Nuclear Brinkmanship, Limited War, and Military Power

Online Appendix

The first step in characterizing the equilibria of the asymmetric-information game is describing the equilibria of the brinkmanship subgame. Let $\Gamma_B(p)$ denote the brinkmanship continuation game given $p$. A pure strategy in this subgame for $D$ is a pair $\{\rho(p), q_D(r|p)\}$ where $\rho(p) \in [r(p), \tau(p)]$ is $D$’s bid and $q_D(r|p) \in \{0, 1\}$ indicates whether $D$ quits ($q_D(r|p) = 1$) or stands firm ($q_D(r|p) = 0$) after $r$. (When we consider mixed strategies, $q_D(r|p) \in [0, 1]$ will be the probability that $D$ quits.) Similarly, a pure strategy for $D'$ is the analogous pair $\{\rho'(p), q_{D'}(r|p)\}$. A strategy for $C$ is a function $q_C(r|p)$ for all $r \in [r, \tau]$ where $q_C(r|p)$ is the probability $C$ quits after bid $r$ given $p$. We ease the notation by suppressing the argument “$p$” when it is not needed for clarity. A belief system for the challenger is a function $\theta$ which is the conditional probability of facing $D'$ after bid $r$. Finally, a PBE of the brinkmanship continuation game is an assessment $\Delta = \{\rho, q_D, \rho', q_{D'}, q_C, t\}$ which is sequentially rational and in which $t$ is derived from $C$’s prior beliefs by Bayes’ rule when possible.

Three lemmas help characterize the PBEs of the brinkmanship game. Lemma 1A demonstrates that neither $D$ nor $D'$ ever bids an $r \in (r, RC(p))$. $C$ is sure to stand firm after such a bid and, consequently, $D$ and $D'$ would have done better by bidding $r$. Lemma 2A shows that at most one $r \in (RC, \tau]$ is played with positive probability in a PBE. Lemma 3A shows no $r \in (RC, \tau]$ is played with positive probability in any PBE satisfying D1. Taken together, these lemmas imply that a PBE satisfying D1 can put positive probability on at most $\underline{r}$ and $RC(p)$.

**Lemma 1A:** Let $\Delta = \{\rho, q_D, \rho', q_{D'}, q_C, t\}$ be a PBE of $\Gamma_B(p)$. Then $\rho \notin (\underline{r}, RC(p))$ and $\rho' \notin (\underline{r}, RC(p))$.

*Proof:* Arguing by contradiction, suppose $D$ bids an $r \in (\underline{r}, RC)$. Since $r < RC$, $C$ strictly prefers to stand firm after $r$ regardless of $C$’s beliefs about the defender’s type. It follows that $D$’s payoff to bidding $r$ is $\max \{-k_D - \underline{r}n_D, (1-p)(1-r)v_D - k_D - rn_D\}$. Given that $p > 0$, $D$ would have done strictly better by bidding $\underline{r}$ and then standing firm to obtain...
(1 - p)(1 - \tau)v_D - k_D - \tau n_D. A similar argument holds for D'.

To ease the proof of Lemma 2A, observe that irresolute defender’s preference over any two distinct bids \( r \geq R_C \) and \( \tilde{r} \geq R_C \) depends solely on the probability that \( C \) backs down after \( r \) and \( \tilde{r} \). Indeed, \( D \) strictly prefers \( r \) to \( \tilde{r} \) if and only if \( q_C(r) > q_C(\tilde{r}) \). To see why, note that \( D \) is bluffing, i.e., sure to quit, whenever at least \( R_C \) since \( R_C > R_D \) by assumption (ii). More specifically, \( R_C(\tilde{p}) - R_D(\tilde{p}) > 0 \) by assumption (ii), and \( R_C(p) - R_D(p) > R_C(\bar{p}) - R_D(\bar{p}) \) for \( p > \bar{p} \) since \( R_C \) is increasing in \( p \) and \( R_D \) is decreasing. Given that \( D \) is sure to quit following any \( r \geq R_C \), \( D \) strictly prefers \( r \) to \( \tilde{r} \) if and only if

\[
q_C(r)[(1 - \tau)v_D - k_D - \tau n_D] + [1 - q_C(\tilde{r})][-k_D - \tau n_C] > q_C(\tilde{r})[(1 - \tau)v_D - k_D - \tau n_D] + [1 - q_C(\tilde{r})][-k_D - \tau n_C] \quad \text{or} \quad q_C(r) > q_C(\tilde{r}).
\]

