Supplementary material - Modeling the fluid mechanics in single-flow batteries with an adjacent channel for improved reactant transport

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Redox flow batteries (RFBs) are an emerging technology envisioned towards storage of renewably generated electricity. A promising sub-class of RFBs utilizes single-flow membraneless architectures in an effort to minimize system cost and complexity. To support multiple functions, including reactant separation and fast reactant transport to electrode surfaces, electrolyte flow must be carefully designed and optimized. In this work, we propose adding a secondary channel adjacent to a permeable battery electrode, and analyze the effects on the single electrolyte flow as well as on the reactant concentration boundary layer at the electrode. We find that an adjacent channel with gradually changing thickness leads to a desired nearly-uniform flow through the electrode to the adjacent channel. Consequently, the thickness of the concentration boundary layer is significantly reduced, increasing reactant transport to the electrode surface to 140% of the rate of a battery with a constant width adjacent channel, and350% of the rate of no adjacent channel. Overall, this theory provides insight into the important role of flow physics on this promising sub-class of flow batteries and will pave the way to create more efficient flow batteries.

*DOI:****

0. Auxiliary Functions

This section is used for the definition of auxiliary modules to be used within the document.

The MakeEquationListFlat[EquationList_] function eliminates the {{}} appearing within list of manipulated equations

```
In[1]:= MakeEquationListFlat[EquationList_] :=
  Module[{FlatEquationList}, NumberOfEquations = Length[EquationList];
   FlatEquationList =
    {Flatten[EquationList[[1, 1]]][[1]] - Flatten[EquationList[[1, 2]]][[1]] == 0};
   For[n = 2, n <= NumberOfEquations, n++,
    FlatEquationList = Append[{FlatEquationList},
      {Flatten[EquationList[[n, 1]]][[1]] - Flatten[EquationList[[n, 2]]][[1]] == 0}]
   ];
   FlatEquationList = Flatten[FlatEquationList]]
```

The first function is the order grabbing function GetEquationListOrder

```
In[2]:= GetEquationListOrder[EquationList_, OrderEps_, SmallParameter_] :=
  Module[{EquationListOrder0}, NumberOfEquations = Length[EquationList];
   EquationListOrder0 = {CoefficientList[EquationList[[1]][[1]],
         SmallParameter, OrderEps + 1][[OrderEps + 1]] - CoefficientList[
         EquationList[[1]][[2]] , SmallParameter, OrderEps + 1][[OrderEps + 1]] == 0};
   For[n = 2, n <= NumberOfEquations, n++,
    EquationListOrder0 = Append[{EquationListOrder0},
      {CoefficientList[EquationList[[n]][[1]], SmallParameter, OrderEps + 1][[
          OrderEps + 1]] - CoefficientList[EquationList[[n]][[2]],
           SmallParameter, OrderEps + 1][[OrderEps + 1]] == 0}]
   ];
   EquationListOrder0 = Flatten[EquationListOrder0]]

In[3]:= clean2[expr_] :=
  HoldForm[expr] /.
   h_Symbol[args___] /; Context@Unevaluated@h =!= "System`" :→ h

In[4]:= pdConv[f_] :=
  TraditionalForm[
   f /. Derivative[inds__][g_][vars__] :→  Apply[Defer[D[g[vars], ##]] &,
   Transpose[{{vars}, {inds}}] /. {{var_, 0} :→
    Sequence[], {var_, 1} :→ {var}}]]

In[5]:= centering := CellPrint[ExpressionCell[#, "Output", TextAlignment → Center]] &

In[6]:= TextbookEq[expr_] :=
  TraditionalForm[{TableForm[expr]  // pdConv }[[1]][[1]] // clean2] // centering
```

1. Problem formulation and governing equations

In[7]:=

We examine the steady viscous Newtonian flow within a system having an inlet on one side and an outlet on the other where the flow is pressure driven.

