Online appendix for the paper

Fuzzy Answer Set Computation via Satisfiability Modulo Theories

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Appendix A Proofs

Proposition 1
For every FASP program $\Pi$, it holds that $\Pi \equiv_{At(\Pi)} simp(\Pi)$, i.e.,

$$|SM(\Pi)| = |SM(simp(\Pi))|$$

and

$$\{I \cap At(\Pi) | I \in SM(\Pi)\} = \{I \cap At(\Pi) | I \in SM(simp(\Pi))\}.$$  

Proof
Since each rule is rewritten independently, we can prove $\Pi \equiv_{At(\Pi)} (\Pi \setminus \{r\}) \cup simp(\{r\})$, where $r$ is some rule in $\Pi$. We use structural induction on $r$. The base case, i.e., $r$ is of the form $\alpha \leftarrow \beta$ with $\alpha \in B$ and $\beta \in B$, is trivial because $simp(\{\alpha \leftarrow \beta\}) = \{\alpha \leftarrow \beta\}$. Now, consider $r$ of the form $\alpha \leftarrow \neg \beta$. We have to show $\Pi \equiv_{At(\Pi)} \Pi'$, where $\Pi' := (\Pi \setminus \{r\}) \cup \{\alpha \leftarrow p, p \leftarrow \beta\}$. For $I \in SM(\Pi)$, define $I'$ such that $I'(p) := I(\beta)$, and $I'(q) := I(q)$ for all $q \in At(\Pi)$. We have that $I' \in SM(\Pi')$. Moreover, for any $J \in SM(\Pi')$ it holds that $J(p) = J(\beta)$ because the only head occurrence of $p$ in $\Pi'$ is in $p \leftarrow \beta$. It turns out that $J \cap At(\Pi)$ belongs to $SM(\Pi)$. The remaining cases are given in (Mushthofa et al. 2014).  

Theorem 1
Checking coherence of FASP programs is $\Sigma^P_2$-hard already in the following cases: (i) all connectives are $\otimes$; (ii) head connectives are $\nabla$, and body connectives are $\bar{\nabla}$ (or $\otimes$); and (iii) head connectives are $\oplus$, and body connectives are $\bar{\nabla}$ (or $\otimes$) and $\oplus$.  

Proof

We start by giving the common properties that will be used to prove each part of the theorem. We reduce the satisfiability problem for 2-QBF formulas to FASP coherence testing. Let \( \phi \) be \( \exists x_1, \ldots, x_m \forall x_{m+1}, \ldots, x_n \bigwedge_{i=1}^{k} L_{k,1} \land L_{k,2} \land L_{k,3} \), where \( n > m \geq 1, k \geq 1 \).

For each \( \odot \in \{ \vee, \oplus, \odot \} \), our aim is to build a FASP program \( \Pi^\circ_{\phi} \) such that \( \phi \) is satisfiable if and only if \( \Pi^\circ_{\phi} \) is coherent.

In the construction of \( \Pi^\circ_{\phi} \) we use the mapping \( \sigma \) such that \( \sigma(x_i) := x_i^T \), and \( \sigma(\neg x_i) := x_i^F \), for all \( i \in [1..n] \). Moreover, \( \Pi^\circ_{\phi} \) will have atoms \( sat \), and \( x_i^T, x_i^F \) for all \( i \in [1..n] \), and its models will satisfy the following properties, for a fixed truth degree \( d \in [0,1] \):

1. \( I \models \Pi^\circ_{\phi} \) implies \( I \models sat \) = 1;
2. \( I \models \Pi^\circ_{\phi} \) implies either \( I(x_i^T) = 1 \land I(x_i^F) = d \), or \( I(x_i^T) = 1 \land I(x_i^F) = d \), for all \( i \in [1..n] \);
3. \( I \models \Pi^\circ_{\phi} \) and \( I \models sat \) = 1 implies \( I(x_i^T) = I(x_i^F) = 1 \), for all \( i \in [m+1..n] \);
4. \( J \subset I \) and \( J \models (\Pi^\circ_{\phi})^J \) implies \( J \models sat \) = \( d \) and either \( I(x_i^T) = 1 \land I(x_i^F) = d \), or \( I(x_i^T) = 1 \land I(x_i^F) = d \), for all \( i \in [1..n] \).