It is also useful to determine when \( D' \) prefers higher bids to lower bids. Suppose \( r > \tilde{r} > R_C \). Assumption (i) ensures that \( R_D'(p) \geq R_D'(\bar{p}) > \tau(\bar{p}) \geq \tau(p) \) and hence that \( D' \) is sure to stand firm after \( r \) or \( \tilde{r} \). Because \( D' \) always stands firm, a higher bid brings a higher cost if \( C \) stands firm, i.e., \( (1 - p)v_D' - k_D - r[(1 - p)v_D' + n_D] \) is decreasing in \( r \). As a result, \( D' \) will only be willing to run risk \( r > \tilde{r} \) if \( C \) is more more likely to quit after \( r \) than after \( \tilde{r} \). To be more precise, \( D' \) strictly prefers \( r \) to \( \tilde{r} \) if and only if

\[
q_C(r)[(1 - \tau)v_D' - k_D - \tau n_D] + [1 - q_C(\tilde{r})][(1 - p)v_D' - k_D - r[(1 - p)v_D' + n_D]] > q_C(\tilde{r})[(1 - \tau)v_D' - k_D - \tau n_D] + [1 - q_C(\tilde{r})][(1 - p)v_D' - k_D - \tilde{r}[(1 - p)v_D' + n_D]].
\]

This is equivalent to \( q_C(r) > q_C(\tilde{r}) \equiv q_C(\tilde{r}) + [1 - q_C(\tilde{r})(r - \tilde{r})][(1 - p)v_D' + n_D]/[(1 - \tau)v_D' - \tilde{r}[(1 - p)v_D' + n_D](R_D' - r)] \). \( D' \) is indifferent when \( q_C(r) = q_C(\tilde{r}) \).

**Lemma 2A:** Let \( \Delta \) be a PBE in which \( r \in (R_C, \tau] \) is played with positive probability. Then no other \( \tilde{r} \in (R_C, \tau] \) is played with positive probability.

**Proof:** The lemma holds vacuously if \( p \leq \bar{p} \) as \( (R_C, \tau] = \emptyset \). Assume \( p > \bar{p} \), and suppose that \( r \) and \( \tilde{r} \) are played with positive probability in \( \Delta \) with \( r > \tilde{r} > R_C \). Then \( D' \) must put positive probability on both offers. Suppose not. If \( D \) alone put positive probability on \( r \), then \( t(r) = 0 \) and \( C \) is sure to stand firm \( (q_C(r) = 0) \) since \( D \) is certain to back down as \( r > R_D \). But if \( q_C(r) = 0 \), \( D \) can profitably deviate from \( r \) to bidding \( \tau \) and then standing firm. Hence \( D' \) must put positive probability on \( r \). Repeating the argument for
\(\hat{r}\) establishes that \(D'\) must also put positive weight on \(\hat{r}\).

In order to put positive weight on \(r\) and \(\hat{r}\), \(D'\) must be indifferent between them. This implies \(1 \geq q_C(r) = \tilde{q}_C(r, \hat{r}) > q_C(\hat{r}) \geq 0\). But this means that \(D\) strictly prefers \(r\) to \(\hat{r}\) as \(C\) is more likely to quit. This leaves \(t(\hat{r}) = 1\) and yields the contradiction \(q_C(\hat{r}) = 1\). ■

**Lemma 3A:** Let \(\Delta\) be a PBE satisfying D1. Then no \(r \in (R_C, \hat{r})\) is played with positive probability \(\Delta\).

**Proof:** The lemma again holds vacuously if \(p \leq \tilde{p}\). Arguing by contradiction when \(p > \tilde{p}\), assume \(r \in (R_C, \hat{r})\) is played with positive probability. Lemmas 1A and 2A imply that the only other bids that might be played with positive probability are \(\underline{r}\) and \(R_C\).