The system, schematically depicted in Figure 1, consists of two channels parted by a permeable separator layer, allowing mass transfer between the them. The Cartesian coordinate system (x,y), is also shown, where the x axis denotes the downstream direction and y is perpendicular thereto.

In[8]:=

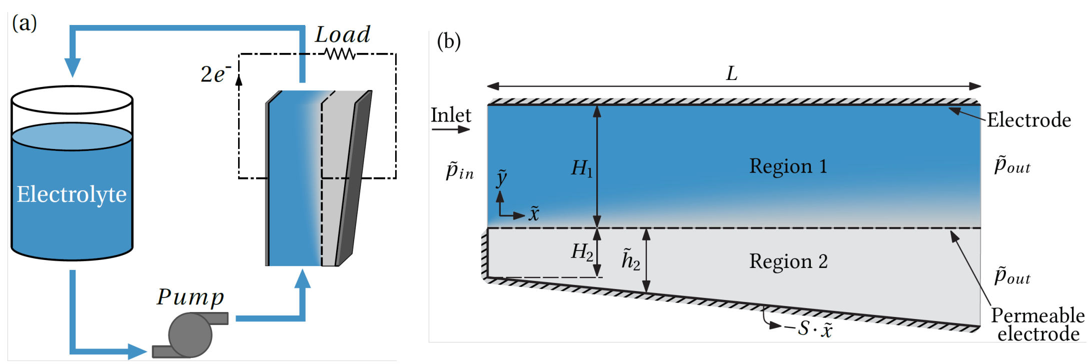


Fig. 1. Schematic illustration of the configuration of the examined system, showing the coordinate system and key parameters used for the analysis. The separator layer (dashed line) allows the steady viscous flow, induced by a pressure gradient in the downstream direction, to cross to the lower channel and visa versa.

The examined system of length l, consists of two parts, the main channel (Region 1) and a secondary channel (Region 2). The former is a channel characterized by height h_1, and the latter having a a bottom wall expanding downstream which is defined by

```
In[9]:= h2Eq = h2[x] == h2 (1 + a1 (x / l)); TextbookEq[h2Eq]
```

$$\mathrm{h2} = \mathrm{h2}\left(1 + \frac{\mathrm{a1}\,x}{l}\right)$$

The regions are parted by a separator layer having permeability k, which is defined as the relative amount of the fluid which is able to pass through the layer, according to the porosity of the layer. I.e., the velocity of the fluid perpendicular to the layer will be governed by

In[10]:=
```
UYKratio = Uy == k / μ ΔP; TextbookEq[UYKratio]
```

$$\mathrm{Uy} = \frac{k\,\Delta\mathrm{P}}{\mu}$$

where Ux_i and Uy_i denote the velocity in the direction of the coordinate and the index indicates the region.

A. Governing equations in dimensional form

Assuming steady Newtonian, incompressible flow with negligible inertial effects, the fluid motion, having a dynamics viscosity, μ, is governed by the Stokes equations. The equations governing the fluid in region 1 are given thus by:

In[11]:=
```
TextbookEq[
 XMomentumConservationEq1 = μ (∂{x} (∂{x} ux1[x, y]) + ∂{y} (∂{y} ux1[x, y])) - ∂{x} p1[x, y] == 0]
TextbookEq[YMomentumConservationEq1 =
  μ (∂{x} (∂{x} uy1[x, y]) + ∂{y} (∂{y} uy1[x, y])) - ∂{y} p1[x, y] == 0]
```

$$-\frac{\partial \mathrm{p1}}{\partial x} + \mu\left(\frac{\partial^2 \mathrm{ux1}}{\partial x^2} + \frac{\partial^2 \mathrm{ux1}}{\partial y^2}\right) = 0$$

$$-\frac{\partial \mathrm{p1}}{\partial y} + \mu\left(\frac{\partial^2 \mathrm{uy1}}{\partial x^2} + \frac{\partial^2 \mathrm{uy1}}{\partial y^2}\right) = 0$$

along with the conservation of mass

In[13]:=
```
TextbookEq[MassConservationEq1 = ∂{x} ux1[x, y] + ∂{y} uy1[x, y] == 0]
```

$$\frac{\partial \mathrm{ux1}}{\partial x} + \frac{\partial \mathrm{uy1}}{\partial y} = 0$$