We will then define a mapping between assignments for \( x_1, \ldots, x_m \) and interpretations of \( \Pi^\circ_{\phi} \). Let \( \nu \) be a Boolean assignment for \( x_1, \ldots, x_m \). Define \( I^\phi_{\nu} \) to be the interpretation such that: \( I^\phi_{\nu}(x_i^T) \) equals 1 if \( \nu(x_i) = 1 \), and \( d \) otherwise, for all \( i \in [1..m] \); \( I^\phi_{\nu}(x_i^F) \) equals 1 if \( \nu(x_i) = 0 \), and \( d \) otherwise, for all \( i \in [1..m] \); \( I^\phi_{\nu}(x_i) = 1 \) for all \( i \in [m+1..n] \); and \( I^\phi_{\nu}(sat) = 1 \). Moreover, for an extended Boolean assignment for \( x_1, \ldots, x_n \), we define \( I^\phi_{\nu} \) to be the interpretation such that: \( I^\phi_{\nu}(x_i^T) \) equals 1 if \( \nu'(x_i) = 1 \), and \( d \) otherwise, for all \( i \in [1..n] \); \( I^\phi_{\nu}(x_i^F) \) equals 1 if \( \nu'(x_i) = 0 \), and \( d \) otherwise, for all \( i \in [1..n] \); and \( I^\phi_{\nu}(sat) = d \). These mappings will allow us to define one-to-one mappings between satisfying assignments of \( \phi \) and stable models of \( \Pi^\circ_{\phi} \), and between unsatisfying assignments of \( \phi \) and minimal models of reducts (counter models of \( \Pi^\circ_{\phi} \)).

Proof of (ii). We adapt the construction by (Eiter and Gottlob 1995). The program \( \Phi^\circ_{\phi} \) is the following:

\[
\begin{align*}
x_i^T \forall x_i^F & \leftarrow 1 \quad \forall i \in [1..n] \\
x_i^T & \leftarrow sat \quad x_i^F & \leftarrow \neg sat \quad 0 & \leftarrow \neg sat \\
\text{sat} & \leftarrow \sigma(L_{k,1}) \bigwedge \sigma(L_{k,2}) \bigwedge \sigma(L_{k,3}) & \forall i \in [1..k]
\end{align*}
\]  

(A1)  

(A2)  

(A3)

The program \( \Phi^\circ_{\phi} \) has the four properties given above for \( d = 0 \). Any model of \( \Phi^\circ_{\phi} \) is of the form \( I^\phi_{\nu} \), for some assignment \( \nu \) for \( x_1, \ldots, x_m \). If we consider the reduct \( (\Phi^\circ_{\phi})^J \), the rule \( 0 \leftarrow \neg sat \) is replaced by \( 0 \leftarrow 0 \). Any minimal model strictly contained in \( I^\phi_{\nu} \) will be of the form \( J^\nu \), for some assignment \( \nu \) extending \( \nu \). Such a \( J^\nu \) would imply that \( \nu'(\psi) = 0 \), and therefore \( \nu(\phi) = 0 \). On the other hand, if such a \( J^\nu \) does not exist, it means that \( sat \) is necessarily 1; if there is \( i \in [1..k] \) such that \( \sigma(L_{k,1}) \bigwedge \sigma(L_{k,2}) \bigwedge \sigma(L_{k,3}) \) is necessarily 1; if all \( \nu \) extending \( \nu \) are such that \( \nu'(\psi) = 1 \); if \( \nu(\psi) = 1 \). Hence, we have that \( \phi \) is satisfiable if \( \Pi^\circ_{\phi} \) is coherent.

To complete this part of the proof, it is enough to replace (A3) by

\[
\text{sat} \leftarrow \sigma(L_{k,1}) \otimes \sigma(L_{k,2}) \otimes \sigma(L_{k,3}) \quad \forall i \in [1..k]
\]  

(A4)
because any model and counter model of \( \Pi_\phi \) give a Boolean interpretation to \( \sigma(L_{k,1}) \otimes \sigma(L_{k,3}) \).