Both \(D\) and \(D'\) must put positive probability \(r\). Observe first that \(q_C(r) > 0\). Otherwise both types prefer to deviate to \(\underline{r}\) and \(r\) would not be played with positive probability. If only \(D\) plays \(r\), \(t(r) = 0\) and this leads to the contradiction \(q_C(r) = 0\). If \(D'\) alone plays \(r\), then \(t(r) = 1\) and \(q_C(r) = 1\). Moreover, \(D'\) must at least weakly prefer \(r\) to \(R_C\), so \(q_C(r) \geq \tilde{q}_C(r, \hat{r}) > q_C(R_C)\). However, \(q_C(r) > q_C(R_C)\) implies that \(D\) strictly prefers \(r\) to \(R_C\). \(D\) must therefore at least weakly prefer \(\underline{r}\) to \(r\). This yields the contradiction \(q_C(r) \leq 1 - p\).

Because both \(D\) and \(D'\) play \(r\) with positive probability, their respective equilibrium payoffs are \(q_C(r)[(1-\underline{r})v_D - k_D - \underline{r}n_D] + [1 - q_C(r)][-k_D - \underline{r}n_C + q_C(r)(1-\underline{r})v_D\) and \(q_C(r)[(1-\underline{r})v'_D - k_D - \underline{r}n_D] + [1 - q_C(r)][(1-p)v'_D - k_D - r[(1-p)v'_D + n_D]]\). Now consider any downward deviation \(z \in (R_C, r)\). We show that D1 requires \(C\) to believe that it is facing \(D'\) for sure (i.e., \(t(z) = 1\)). \(C\)'s best response given this belief is to quit with \(q_C(z) = 1\). But this would make \(z\) a profitable deviation for both \(D\) and \(D'\), and this contradiction would establish the lemma.

To see that D1 eliminates \(D\) at \(z\), observe first that \(C\) is indifferent between standing firm and quitting after \(z\) if it believes it is facing \(D'\) with probability \((pv_C + n_C)(z - R_C)/[(1-\underline{r})v_C + (pv_C + n_C)(z - R_C)]\). Hence, any \(q_C(z) \in [0, 1]\) can be rationalized as a best response to some beliefs about the deviator’s type.

Moreover, \(D'\) strictly prefers deviating to \(z\) if \(q_C(z) > \tilde{q}_C(z, r)\) where \(\tilde{q}_C(z, r) < q_C(r)\) when \(z < r\). \(D\) weakly prefers bluffing at \(z\) to bluffing at \(r\) when \(q_C(z) \geq q_C(r)\). The set

\[\hat{r}\]
of C’s weakly profitable deviations for D is a strict subset of the set of deviations that are strictly profitable for D’. Hence, D1 eliminates D.

The previous lemmas make it easy to specify a PBE satisfying D1. Lemma 1A implies that both D and D’ bid \( r \) and all states subsequently stand firm whenever \( p < \bar{p} \) as this implies \( \bar{\pi} < R_C \). Proposition 1A describes a separating PBE satisfying D1 when \( p \in (\bar{p}, \bar{\pi}] \). D and D’ respectively bid \( r \) and \( R_C \), and both types then stand firm. C stands firm after \( r \) and does so with probability \( p \) after \( R_C \). More precisely, define the assessment \( \Delta_0 \) in which D plays according to \( \rho = r \), \( q_D(r) = 0 \) for \( r \leq R_D \) and \( q_D(r) = 1 \) for \( r > R_D \); D’ plays according to \( \rho' = R_C \), \( q_D(r) = 0 \) for \( r \leq R'_D \) and \( q_D(r) = 1 \) for \( r > R'_D \); and C follows \( q_C(r) = 0 \) for \( r \neq R_C \) and \( q_C(R_C) = 1 - p \). C’s beliefs are \( t(r) = 0 \) if \( r = r \), \( t(r) = \tau \) for \( r \in (r, R_C) \), \( t(R_C) = 1 \), and \( t(r) = 0 \) for \( r \in (R_C, \bar{\pi}] \).