The suitable boundary conditions are no slip at the top and bottom of the region,

In[14]:=
```
TextbookEq[NoSlipUP1 = ux1[x, h1] == 0]
TextbookEq[NoSlipDN1 = ux1[x, 0] == 0]
```

$$\mathrm{ux1} = 0$$

$$\mathrm{ux1} = 0$$

no penetration at the top of the channel,

In[16]:=
```
TextbookEq[NoPenetrationUP1 = uy1[x, h1] == 0]
```

$$\mathrm{uy1} = 0$$

and a velocity perpendicular to the channel at the separator layer

In[17]:=
```
TextbookEq[FluxDN1 = uy1[x, 0] == k (p2[x, 0] - p1[x, 0]) / (w μ)]
```

$$\mathrm{uy1} = \frac{k\,(-\mathrm{p1} + \mathrm{p2})}{w\,\mu}$$

The pressure at the inlet and the outlet is

In[18]:=
```
TextbookEq[P1Inlet = p1[0, y] == pin]
TextbookEq[P1Outlet = p1[l, y] == 0]
```

$$\text{p1} = \text{pin}$$

$$\text{p1} = 0$$

In[20]:=
```
MassConservationEq2 = ∂{x} ux2[x, y] + ∂{y} uy2[x, y] == 0;
XMomentumConservationEq2 = μ (∂{x} (∂{x} ux2[x, y]) + ∂{y} (∂{y} ux2[x, y])) - ∂{x} p2[x, y] == 0;
YMomentumConservationEq2 = μ (∂{x} (∂{x} uy2[x, y]) + ∂{y} (∂{y} uy2[x, y])) - ∂{y} p2[x, y] == 0;
```

The governing equations for region 2 are derived in a similar fashion, with the following boundary conditions

In[21]:=
```
NoSlipUP2 = ux2[x, 0] == 0;
NoSlipDN2 = ux2[x, - h2 (1 + a1 (x / l) + a2 (x / l) ^2 + a3 (x / l) ^3)] == 0;
NoPenetrationDN2 = uy2[x, - h2 (1 + a1 (x / l) + a2 (x / l) ^2 + a3 (x / l) ^3)] == 0;
FluxUP2 = uy2[x, 0] == k (p2[x, 0] - p1[x, 0]);
TextbookEq[P2Inlet = D[p2[x, y], x] == 0]
TextbookEq[P2Inlet = P2Inlet //. x → 0]
TextbookEq[P2Outlet = p2[l, y] == 0]
```

$$\frac{\partial \text{p2}}{\partial x} = 0$$

$$\frac{\partial \text{p2}}{\partial 0} = 0$$

$$\text{p2} = 0$$

The full list of equations, initial conditions and boundary conditions, for regions 1 and 2 are thus

In[24]:=
```
TextbookEq[EquationList1 =
  List[MassConservationEq1, XMomentumConservationEq1, YMomentumConservationEq1,
   NoSlipUP1, NoSlipDN1, NoPenetrationUP1, FluxDN1, P1Inlet, P1Outlet]]
```