**Proof of (iii).** This is essentially folklore. Having \( \oplus \) in rule bodies allows to crispify a variable \( p \) by means of the common pattern \( p \leftarrow p \oplus p \). The program \( \Pi_\phi \) is thus

\[
\begin{align*}
x_i^T \oplus x_i^F &\leftarrow 1 & \quad \forall i \in [1..n] \\
x_i^T &\leftarrow x_i^T \oplus x_i^T & \quad \forall i \in [1..n] \\
x_i^F &\leftarrow x_i^F \oplus x_i^F & \quad \forall i \in [1..n] \\
x_i^T &\leftarrow \text{sat} & \quad \forall i \in [0..n] \\
x_i^F &\leftarrow \text{sat} & \quad \forall i \in [0..n] \\
0 &\leftarrow \sim\text{sat} & \quad \forall i \in [0..n] \\
0 &\leftarrow \text{sat} \oplus \text{sat} & \quad \forall i \in [0..n] \\
\text{sat} &\leftarrow \sigma(L_{k,1}) \oplus \sigma(L_{k,2}) \oplus \sigma(L_{k,3}) & \quad \forall i \in [1..k]
\end{align*}
\]

(A5)
(A6)
(A7)

The same argument used for (ii) proves that \( \phi \) is satisfiable iff \( \Pi_\phi \) is coherent. The same holds if (A7) is replaced by (A4).

**Proof of (i).** This is the most sophisticated construction. The program \( \Pi_\phi \) is

\[
\begin{align*}
x_i^T \oplus x_i^F &\leftarrow 0.5 & \quad \forall i \in [1..n] \\
x_i^T \otimes x_i^T \otimes x_i^T &\leftarrow x_i^T \otimes x_i^T & \quad \forall i \in [1..n] \\
x_i^F \otimes x_i^F \otimes x_i^F &\leftarrow x_i^F \otimes x_i^F & \quad \forall i \in [1..n] \\
x_i^T &\leftarrow \text{sat} & \quad \forall i \in [0..n] \\
x_i^F &\leftarrow \text{sat} & \quad \forall i \in [0..n] \\
0 &\leftarrow \sim\text{sat} & \quad \forall i \in [0..n] \\
0 &\leftarrow \text{sat} \oplus \text{sat} & \quad \forall i \in [0..n] \\
\text{sat} &\leftarrow \sigma(L_{k,1}) \otimes \sigma(L_{k,2}) \otimes \sigma(L_{k,3}) & \quad \forall i \in [1..k]
\end{align*}
\]

(A8)
(A9)
(A10)
(A11)
(A12)

This program \( \Pi_\phi \) has the four properties given at the beginning of this proof, but for \( d = 0.5 \). (Note that rule \( \text{sat} \leftarrow 0.5 \) was added to have a uniform proof with the previous parts, but the construction would work also without such a rule.) In fact, all atoms must be assigned a truth degree of 0.5 or 1. Hence, the interpretation of \( \sigma(L_{k,1}) \otimes \sigma(L_{k,2}) \otimes \sigma(L_{k,3}) \) will be 1 if \( \sigma(L_{k,1}), \sigma(L_{k,2}), \sigma(L_{k,3}) \) are 1, and less than or equal to 0.5 otherwise. We can thus rely on the argument given in the proof of (ii). \( \square \)

**Theorem 2**

Let \( \Pi \) be FASP program. If \( \Pi \) is HCF then \( \Pi \equiv_{A_{t}(\Pi)} \text{shift}(\Pi) \).

**Proof**

Since the shift is performed independently on each rule of \( \Pi \), it suffices to show \( \Pi' \cup \{p_1 \odot \cdots \odot p_n \leftarrow \beta\} \equiv_{A_{t}(\Pi)} \Pi' \cup \text{shift}(\{p_1 \odot \cdots \odot p_n \leftarrow \beta\}) \), where \( \Pi' \cup \{p_1 \odot \cdots \odot p_n \leftarrow \beta\} = \Pi \), \( n \geq 2 \), and \( \odot \in \{\odot, \otimes\} \). To simplify the presentation, \( \beta \) is assumed to be a propositional atom. Moreover, since \( \Pi \) is HCF, w.l.o.g. we can assume that, for \( 1 \leq i < j \leq n \), \( p_i \) does not reach \( p_j \) in \( \Pi \). In each part of the proof, we will provide a one-to-one mapping between the (minimal) models of the original program and the models of shifted program. Moreover, we will give a mapping of the counter model of the original program into the counter models of the shifted program, and vice versa.