Proposition 1A: If \( p \in (\bar{p}, \bar{\pi}] \), \( \Delta_0 \) is a PBE satisfying D1.

Proof: Verifying that \( \Delta_0 \) is a PBE is straightforward. Given \( q_C(r) = 0 \) for \( r \neq R_C \), D and D’ will bid either \( r \) or \( R_C \) since the payoff to \( r \) is strictly better than the payoff to bidding any \( r \notin \{r, R_C\} \). At \( q_C(R_C) = 1 - p \), D is indifferent between \( r \) and \( R_C \), so \( r \) is a best reply. D’ strictly prefers \( R_C \) to \( r \). C in turn strictly prefers standing firm after \( r \) and is indifferent after \( R_C \) given that it believes it is facing D’ for sure (\( t(R_C) = 1 \)). Accordingly, \( q_C(r) = 0 \) and \( q_C(R_C) = 1 - p \) are best replies. C’s beliefs are also clearly consistent with Bayes’ rule.

To demonstrate that \( \Delta_0 \) satisfies D1 consider any deviation \( r > R_C \). As shown in the proof of Lemma 3A, any \( q_C(r) \in [0, 1] \) can be rationalized as a best response to some beliefs about the deviator’s type. Moreover, D’ stands firm after bidding \( r \) since

\[
R'_D(p) \geq R'_D(\bar{p}) > \bar{\pi}(p) \geq r.
\]

This implies the responses \( q_C(r) \) for which \( r \) is weakly profitable are defined by

\[
q_C(r)[(1 - r)v'_D - \mathbb{M} - \mathbb{N}D] + [1 - q_C(r)][(1 - p)v'_D - k_D - r'[1 - (1 - p)v'_D + \mathbb{M}D] \geq (1 - p)[(1 - r)v'_D - k_D - \mathbb{M}D] + p[(1 - p)v'_D - k_D - R_C[(1 - p)v'_D + \mathbb{M}D]].
\]

This simplifies to

\[
q_C(r) \geq \tilde{q}_0(r) \equiv 1 - p + p[(1 - p)v'_D + \mathbb{M}D](r - R_C).
\]

As for D, assumption (ii) ensures that D is certain to quit if it bids \( r \) and C stands firm. That is, \( r > R_C(\bar{p}) > R_C(\bar{p}) \) and \( R_D(\bar{p}) > R_D(p) \) since \( R_C \) is increasing and \( R_D \) is decreasing and, by assumption (ii), \( R_C(\bar{p}) > R_D(\bar{p}) \). According, deviating to \( r \) is strictly
profitable for $D$ if $q_C(r)[(1 - \rho)v_D - k_D - \rho n_D] + [1 - q_C(r)](-k_D - \rho n_D) > (1 - p)(1 - \rho)v_D - k_D - \rho n_D$ or $q_C(r) > 1 - p$.

The set of $C$’s responses to $r$ for which $r$ is weakly profitable for $D'$ is a strict subset of the set of responses for which $r$ is strictly profitable for $D$. That is, \{$(q_C(r) : q_C(r) \geq \tilde{q}_0(r)) \subset \{q_C(r) : q_C(r) > 1 - p\}$. D1 therefore eliminates $D'$ and requires $C$ to put probability one on $D$ after $r > R_C$. $\Delta_0$ does this as $t(r) = 0$ for $r > R_C$.

For $r \in (\underline{r}, R_C)$, $C$’s unique best response is $q_C(r) = 0$ regardless of its beliefs. As a result, D1 has no bite, and any $t(r) \in [0, 1]$ is consistent with D1. Hence, $C$’s out-of-equilibrium beliefs in $\Delta_0$ satisfy D1.\footnote{Other PBEs exist as well, e.g., at $q_C(R_C) = q_b$ where $D'$ is indifferent between bidding $R_C$ and $\underline{r}$.}