$$\frac{\partial \text{ux1}}{\partial x} + \frac{\partial \text{uy1}}{\partial y} = 0$$
$$-\frac{\partial \text{p1}}{\partial x} + \mu \left(\frac{\partial^2 \text{ux1}}{\partial x^2} + \frac{\partial^2 \text{ux1}}{\partial y^2}\right) = 0$$
$$-\frac{\partial \text{p1}}{\partial y} + \mu \left(\frac{\partial^2 \text{uy1}}{\partial x^2} + \frac{\partial^2 \text{uy1}}{\partial y^2}\right) = 0$$
$$\text{ux1} = 0$$
$$\text{ux1} = 0$$
$$\text{uy1} = 0$$
$$\text{uy1} = \frac{k\,(-\text{p1} + \text{p2})}{w\,\mu}$$
$$\text{p1} = \text{pin}$$
$$\text{p1} = 0$$

and

In[25]:=
```
TextbookEq[EquationList2 =
  List[MassConservationEq2, XMomentumConservationEq2, YMomentumConservationEq2,
   NoSlipUP2, NoSlipDN2, NoPenetrationDN2, FluxUP2, P2Inlet, P2Outlet]]
```

$$\frac{\partial \mathrm{ux2}}{\partial x} + \frac{\partial \mathrm{uy2}}{\partial y} = 0$$
$$-\frac{\partial \mathrm{p2}}{\partial x} + \mu\left(\frac{\partial^2 \mathrm{ux2}}{\partial x^2} + \frac{\partial^2 \mathrm{ux2}}{\partial y^2}\right) = 0$$
$$-\frac{\partial \mathrm{p2}}{\partial y} + \mu\left(\frac{\partial^2 \mathrm{uy2}}{\partial x^2} + \frac{\partial^2 \mathrm{uy2}}{\partial y^2}\right) = 0$$
$$\mathrm{ux2} = 0$$
$$\mathrm{ux2} = 0$$
$$\mathrm{uy2} = 0$$
$$\mathrm{uy2} = k(-\mathrm{p1} + \mathrm{p2})$$
$$\frac{\partial \mathrm{p2}}{\partial 0} = 0$$
$$\mathrm{p2} = 0$$

We proceed to perform order-of-magnitude analysis of these equations and boundary conditions.

B.Scaling analysis and normalization of the governing equations

In[26]:=
```
AssumptionOnScaling = {l > 0, h1 > 0, h2 > 0, pC1 > 0,
   uxC > 0, uyC > 0, ϵ1 > 0, ϵ2 > 0, μ > 0, K1 > 0, K2 > 0, a1 > 0, Pin > 0};
```

In order to simplify the governing equations, the following geometrical and physical parameters were assumed to be positive:

In[27]:=
```
TextbookEq[{l > 0, h1 > 0, h2 > 0, μ > 0, K > 0, a1 > 0, Pin > 0}]
```

$$l > 0$$
$$\mathrm{h1} > 0$$
$$\mathrm{h2} > 0$$
$$\mu > 0$$
$$K > 0$$
$$\mathrm{a1} > 0$$
$$\mathrm{Pin} > 0$$

Scaling by the characteristic dimensions, which are assumed to be positive

In[28]:=
```
TextbookEq[{uxC > 0, uyC > 0, pC1 > 0}]
```

$$\mathrm{uxC} > 0$$
$$\mathrm{uyC} > 0$$
$$\mathrm{pC1} > 0$$

we introduce the non-dimensional variables

In[29]:=
```
X[x_] = x / l; Y1[y_] = y / h1; Y2[y_] = y / h2;
```

In[30]:=
```
TextbookEq[{X == x / l, Y1 == y / h1, Y2 == y / h2}]
```

$$X = \frac{x}{l}$$
$$\mathrm{Y1} = \frac{y}{\mathrm{h1}}$$
$$\mathrm{Y2} = \frac{y}{\mathrm{h2}}$$

In[31]:=

and define the slenderness ratio

In[32]:=
```
ShallowRelation1 = ϵ1 == h1 / l; ShallowRelation2 = ϵ2 == h2 / l;
```

In[33]:=
```
TextbookEq[{ShallowRelation1, ShallowRelation2}]
```

$$\epsilon 1 = \frac{\text{h1}}{l}$$
$$\epsilon 2 = \frac{\text{h2}}{l}$$

Substituting the above scaled coordinates as well as ϵ1 and ϵ2, to get the normalized set of equations as follows