**Proof of \( \oplus \).** \( I \models \Pi' \cup \{p_1 \oplus \cdots \oplus p_n \leftarrow \beta\} \) iff \( I \models \Pi' \cup \text{shift}(\{p_1 \oplus \cdots \oplus p_n \leftarrow \beta\}) \) holds because \( I(p_1) + \cdots + I(p_n) \geq I(\beta) \) iff

\[
I(p_i) \geq I(\beta) + \sum_{j \in [1..n], j \neq i} (1 - I(p_j)) - (n - 1) = I(\beta) - \sum_{j \in [1..n], j \neq i} I(p_j)
\]
for all $i \in [1..n]$. Let $I$ be a model of the two programs.

For all $J \subset I$, it holds that $J \models (\Pi')^I \cup \{p_1 + \cdots + p_n \leftarrow \beta\}^I$ implies that $J \models (\Pi')^I \cup \text{shift}(\{p_1 + \cdots + p_n \leftarrow \beta\})^I$ because $J(p_1) + \cdots + J(p_n) \geq J(\beta)$ if $J(p_i) \geq J(\beta) + \sum_{j \in [1..n], j \neq i}(1 - J(p_j))$ for all $i \in [1..n]$, which implies

$$J(p_i) \geq J(\beta) + \sum_{j \in [1..n], j \neq i}(1 - J(p_j)) - (n - 1) = J(\beta) - \sum_{j \in [1..n], j \neq i} I(p_j),$$

by assumption $J(p_j) \leq I(p_j)$ for all $p_j \in [1..n]$.

For the converse direction, we show that for any interpretation $J \subset I$ such that $J \models (\Pi')^I \cup \text{shift}(\{p_1 + \cdots + p_n \leftarrow \beta\})^I$, there is $K$ such that $J \subseteq K \subset I$ and $K \models (\Pi')^I \cup \{p_1 + \cdots + p_n \leftarrow \beta\}^I$. Let us assume that $\{p_1 + \cdots + p_n \leftarrow \beta\} \neq \emptyset$, and that $J(p_i) < I(p_i)$ for some $i \in [1..n]$, otherwise the proof is immediate. We define the following non-deterministic sequence: $K_0 := J$; for $i \in [0..n-1]$, $K_{i+1}$ is any subset minimal model of $(\Pi')^I$ such that $K_i \subseteq K_{i+1} \subset I$, and $K_{i+1} = \text{min}(I(p_{n-i}), m)$, where $m = \max(K_i(p_{n-i}), K_i(\beta) - \sum_{j \in [1..n], j \neq i} K_i(p_j))$. The sequence is well defined because in $K_{i+1}$ we are possibly increasing the truth degree of $p_{n-i}$, which cannot cause an increase of any $p_j$ with $j < n - i$ by assumption. Intuitively, we possibly increase the truth degree of $p_1, \ldots, p_n$ in order to satisfy the original rule $p_1 + \cdots + p_n \leftarrow \beta$, and we do this by preferring atoms with higher indices. Hence, we have $K_n \subset I$ and $K_n \models (\Pi')^I \cup \{p_1 + \cdots + p_n \leftarrow \beta\}^I$.

**Proof for $\otimes$.** For an interpretation $I$, define $I'$ to be such that: $I'(p) = I(p)$ for all $p \in \text{At}(\Pi)$; $I'(q) = 1$ if $I(\beta) > 0$, and 0 otherwise. We follow the line of the previous proof. Let $I$ be an interpretation such that $I(\beta) > 0$, otherwise the proof is immediate. Then, $I(q) = 1$, and $I$ is a minimal model of $\Pi'' \cup \{p_1 + \cdots + p_n \leftarrow \beta\}$ if and only if $I'$ is a minimal model of $\Pi'' \cup \text{shift}(\{p_1 + \cdots + p_n \leftarrow \beta\})$ because $I(p_1) + \cdots + I(p_n) - (n - 1) \geq I(\beta)$ if $I(p_i) \geq I(\beta) + \sum_{j \in [1..n], j \neq i}(1 - I(p_j))$, for all $i \in [1..n]$. Let $I$ be a minimal model of $\Pi$ with $I(\beta) > 0$.

For all $J \subset I$, we have that $J \models (\Pi'')^I \cup \{p_1 + \cdots + p_n \leftarrow \beta\}^I$ implies that $J' \models (\Pi'')^I \cup \text{shift}(\{p_1 + \cdots + p_n \leftarrow \beta\})^I$ because $J(p_1) + \cdots + J(p_n) - (n - 1) \geq J(\beta)$ if $J(p_i) \geq J(\beta) + \sum_{j \in [1..n], j \neq i}(1 - J(p_j))$ for all $i \in [1..n]$, which itself implies $J'(p_i) \geq J(\beta) + \sum_{j \in [1..n], j \neq i}(1 - I(p_j))$ since by assumption $J'(p_j) = J(p_j) \leq I(p_j)$ for all $p_j \in [1..n]$.