D1 does not pin down a unique PBE. As Corollary 1A shows, other equilibria satisfying D1 exist. This multiplicity of equilibria arises from $C$’s indifference between quitting and standing firm following a bid of $r = R_C$. To establish that equilibria other than $\Delta_0$ exist and satisfy D1, observe first that $C$ is indifferent only if $C$ believes that it is facing $D'$ with probability one after a bid of $R_C$. Any positive probability of facing $D$ breaks $C$’s indifference and leads $C$ to stand firm. It follows that $C$ must quit after a bid of $R_C$ with a high enough probability that $D'$ is willing to bid $R_C$ rather than $\underline{r}$ but not so high that $D$ prefers bidding $R_C$ to $\underline{r}$. To define this range, let $q_b$ be the smallest probability of quitting for which $D'$ is willing to bid $R_C$. Then $q_b$ is the smallest $q$ satisfying $(1 - \underline{r})[(1 - p)v'_D - k_D - \underline{r}\rho k_D + \rho n_D] \leq q[-k_D - \rho n_D] + (1 - q)[(1 - R_C)(1 - p)v'_D - k_D] - R_C[k_D + n_D]$.

Corollary 1A shows that a PBE satisfying D1 exists for all $q_C(R_C) \in [q_b, 1 - p]$. Define the PBE $\Delta_q$ to be the same as $\Delta_0$ except that $q_C(R_C) = q$ for any $q \in [q_b, 1 - p]$. Then

**Corollary 1A:** $\Delta_q$ for $q \in [q_b, 1 - p]$ are PBEs satisfying D1.\footnote{Other PBEs exist as well, e.g., at $q_C(R_C) = q_b$ where $D'$ is indifferent between bidding $R_C$ and $\underline{r}$.}

**Proof:** By construction, $D'$ at least weakly prefers bidding $R_C$ and then standing firm to bidding $\underline{r}$ for $q \in [q_b, 1 - p]$. Verifying that $\Delta_q$ is a PBE is straightforward. Repeating the argument in Proposition 1A also shows that D1 eliminates $D'$ for all $r > R_C$. D1 thus requires $t(r) = 0$ at $r > R_C$ as is the case in $\Delta_q$.\footnote{Other PBEs exist as well, e.g., at $q_C(R_C) = q_b$ where $D'$ is indifferent between bidding $R_C$ and $\underline{r}$.}
The main idea underlying this criterion is that if the set of bids was discrete, it would be very unlikely that the defender could bid exactly $R_C$. The discrete-bid criterion imposes this condition.

To define this criterion, consider a discrete-bid analogue of the brinkmanship game when $p > \tilde{p}$. $C$ must now select a bid from a finite set of offers $r_0, ..., r_n$ such that $\underline{r} = r_0 < \cdots < r_{m-1} < R_C < r_m < \cdots < r_n = \tau$ with $r_j - r_{j-1} \leq \delta$ for all $j$ and a $\delta > 0$. (The models of brinkmanship in Powell 1990 have this structure. Every step toward the brink raises the risk of disaster by a fixed amount $\delta$.) The key features of this discrete set of offers is that the defender is no longer able to bid exactly $R_C$ and there is a well defined next highest bid above $R_C$, namely, $r_m$. At $r_m$, $C$ is no longer indifferent between quitting and standing firm if it is certain that it is facing $D'$. Rather, $C$ strictly prefers to quit.

Call the discrete-bid brinkmanship game described above $\Gamma_B^\delta(p)$. Define the assessment $\Delta_{\delta}$ to be: $D$ plays $r_m$ with probability $\mu_\delta$ and $\underline{r}$ with probability $1 - \mu_\delta$ where $\mu_\delta \equiv \tau(pv_C + n_C)(r_m - R_C)/[(1 - \tau)(1 - \underline{r})v_C]$, $q_D(r_j) = 0$ for any $r_j < R_D$ and $q_D(r_j) = 1$ for $r_j > R_D$. $D'$ plays according to $\rho' = r_m$, $q'_D(r_j) = 0$ for $r_j < R_D'$ and $q'_D(r_j) = 1$ for $r_j > R_D'$. $C$ follows $q_C(r_j) = 0$ for $r \neq r_m$ and $q_C(r_m) = 1 - p$. $C$’s beliefs are $t(\underline{r}) = 0$, $t(r) = \tau$ for $\underline{r} < r_j \leq r_{m-1}$, $t(r_m) = \tau/[(\tau + (1 - \tau)\mu_\delta]$, and $t(r_j) = 0$ for $r_j > r_m$.