In[34]:=
```
EquationList1 =
  Simplify[EquationList1 //. {ux1 → (ux1C UX1[X[#1], Y1[#2]] &), uy1 → (uyC UY1[X[#1], Y1[#2]] &),
     p1 → (pC1 P1[X[#1], Y1[#2]] &), p2 → (pC1 P2[X[#1], 0] &), pin → (Pin pC1 ),
     x → (X l), y → (Y1 h1)}, AssumptionOnScaling];
```

In[35]:=
```
TextbookEq[EquationList1]
```

$$\text{h1 ux1C} \frac{\partial \text{UX1}}{\partial X} + l \, \text{uyC} \frac{\partial \text{UY1}}{\partial \text{Y1}} = 0$$
$$\text{ux1C}\, \mu \left(\text{h1}^2 \frac{\partial^2 \text{UX1}}{\partial X^2} + l^2 \frac{\partial^2 \text{UX1}}{\partial \text{Y1}^2}\right) = \text{h1}^2\, l \, \text{pC1} \frac{\partial \text{P1}}{\partial X}$$
$$\text{uyC}\, \mu \left(\text{h1}^2 \frac{\partial^2 \text{UY1}}{\partial X^2} + l^2 \frac{\partial^2 \text{UY1}}{\partial \text{Y1}^2}\right) = \text{h1}\, l^2 \, \text{pC1} \frac{\partial \text{P1}}{\partial \text{Y1}}$$
$$\text{ux1C UX1} = 0$$
$$\text{ux1C UX1} = 0$$
$$\text{UY1} = 0$$
$$\text{uyC UY1} = \frac{k \, \text{pC1} (-\text{P1} + \text{P2})}{w \, \mu}$$
$$\text{Pin} = \text{P1}$$
$$\text{P1} = 0$$

In[36]:=
```
EquationList1 = Flatten[Simplify[EquationList1 //. {Solve[ShallowRelation1, h1]}, AssumptionOnScaling], 6];
```

Examining the scales of the terms of the continuity equation and assuming that $\frac{\partial \text{UX1}}{\partial X}$ is of the same order as $\frac{\partial \text{UY1}}{\partial \text{Y1}}$

In[37]:=
```
TextbookEq[EquationList1[[1]]]
```

$$\text{ux1C}\, \epsilon 1 \frac{\partial \text{UX1}}{\partial X} + \text{uyC} \frac{\partial \text{UY1}}{\partial \text{Y1}} = 0$$

yields the first scaling relation

In[38]:=
```
TextbookEq[FirstScalingRelation1 = uyC/ux1C == ϵ1]
```

$$\frac{\text{uyC}}{\text{ux1C}} = \epsilon 1$$

The relation is used to express uyC using the following parameters

In[39]:=
```
TextbookEq[Solve[FirstScalingRelation1, uyC][[1]][[1]]]
```

$$\text{uyC} \rightarrow \text{ux1C}\, \epsilon 1$$

which is then substituted into the governing equations

In[40]:= **EquationList1 =
Flatten[Simplify[EquationList1 //. {Solve[FirstScalingRelation1, uyC]}, AssumptionOnScaling], 6];**

