameter \( \sigma \),

\[
\frac{\partial p_k(\tau)}{\partial y_0} = \frac{\partial Q_{ky}(\tau)}{\partial y_0} Q_{kx},
\]

where

\[
\frac{\partial Q_{ky}}{\partial y_0} = \frac{1}{\sqrt{2\pi\sigma}} \left( e^{-\frac{(y_{k1}-y_0-y \tau)^2}{2\sigma^2}} - e^{-\frac{(y_{k2}-y_0-y \tau)^2}{2\sigma^2}} \right)
\]

\[
= -\frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(y_{k1}-y_0-y \tau)^2}{2\sigma^2}} \left( e^{\frac{2\Delta y(y_0+y \tau-y_{k1})}{2\sigma^2}} - 1 \right).
\]

• Partial derivative with respect to \( v_x \):

\[
\frac{\partial L(\Theta)}{\partial v_x} = \sum_{k=1}^{N} \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \frac{\partial \Lambda_k(T)}{\partial v_x},
\]

where

\[
\frac{\partial \Lambda_k(T)}{\partial v_x} = \lambda_0 \int_{0}^{T} \frac{\partial p_k(\tau)}{\partial v_x} d\tau,
\]

and

\[
\frac{\partial p_k(\tau)}{\partial v_x} = \tau \int_{x_{k1}-x_0-v_x \tau}^{x_{k2}-x_0-v_x \tau} (g(x_{k1} - x_0 - v_x \tau, y) - g(x_{k2} - x_0 - v_x \tau, y)) dy.
\]

We note that

\[
\frac{\partial p_k(\tau)}{\partial v_x} = \tau \frac{\partial p_k(\tau)}{\partial x_0}.
\]
Partial derivative with respect to $v_y$ can be similarly obtained as:

$$
\frac{\partial L(\Theta)}{\partial v_y} = \sum_{k=1}^{N} \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \frac{\partial \Lambda_k(T)}{\partial v_y},
$$

(16)

where

$$
\frac{\partial \Lambda_k(T)}{\partial v_y} = \lambda_0 \int_0^T \frac{\partial p_k(\tau)}{\partial v_y} d\tau,
$$

(17)

and

$$
\frac{\partial p_k(\tau)}{\partial v_y} = \tau \int_{x_{k2} - x_{0y} - \tau}^{x_{k1} - x_{0y} - \tau} (g(x, y_{k1} - y_0 - v_y \tau) - g(x, y_{k2} - y_0 - v_y \tau)) dx.
$$

Again, we note that

$$
\frac{\partial p_k(\tau)}{\partial v_y} = \tau \frac{\partial p_k(\tau)}{\partial y}.
$$

(18)

- Partial derivative with respect to $\lambda_0$:

$$
\frac{\partial L(\Theta)}{\partial \lambda_0} = \sum_{k=1}^{N} \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \frac{\partial \Lambda_k(T)}{\partial \lambda_0},
$$

(19)

where

$$
\frac{\partial \Lambda_k(T)}{\partial \lambda_0} = \int_0^T p_k(\tau) d\tau.
$$

(20)

Therefore,

$$
\frac{\partial L(\Theta)}{\partial \lambda_0} = \sum_{k=1}^{N} \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \int_0^T p_k(\tau) d\tau.
$$

(21)

- Partial derivative with respect to $\lambda_{bg}$:

$$
\frac{\partial L(\Theta)}{\partial \lambda_{bg}} = \sum_{k=1}^{N} \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \frac{\partial \Lambda_k(T)}{\partial \lambda_{bg}},
$$

(22)

where

$$
\frac{\partial \Lambda_k(T)}{\partial \lambda_{bg}} = T.
$$

(23)

Therefore,

$$
\frac{\partial L(\Theta)}{\partial \lambda_{bg}} = T \sum_{k=1}^{N} \frac{m_k}{\Lambda_k(T)} - TN.
$$

(24)
Second Order Derivatives:

A general expression for the second order partial derivative of the log-likelihood function with respect to the components of $\Theta$ can be written as:

$$\frac{\partial^2 L(\Theta)}{\partial \Theta_i \partial \Theta_j} = \sum_{k=1}^{N} \left( \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \frac{\partial^2 \Lambda_k(T)}{\partial x_0^2} - \frac{m_k}{\Lambda_k^2(T)} \left( \frac{\partial \Lambda_k(T)}{\partial x_0} \right)^2 \right).$$ \hspace{1cm} (25)

We will only consider the case $\Theta_i = \Theta_j$, as the algorithm will be using only the diagonal terms of the Hessian matrix.