The next lemma shows that all the PBEs of $\Gamma_B^\delta$ satisfying D1 are the same as $\Delta_{\delta}$ except possibly at $C$’s beliefs following an out of equilibrium offer less than $R_C$. These beliefs have no effect on subsequent play. $C$ and both $D$ and $D'$ stand firm after this bid. Proposition 2A demonstrates that if we let the maximal distance between adjacent offers $\delta$ go to zero, then the limit of $\Delta_{\delta}$ is identical to $\Delta_0$ except possibly for $C$’s beliefs at $t(r)$ for $r \in (\underline{r}, R_C)$. In this sense, D1 and the limit-criterion uniquely select $\Delta_0$.

**Lemma 4A:** Assume $p > \tilde{p}$ and let $\Delta$ be any PBE of $\Gamma_B^\delta$ satisfying D1. Then $\Delta$ is identical to $\Delta_{\delta}$ except possibly for $C$’s beliefs $t(r_j)$ for $\underline{r} < r_j \leq r_{m-1}$.

**Proof:** Let $\Delta$ be a PBE satisfying D1. Repeating the argument in the proofs of Lemmas 1A and 2A shows that $\Delta$ can only put positive weight on $\underline{r}$ and on one $r_j \geq r_m$. If $r_j > r_m$, repeating the argument in the proof of Lemma 3A shows that $C$ must believe
that it is facing $D'$ after the downward deviation to $r_{j-1} \geq r_m$. That is, $t(r_{j-1}) = 1$.

$C$’s best response given this belief and $r_m > R_C$ is to quit, $q_C(r_{j-1}) = 1$. This, however, makes $r_{j-1}$ a profitable deviation. Hence, putting positive weight on $r_j > r_m$ yields a contradiction. As a result, $\Delta$ can put positive probability on at most $\underline{r}$ and $r_m$.

In fact, $\Delta$ must put positive weight on $r_m$. Suppose not. Then $r_m$ is an out-of-equilibrium bid, and $t(r_m) = 1$ by D1. To see that D1 eliminates $D$, note that $D$ quits after $r_m$ if $C$ stands firm. $D$ therefore weakly prefers to deviate to $r_m$ from $\underline{r}$ if $q_C(r_m) \geq 1 - p$. Algebra shows that $D'$ strictly prefers bidding $r_m$ and standing firm when $q_C(r_m) \geq 1 - p$. D1 therefore eliminates $D$ and leaves $t(r_m) = 1$.

A contradiction follows. If $t(r_m) = 1$, $C$’s best reply is $q_C(r_m) = 1$. This, however, makes $r_m$ a profitable deviation, and this contradiction ensures that $\Delta$ must put positive weight on $r_m$.

$\Delta$ must put positive probability on $\underline{r}$ as well. Arguing again by contradiction, suppose $D$ and $D'$ pool on $r_m$. Then $t(r_m) = \tau$. $D$’s weak preference for $r_m$ also implies $q_C(r_m) \geq 1 - p$. But as shown below, $C$ strictly prefers to stand firm if $D$ and $D'$ pool on $r_m$ and $\delta$ is sufficiently small. This yields the contradiction $q_C(r_m) = 0$.

To establish that $C$ stands firm after $r_m$ if $D$ and $D'$ pool on this bid and $\delta$ is sufficiently small, note that $C$ stands firm if $(1 - \tau)[v_C - R_m] + \tau[(1 - r_m)pv_C - r_m n_C] > -R_m n_C$. This is equivalent to $\tau < v_C/[v_C + (pv_C + n_C)(r_m - R_C(p))]$. As $\delta$ goes to zero, this constraint goes to $\tau < 1$ and is sure to hold.