For the converse direction, we only change the non-deterministic sequence from the previous proof as follows: $K_0 := J$; for $i \in [0..n-1]$, $K_{i+1}$ is any subset minimal model of $(\Pi'')^I$ such that $K_i \subseteq K_{i+1} \subset I$, and $K_{i+1} = \text{min}(I(p_{n-i}), m)$, where $m = \max(K_i(p_{n-i}), K_i(\beta) + \sum_{j \in [1..n], j \neq i}(1 - K_i(p_j)))$. We have $K_n \subset I'$.

**Proof for $\forall$.** Given an interpretation $I$, define $I'$ to be such that: $I'(p) = I(p)$ for every $p \in \text{At}(\Pi)$; $I'(q_i) = 1$; and for $i \in [1..n-1]$, $I'(q_i)$ is equal to 1 if $I(p_i) > \max(I(p_j) | j \in [i + 1..n])$, and 0 otherwise. Following the line of the previous two proofs, $I$ is a minimal model of $\Pi'' \cup \{p_1 \otimes \cdots \otimes p_n \leftarrow \beta\}$ if and only if $I'$ is a minimal model of
Theorem 3

Let $\Pi$ be a FASP program. $I \in SM(\Pi)$ if and only if $A_I \models smt(\Pi)$.

Proof

We use structural induction to prove that $I(\alpha) = f(\alpha)^{A_I}$ holds for any expression or term $\alpha$, and for $f \in \{out, inn\}$.

- The base cases are immediate: for $c \in [0, 1]$, $I(c) = c^{A_I} = f(c)^{A_I}$ by definition; for $p \in At(\Pi)$, $I(p) = p^{A_I} = f(p)^{A_I}$ by definition.
- For $\sim$, assuming that the claim holds for $\alpha$, we have $I(\sim \alpha) = 1 - I(\alpha) = 1 - out(\alpha)^{A_I} = f(\sim \alpha)^{A_I}$.
- For $\oplus$, assuming that the claim holds for $\alpha$ and $\beta$, we have
  \[ I(\alpha \oplus \beta) = \min(I(\alpha) + I(\beta), 1) = \min(f(\alpha)^{A_I} + f(\beta)^{A_I}, 1) = \text{ite}(f(\alpha) + f(\beta) \leq 1, f(\alpha) + f(\beta))^{A_I} = f(\alpha \oplus \beta)^{A_I}. \]
- For $\otimes$, assuming that the claim holds for $\alpha$ and $\beta$, we have
  \[ I(\alpha \otimes \beta) = \max(I(\alpha) + I(\beta) - 1, 0) = \max(f(\alpha)^{A_I} + f(\beta)^{A_I} - 1, 0) = \text{ite}(f(\alpha) + f(\beta) - 1 \geq 0, f(\alpha) + f(\beta) - 1, 0)^{A_I} = f(\alpha \otimes \beta)^{A_I}. \]
- For $\forall$, assuming that the claim holds for $\alpha$ and $\beta$, we have
  \[ I(\alpha \forall \beta) = \max(I(\alpha), I(\beta)) = \max(f(\alpha)^{A_I}, f(\beta)^{A_I}) = \text{ite}(f(\alpha) \geq f(\beta), f(\alpha), f(\beta))^{A_I} = f(\alpha \forall \beta)^{A_I}. \]
- For $\exists$, assuming that the claim holds for $\alpha$ and $\beta$, we have
  \[ I(\alpha \exists \beta) = \min(I(\alpha), I(\beta)) = \min(f(\alpha)^{A_I}, f(\beta)^{A_I}) = \text{ite}(f(\alpha) \leq f(\beta), f(\alpha), f(\beta))^{A_I} = f(\alpha \exists \beta)^{A_I}. \]
We can thus conclude that $I \models \Pi$ if and only if $A_I$ is a $\Sigma$-model of the theory $\{ p \in [0, 1] \mid p \in At(\Pi) \cup \{ out(r) \mid r \in \Pi \}$ Moreover, if $I \in SM(\Pi)$ then there is no $J \subset I$ such that $J \models \Pi^I$, which is the case if and only if $A_I$ also satisfies formula $\phi_{inn}$. 