- The second order partial derivative with respect to $x_0$:

$$\frac{\partial^2 L(\Theta)}{\partial x_0^2} = \sum_{k=1}^{N} \left( \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \frac{\partial^2 \Lambda_k(T)}{\partial x_0^2} - \frac{m_k}{\Lambda_k^2(T)} \left( \frac{\partial \Lambda_k(T)}{\partial x_0} \right)^2 \right),$$ \hspace{1cm} (26)

where

$$\frac{\partial^2 \Lambda_k(T)}{\partial x_0^2} = \lambda_0 \int_0^T \frac{\partial^2 p_k(\tau)}{\partial x_0^2} d\tau. \hspace{1cm} (27)$$

If the point spread function is Gaussian with parameter $\sigma$, we can write

$$\frac{\partial^2 p_k(\tau)}{\partial x_0^2} = \frac{1}{\sqrt{2\pi}\sigma^3} \left( (x_{k1} - x_0 - v_x \tau)e^{-\frac{(x_{k1} - x_0 - v_x \tau)^2}{2\sigma^2}} - (x_{k2} - x_0 - v_x \tau)e^{-\frac{(x_{k2} - x_0 - v_x \tau)^2}{2\sigma^2}} \right) \times Q_{xy}(\tau).$$

- The second order partial derivative with respect to $y_0$:

$$\frac{\partial^2 L(\Theta)}{\partial y_0^2} = \sum_{k=1}^{N} \left( \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \frac{\partial^2 \Lambda_k(T)}{\partial y_0^2} - \frac{m_k}{\Lambda_k^2(T)} \left( \frac{\partial \Lambda_k(T)}{\partial y_0} \right)^2 \right),$$ \hspace{1cm} (28)

where

$$\frac{\partial^2 \Lambda_k(T)}{\partial y_0^2} = \lambda_0 \int_0^T \frac{\partial^2 p_k(\tau)}{\partial y_0^2} d\tau. \hspace{1cm} (29)$$

If the point spread function is Gaussian, with parameter $\sigma$, we can write

$$\frac{\partial^2 p_k(\tau)}{\partial y_0^2} = \frac{1}{\sqrt{2\pi}\sigma^3} \left( (y_{k1} - y_0 - y_x \tau)e^{-\frac{(y_{k1} - y_0 - y_x \tau)^2}{2\sigma^2}} - (y_{k2} - y_0 - y_x \tau)e^{-\frac{(y_{k2} - y_0 - y_x \tau)^2}{2\sigma^2}} \right) \times Q_{xy}(\tau).$$
• The second order partial derivative with respect to $v_x$:

$$\frac{\partial^2 L(\Theta)}{\partial v_x^2} = \sum_{k=1}^{N} \left( \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \frac{\partial^2 \Lambda_k(T)}{\partial v_x^2} - m_k \Lambda_k(T) \left( \frac{\partial \Lambda_k(T)}{\partial v_x} \right)^2 \right),$$

(30)

where

$$\frac{\partial^2 \Lambda_k(T)}{\partial v_x^2} = \lambda_0 \int_0^T \frac{\partial^2 p_k(\tau)}{\partial v_x^2} d\tau.$$  

(31)

If the point spread function is Gaussian, with parameter $\sigma$, we can write

$$\frac{\partial^2 p_k(\tau)}{\partial v_x^2} = \tau^2 \frac{\partial^2 p_k(\tau)}{\partial x_0^2}. $$

• The second order partial derivative with respect to $v_y$:

$$\frac{\partial^2 L(\Theta)}{\partial v_y^2} = \sum_{k=1}^{N} \left( \left( \frac{m_k}{\Lambda_k(T)} - 1 \right) \frac{\partial^2 \Lambda_k(T)}{\partial v_y^2} - m_k \Lambda_k(T) \left( \frac{\partial \Lambda_k(T)}{\partial v_y} \right)^2 \right),$$

(32)

where

$$\frac{\partial^2 \Lambda_k(T)}{\partial v_y^2} = \lambda_0 \int_0^T \frac{\partial^2 p_k(\tau)}{\partial v_y^2} d\tau,$$

(33)

If the point spread function is Gaussian, with parameter $\sigma$, we can write

$$\frac{\partial^2 p_k(\tau)}{\partial v_y^2} = \tau^2 \frac{\partial^2 p_k(\tau)}{\partial y_0^2}. $$

• The second order partial derivative with respect to $\lambda_0$:

$$\frac{\partial^2 L(\Theta)}{\partial \lambda_0^2} = - \sum_{k=1}^{N} m_k \Lambda_k(T) \left( \frac{\partial \Lambda_k(T)}{\partial \lambda_0} \right)^2,$$

(34)

• The second order partial derivative with respect to $\lambda_{bg}$:

$$\frac{\partial^2 L(\Theta)}{\partial \lambda_{bg}^2} = - \sum_{k=1}^{N} m_k \Lambda_k(T) \left( \frac{\partial \Lambda_k(T)}{\partial \lambda_{bg}} \right)^2,$$

(35)