That both $\underline{r}$ and $r_m$ are played with positive probability implies that $D'$ bids $r_m$ and $D$ mixes between $\underline{r}$ and $r_m$. Clearly the types cannot separate. If $D$ plays $\underline{r}$ and $D'$ plays $r_m$, then $q_C(r_m) = 1$ and $D$ prefers to deviate. If $D$ plays $r_m$ and $D'$ plays $\underline{r}$, then $q_C(r_m) = 0$ and $D$ prefers to deviate to $\underline{r}$. Given that the types cannot separate and that $D'$ strictly prefers $r_m$ whenever $D$ weakly prefers $r_m$, $D'$ must play $r_m$, i.e., $\rho' = r_m$, and $D$ must mix between $\underline{r}$ and $r_m$.

In order for $D$ to mix, it must be indifferent. This implies that $C$ must mix after $r_m$ with $q_C(r_m) = 1 - p$. Because $C$ is mixing, it must be indifferent between quitting and standing firm. Let $\mu_\delta$ be the probability that $D$ bids $r_m$. Then $C$’s indifference
gives $-k_C - \sum n_C = (1 - \tau \mu_\delta)/(1 - \tau)(1 - \mu_\delta + \tau)[(1 - \tau)v_C - k_C - \sum n_C] + \tau/[(1 - \tau)(1 - \mu_\delta + \tau)]r_m(p_C + n_C)$ where $(1 - \tau \mu_\delta)/(1 - \tau)(1 - \mu_\delta + \tau)$ and $\tau/[(1 - \tau)(1 - \mu_\delta + \tau)]$ are the posteriors of facing $D$ and $D'$ given $r_m$. This yields $\mu_\delta = (p_C + n_C)(r_m - R_C)/(1 - \tau)((1 - \tau)v_C + (p_C + n_C)(r_m - R_C))$.

Finally consider C’s out-of-equilibrium beliefs and actions $t(r_j)$ and $q_C(r_j)$ for $j > m$. Deviation $r_j$ is strictly profitable for $D$ if $q_C(r_j) > 1 - p$. $D'$ weakly prefers to deviate if $q_C(r_j) \geq q_C(r_m) = 1 - p$. Thus, D1 eliminates $D'$ at $r_j$. This leaves $t(r_j) = 0$, $q_C(r_j) = 0$, and establishes the lemma.

It immediately follows that $\Delta_\delta$ converges to $\Delta_0$ except possibly for C’s beliefs at $r \in (r, R_C)$. Since $\delta \geq r_m - r_{m-1} > r_m - R_C$, we have $\mu_\delta \rightarrow 0$ and $t(r_m) \rightarrow 0$ as $\delta \rightarrow 0$. This leaves

**Proposition 2A:** Assume $p > \tilde{p}$. Then the assessment $\lim_{\delta \rightarrow 0} \Delta_\delta$ is identical to $\Delta_0$ except possibly at the irrelevant beliefs $t(r)$ for $r \in (\bar{r}, R_C)$.

Turning to a determination of C’s choice of $p$ at the outset of the game, Lemma 1A and Proposition 2A imply that the challenger’s payoff to bring $p$ to bear is:

$$U_C(p) = \left\{ \begin{array}{ll} pv_C - k_C - r(p)n_C & \text{if } 0 < p < \tilde{p} \\ (1 - \tau)[1 - r(p)]pv_C - k_C - r(p)n_C & \text{if } \tilde{p} < p \leq \bar{p}. \end{array} \right.$$  

As for the optimal $p$, define $U_C(0) = 0$ and $U_C(\tilde{p}) = \lim_{p \uparrow \tilde{p}} U_C(p) = [1 - r(\tilde{p})]pv_C - k_C - r(\tilde{p})n_C$. We justify this specification of $U_C(\tilde{p})$ below. For now, observe that $U_C(p)$ defined in this way over $[0, \bar{p}]$ has a well defined global maximizer. This follows from the fact that $U_C$ is weakly concave over $(0, \tilde{p})$ and strictly concave over $(0, \bar{p}]$ with $U_C(0) = 0 > \lim_{p \rightarrow 0} U_C(p) = -k_C$ and $\lim_{p \uparrow \tilde{p}} U_C(p) > \lim_{p \downarrow \tilde{p}} U_C(p)$. Moreover, this maximizer is generically unique. That is, the set of feasible parameter values $v_C$ and $n_C$ for which there are multiple maximizers is a set of measure zero. Let $p^{**}$ denote this maximizer.