Theorem 4

Let $\Pi$ be a program such that $\Pi \setminus \text{bool}(\Pi)$ is acyclic. Then, $I \in SM(\Pi)$ if and only if $A_I \models rcomp(shift(simp(\Pi)))$.

Proof

Let $\Pi'$ be $shift(simp(\Pi))$, and $\Pi'' = \text{bool}^+(\Pi')$. By Proposition 1 and Theorem 2, we know that $\Pi \equiv_{At(\Pi)} \Pi'$. Moreover, if $\Pi \setminus \text{bool}(\Pi)$ is acyclic then $\Pi''$ is acyclic. From the correctness of the completion proved by Janssen et al. (2012), and since $\text{supp}(p, heads(\Pi''))^A_I = \max \{ \beta^A \mid p \leftarrow \beta \in \Pi'' \} = \max \{ I(\beta) \mid p \leftarrow \beta \in \Pi'' \}$ captures the notion of support of $p$, we have that $I \in SM(\Pi'')$ iff $A_I \models \text{comp}(\Pi'')$. Hence, the models of $\text{rcomp}(\Pi)$ are the structures $A_I$ such that $J \in SM(\Pi'')$ satisfying the following condition: $I(b_i)$ equals 1 if $I(p) > 0$, and 0 otherwise. These are exactly the stable models of $\Pi''$, which concludes the proof. 

Lemma 1

Let $\Pi$ be such that $\Pi$ has atomic heads and non-recursive $\oplus, \forall$ in rule bodies. Let $I$ be an interpretation for $\Pi$. The least fixpoint of $T_{HI'}$ is reached in $|At(\Pi)|$ steps.

Proof

We first prove the claim for programs without $\oplus$. Let $J_0$ be the interpretation mapping everything to 0, and $J_{i+1} := T_{HI'}(J_i)$, for all $i \geq 0$. For every $i \geq 0$ and $p \in At(\Pi)$, if $J_i(p) < J_{i+1}(p)$, then there is a rule $p \leftarrow \beta \in \Pi^I$ with $J_{i+1}(p) = J_i(\beta)$. In this case, for each atom $q$ (including numeric constants) occurring $\beta$, we say that $p$ is inferred by $q$. In particular, since $\beta$ can only contain $\overline{a}$ and $\forall y$, we have the following property: ($*$) $J_{i+1}(p) \leq J_i(q)$. Let $n = |At(\Pi)|$ be the number of atoms in $\Pi$. We prove that any chain of inferred atoms has length at most $n+1$, which implies that $n$ applications of $T_{HI'}$ give the fixpoint of the operator. Suppose on the contrary that there are $p_0, \ldots, p_{n+1}$ such that $p_0$ is a numeric constant and $p_{i+1} \in At(\Pi)$ is inferred by $p_i \in At(\Pi)$, for all $i \in [0..n]$. Since $n = |At(\Pi)|$, there exist $1 \leq j < k \leq n+1$ such that $p_j = p_k$. Hence, from $J_i(p) < J_{i+1}(p)$ we have $J_{i+1}(p_{i+1}) > J_i(p_{i+1})$ for $i \in [0..n]$, and thus $J_k(p_k) > J_{k-1}(p_k) \geq J_j(p_k)$ (where the last inequality is due to the monotonicity of $T_{HI'}$). From ($*$) we have $J_{i+1}(p_{i+1}) \leq J_i(p_i)$ for $i \in [0..n]$, and thus $J_k(p_k) \leq J_j(p_j) = J_j(p_k)$. Therefore, we have $J_k(p_k) > J_j(p_k)$ and $J_k(p_k) < J_j(p_k)$, that is, a contradiction.

Let us now add non-recursive $\oplus$ in rule bodies. If there is $i \in [0..n]$ such that $p_{i+1}$ and $p_i$ do not satisfy ($*$), i.e., $J_{i+1}(p_{i+1}) > J_i(p_i)$, then $\beta$ must contain some occurrence of $\oplus$. Since $\oplus$ is non-recursive by assumption, $\{ p_j \mid i \in [1..j] \}$ and $\{ p_j \mid [i+1..n+1] \}$ are disjoint sets. Either $p_1, \ldots, p_i$ or $p_{i+1}, \ldots, p_{n+1}$ must have a repeated atom, and argument used before gives a contradiction. 