In[41]:= **TextbookEq[EquationList1]**

$$\mathrm{ux1C}\left(\frac{\partial \mathrm{UX1}}{\partial X}+\frac{\partial \mathrm{UY1}}{\partial \mathrm{Y1}}\right)=0$$
$$\epsilon 1^2\left(l\,\mathrm{pC1}\,\frac{\partial \mathrm{P1}}{\partial X}-\mathrm{ux1C}\,\mu\,\frac{\partial^2 \mathrm{UX1}}{\partial X^2}\right)=\mathrm{ux1C}\,\mu\,\frac{\partial^2 \mathrm{UX1}}{\partial \mathrm{Y1}^2}$$
$$l\,\mathrm{pC1}\,\frac{\partial \mathrm{P1}}{\partial \mathrm{Y1}}=\mathrm{ux1C}\,\mu\left(\epsilon 1^2\,\frac{\partial^2 \mathrm{UY1}}{\partial X^2}+\frac{\partial^2 \mathrm{UY1}}{\partial \mathrm{Y1}^2}\right)$$
$$\mathrm{ux1C}\,\mathrm{UX1}=0$$
$$\mathrm{ux1C}\,\mathrm{UX1}=0$$
$$\mathrm{UY1}=0$$
$$\mathrm{ux1C}\,\epsilon 1\,\mathrm{UY1}=\frac{k\,\mathrm{pC1}\,(-\mathrm{P1}+\mathrm{P2})}{w\,\mu}$$
$$\mathrm{Pin}=\mathrm{P1}$$
$$\mathrm{P1}=0$$

By examining the scales of the streamwise momentum equation, given by

In[42]:= **TextbookEq[EquationList1[[2]]]**

$$\epsilon 1^2\left(l\,\mathrm{pC1}\,\frac{\partial \mathrm{P1}}{\partial X}-\mathrm{ux1C}\,\mu\,\frac{\partial^2 \mathrm{UX1}}{\partial X^2}\right)=\mathrm{ux1C}\,\mu\,\frac{\partial^2 \mathrm{UX1}}{\partial \mathrm{Y1}^2}$$

by expecting a balance between the viscous terms and pressure related terms, a pressure-dissipation scaling relation is obtained

In[43]:= **TextbookEq[PressureScale1 = pC1 / l == ux1C μ / (ϵ1 ϵ1 l l)]**

$$\frac{\mathrm{pC1}}{l}=\frac{\mathrm{ux1C}\,\mu}{l^2\,\epsilon 1^2}$$

Using this relation to express ux1C as follows

In[44]:= **TextbookEq[Solve[PressureScale1, ux1C][[1]][[1]]]**

$$\mathrm{ux1C}\to\frac{l\,\mathrm{pC1}\,\epsilon 1^2}{\mu}$$

and substitute it into the governing equations

In[45]:= **EquationList1 = Flatten[Simplify[EquationList1 //. {Solve[PressureScale1, ux1C]}, AssumptionOnScaling], 6];**

In[46]:= **TextbookEq[EquationList1]**

$$\frac{\partial \mathrm{UX1}}{\partial X}+\frac{\partial \mathrm{UY1}}{\partial \mathrm{Y1}}=0$$
$$\epsilon 1^2\,\frac{\partial^2 \mathrm{UX1}}{\partial X^2}+\frac{\partial^2 \mathrm{UX1}}{\partial \mathrm{Y1}^2}=\frac{\partial \mathrm{P1}}{\partial X}$$
$$\frac{\partial \mathrm{P1}}{\partial \mathrm{Y1}}=\epsilon 1^4\,\frac{\partial^2 \mathrm{UY1}}{\partial X^2}+\epsilon 1^2\,\frac{\partial^2 \mathrm{UY1}}{\partial \mathrm{Y1}^2}$$
$$\mathrm{UX1}=0$$
$$\mathrm{UX1}=0$$
$$\mathrm{UY1}=0$$
$$\frac{k\,\mathrm{P1}-k\,\mathrm{P2}+l\,w\,\epsilon 1^3\,\mathrm{UY1}}{w}=0$$
$$\mathrm{Pin}=\mathrm{P1}$$
$$\mathrm{P1}=0$$

From the boundary condition expressing the flux at the separator layer, i.e.