To justify the specification of $U_C(\tilde{p})$, we show below that the brinkmanship game following $\tilde{p}$ has multiple equilibria satisfying D1, and C’s equilibrium payoffs vary across these equilibria. However, these payoffs are above below by $U_C(\tilde{p})$, and a unique equilibrium path yields $U_C(\tilde{p})$. Hence which equilibrium is played after $\tilde{p}$ has no effect on the optimal choice of $p$ if $p^{**} \neq \tilde{p}$. If $p^{**} = \tilde{p}$, the states must play an equilibrium with the
unique path giving C a payoff of \( U_C(\bar{p}) \). Otherwise C’s payoff to \( p \) would discontinuously jump down at \( \bar{p} \) and C would not have a best reply to the defender’s strategy.

To see that \( \Delta(\bar{p}) \) has multiple equilibria satisfying D1, let \( \Delta_{\bar{p}} \) be the assessment in which both \( D \) and \( D' \) bid \( \bar{r} \), and all three states subsequently stand firm after any \( r < R_C \). C quits after \( r = R_C \) with any \( q_C(R_C) \leq q_b \). C’s beliefs at \( r \in (\bar{r}, R_C) \) can be anything, and \( t(R_C) = 1 \). These equilibria have the same equilibrium path and give a payoff of \( U_C(\tilde{p}) \) to C. By contrast, \( \Delta_0 \) is also an equilibrium satisfying D1 and yields a payoff of \( \lim_{p \uparrow \tilde{p}} U_C(p) \) to C.

To establish that C’s equilibrium payoffs are bounded above by \( U_C(\bar{p}) \), let \( z \) be the equilibrium probability that \( D \) or \( D' \) bids \( \bar{r} \). Then C’s equilibrium payoff is \( z[\bar{p}v_C - k_C - \bar{r}(\bar{p})(\bar{p}v_C + n_C)] - (1 - z)[k_C + \bar{r}(\bar{p})n_C] \). This payoff is increasing in \( z \) and equal to \( U_C(\bar{p}) \) at \( z = 1 \). Finally, it is easy to see that the equilibrium path of \( \Delta_{\bar{p}} \) is the unique path giving C the payoff \( U_C(\bar{p}) \). Since \( z = 1 \), \( D \) and \( D' \) must bid \( \bar{r} \) which corresponds to the path in \( \Delta_{\bar{p}} \).

In determining the comparative statics, assume an interior solution, \( U'_C(p^{**}) = 0 \). Using \( U''_C < 0 \) gives \( \text{sgn}\{\partial p^{**}/\partial n_C\} = \text{sgn}\{\partial^2 U_C(p^{**})/\partial n_C \partial p\} \). Differentiation gives \( \partial U_C/\partial p = -\bar{r}'n_C + (1 - \tau)[(1 - \bar{r})v_C - \bar{r}'pv_C] \). Trivially, \( \partial^2 U_C/\partial n_C \partial p = -\bar{r}' < 0 \) and \( \partial p^{**}/\partial n_C < 0 \). Further, \( \partial^2 U_C/\partial v_C \partial p = (1 - \tau)[1 - \bar{r} - \bar{r}'p] \). But \( U'_C(p^{**}) = 0 \) ensures \( (1 - \tau)[1 - \bar{r} - \bar{r}'p] = \bar{r}'n_C/v_C > 0 \). Thus \( \partial p^{**}/\partial v_C > 0 \). And, \( \partial^2 U_C/\partial \tau \partial p = \bar{r}'pv_C - (1 - \bar{r})v_C = -\bar{r}'n_C/(1 - \tau) < 0 \), so \( \partial p^{**}/\partial \tau < 0 \). As for \( \partial R_C(p^{**})/\partial v_C \) and \( \partial R_C(p^{**})/\partial n_C \), write \( R_C(p^{**}) = 1 - (1 - \bar{r}(p^{**}))/[1 + p^{**}v_C/n_C] \). The results follow immediately.