Theorem 5

Let Π be an HCF program with non-recursive ⊕ in rule bodies, and whose head connectives are \( \overline{\cdot}, \oplus \). If \( I \in SM(\Pi) \) then \( A^q_I \models ocomp(shift(simp(\Pi))) \). Dually, if \( A \models ocomp(shift(simp(\Pi))) \) then \( I_A \in SM(\Pi) \).

Proof

Let \( \Pi' \) be \( shift(\, simp(\Pi)\, \)\). From Proposition 1 and Theorem 2 we have \( \Pi \equiv_{At(\Pi)} \Pi' \). Moreover, \( \Pi' \) has atomic heads and non-recursive ⊕ in rule bodies. We show that stable models of \( \Pi' \) and \( \Sigma \)-models of \( ocomp(\Pi') \) are related.

First, notice that for any structure \( A \) and set of atoms \( A, \) rank\( (A) \)\( ^A \) equals max\( \{ r_p^A \mid p \in A \} \) if \( A \neq \emptyset \), and 0 otherwise. Moreover, osupp(\( p, heads(p, \Pi')\))\( ^A \) = 1 if there is \( p \leftarrow \beta \in heads(p, \Pi) \) such that \( p^A = \beta^A \) and \( r_p^A = 1 + \text{rank}(\text{pos}(\beta))^A \).

\((\Rightarrow)\) Let \( I \in SM(\Pi') \). Let \( J_0 \) be the interpretation mapping everything to 0, and \( J_{i+1} \) be \( T_{\Pi'}(J_i) \), for \( i \geq 0 \). By Lemma 1, \( J_{n+1} = J_n \). Let \( r \) be the ranking associated with \( I \), i.e., \( r(p) \) equals the minimum index \( i \in [1..n] \) such that \( J_i(p) = J_n(p) \).

We now use induction on the rank of inferred atoms to prove the following: \( A^q_I \models p = out(\beta) \wedge r_p = 1 + \text{rank}(\text{pos}(\beta)) \). For all \( p \in At(\Pi) \) such that \( J_n(p) > 0 \) and \( r(p) = 1 \), there is a rule \( p \leftarrow \beta \in \Pi' \) such that \( J_n(\beta) = J_n(p) \) and \( \beta \) only contains numeric constants; in this case \( A^q_I \models p = out(\beta) \wedge r_p = 1 + \text{rank}(\text{pos}(\beta)) \). For \( m \in [1..n-1] \), and for all \( p \in At(\Pi) \) such that \( J_n(p) > 0 \) and \( r(p) = m + 1 \), there is a rule \( p \leftarrow \beta \in \Pi' \) such that \( J_n(\beta) = J_n(p) \) and at least one of them must satisfy \( r(q) = m \), we have \( A^q_I \models p = out(\beta) \wedge r_p = 1 + \text{rank}(\text{pos}(\beta)) \).

That \( A^q_I \models \text{comp}(\Pi') \) follows by the fact that the completion captures the notion of supported model. Hence, \( A^q_I \models ocomp(\Pi') \).

\((\Leftarrow)\) Let \( A \) be a \( \Sigma \)-model of \( ocomp(\Pi') \), and let \( I := I_A \). We shall show that \( I_A \in SM(\Pi') \). Let \( J_0 \) be the interpretation mapping everything to 0, and \( J_{i+1} = T_{\Pi'}(J_i) \), for \( i \geq 0 \).

We use induction on \( r_p^A \) to show that \( J_{r_p^A}(p) = I(p) \). If \( p^A > 0 \) and \( r_p^A = 1 \), then there is \( p \leftarrow \beta \in \Pi' \) such that \( p^A = \beta^A \) and \( \text{pos}(\beta) = 0 \); in this case \( J_1(p) = I(p) \). If \( p^A > 0 \) and \( r_p^A = m + 1 \) for some \( m \in [1..n-1] \), then \( p \leftarrow \beta \in \Pi' \) such that \( p^A = \beta^A \) and \( \max\{ r_q^A \mid q \in \text{pos}(\beta) \} = m \); since \( J_m(q) = I(q) \) for all \( q \in \text{pos}(\beta) \) by the induction hypothesis, we have \( J_{m+1}(p) = I(\beta) = I(p) \).

The proof is thus complete.

\[ \square \]

References