In[47]:= **TextbookEq[EquationList1[[7]]]**

$$\frac{k\,\mathrm{P1} - k\,\mathrm{P2} + l\,w\,\epsilon 1^3\,\mathrm{UY1}}{w} = 0$$

an additional dimensional parameter can be extracted

In[48]:=
```
TextbookEq[Kparameter1 = K1 == k / (w l ϵ1^3)]
```

$$\mathrm{K1} = \frac{k}{l\,w\,\epsilon 1^3}$$

resulting in an expression for the permeability K

In[49]:= **TextbookEq[Solve[Kparameter1, *k*][[1]][[1]]]**

$$k \to \mathrm{K1}\,l\,w\,\epsilon 1^3$$

which again is substituted to the governing equations

In[50]:= **EquationList1 = Flatten[Simplify[EquationList1 //. {Solve[Kparameter1, *k*]}, AssumptionOnScaling], 6];**

In[51]:= **TextbookEq[EquationList1]**

$$\begin{aligned}
&\frac{\partial \mathrm{UX1}}{\partial X} + \frac{\partial \mathrm{UY1}}{\partial \mathrm{Y1}} = 0\\
&\epsilon 1^2\,\frac{\partial^2 \mathrm{UX1}}{\partial X^2} + \frac{\partial^2 \mathrm{UX1}}{\partial \mathrm{Y1}^2} = \frac{\partial \mathrm{P1}}{\partial X}\\
&\frac{\partial \mathrm{P1}}{\partial \mathrm{Y1}} = \epsilon 1^4\,\frac{\partial^2 \mathrm{UY1}}{\partial X^2} + \epsilon 1^2\,\frac{\partial^2 \mathrm{UY1}}{\partial \mathrm{Y1}^2}\\
&\mathrm{UX1} = 0\\
&\mathrm{UX1} = 0\\
&\mathrm{UY1} = 0\\
&\mathrm{K1}\,\mathrm{P1} + \mathrm{UY1} = \mathrm{K1}\,\mathrm{P2}\\
&\mathrm{Pin} = \mathrm{P1}\\
&\mathrm{P1} = 0
\end{aligned}$$

Similar scaling is performed for region 2

In[52]:=
```
EquationList2 = Simplify[
   EquationList2 //. {ux2 → (ux2C UX2[X[#1], Y2[#2]] &), uy2 → (uyC UY2[X[#1], Y2[#2]] &),
     p2 → (pC2 P2[X[#1], Y2[#2]] &), p1 → (pC2 P1[X[#1], 0] &),
     pin → (Pin pC2), x → (X l), y → (Y2 h2)}, AssumptionOnScaling];
```

In[53]:= **EquationList2 = Flatten[Simplify[EquationList2 //. {Solve[ShallowRelation2, h2]}, AssumptionOnScaling], 6];**

similarly, scaling relations for the second region result in

In[54]:=
```
TextbookEq[FirstScalingRelation2 = uyC/ux2C == ϵ2]
```

$$\frac{\mathrm{uyC}}{\mathrm{ux2C}} = \epsilon 2$$

In[55]:=
```
TextbookEq[Kparameter2 = K2 == k / (w l ϵ2^3)]
```

$$\mathrm{K2} = \frac{k}{l\,w\,\epsilon 2^3}$$

and

In[56]:= `TextbookEq[PressureScale2 = pC2 / l == uyC μ / (ϵ2 ϵ2 ϵ2 l l)]`

$$\frac{\text{pC2}}{l} = \frac{\text{uyC}\,\mu}{l^2\,\epsilon 2^3}$$

These scaling relations are substituted to the governing equations of region 2 as follows

In[57]:= **TextbookEq[Solve[FirstScalingRelation2, ux2C][[1]][[1]]]**

$$\text{ux2C} \to \frac{\text{uyC}}{\epsilon 2}$$

In[58]:= **EquationList2 =
Flatten[Simplify[EquationList2 //. {Solve[FirstScalingRelation2, ux2C]}, AssumptionOnScaling], 6];**

In[59]:= **TextbookEq[Solve[PressureScale2, pC2][[1]][[1]]]**

$$\text{pC2} \to \frac{\text{uyC}\,\mu}{l\,\epsilon 2^3}$$

In[60]:= **EquationList2 = Flatten[Simplify[EquationList2 //. {Solve[PressureScale2, pC2]}, AssumptionOnScaling], 6];**

In[61]:= **TextbookEq[Solve[Kparameter2, *k*][[1]][[1]]]**

$$k \to \text{K2}\,l\,w\,\epsilon 2^3$$

In[62]:= **EquationList2 = Flatten[Simplify[EquationList2 //. {Solve[Kparameter2, *k*]}, AssumptionOnScaling], 6];**

Using the scaling relation for region 2 and replacing h2→l ϵ2 gives

In[63]:= **TextbookEq[h2 → ϵ2 *l*]**

$$\text{h2} \to l\,\epsilon 2$$

In[64]:= **EquationList2 = Flatten[Simplify[EquationList2 //. h2 → ϵ2 *l*, AssumptionOnScaling], 6];**

In[65]:= `TextbookEq[EquationList2]`

$$\begin{aligned}
&\frac{\partial \text{UX2}}{\partial X} + \frac{\partial \text{UY2}}{\partial \text{Y2}} = 0\\
&\epsilon 2^2\,\frac{\partial^2 \text{UX2}}{\partial X^2} + \frac{\partial^2 \text{UX2}}{\partial \text{Y2}^2} = \frac{\partial \text{P2}}{\partial X}\\
&\frac{\partial \text{P2}}{\partial \text{Y2}} = \epsilon 2^2\left(\epsilon 2^2\,\frac{\partial^2 \text{UY2}}{\partial X^2} + \frac{\partial^2 \text{UY2}}{\partial \text{Y2}^2}\right)\\
&\text{UX2} = 0\\
&\text{UX2} = 0\\
&\text{UY2} = 0\\
&\text{K2}\,w\,\mu\,\text{P1} + \text{UY2} = \text{K2}\,w\,\mu\,\text{P2}\\
&\frac{\partial \text{P2}}{\partial 0} = 0\\
&\text{P2} = 0
\end{aligned}$$

To summarize, the normalized system of equations are given by

In[66]:=
```
TextbookEq[EquationList1]
TextbookEq[EquationList2]
```

$$\frac{\partial UX1}{\partial X}+\frac{\partial UY1}{\partial Y1}=0$$
$$\epsilon1^2\frac{\partial^2 UX1}{\partial X^2}+\frac{\partial^2 UX1}{\partial Y1^2}=\frac{\partial P1}{\partial X}$$
$$\frac{\partial P1}{\partial Y1}=\epsilon1^4\frac{\partial^2 UY1}{\partial X^2}+\epsilon1^2\frac{\partial^2 UY1}{\partial Y1^2}$$
$$UX1=0$$
$$UX1=0$$
$$UY1=0$$
$$K1\,P1+UY1=K1\,P2$$
$$Pin=P1$$
$$P1=0$$

$$\frac{\partial UX2}{\partial X}+\frac{\partial UY2}{\partial Y2}=0$$
$$\epsilon2^2\frac{\partial^2 UX2}{\partial X^2}+\frac{\partial^2 UX2}{\partial Y2^2}=\frac{\partial P2}{\partial X}$$
$$\frac{\partial P2}{\partial Y2}=\epsilon2^2\left(\epsilon2^2\frac{\partial^2 UY2}{\partial X^2}+\frac{\partial^2 UY2}{\partial Y2^2}\right)$$
$$UX2=0$$
$$UX2=0$$
$$UY2=0$$
$$K2\,w\,\mu\,P1+UY2=K2\,w\,\mu\,P2$$
$$\frac{\partial P2}{\partial 0}=0$$
$$P2=0$$

Asymptotic Scheme

Expanding a regular asymptotic series for velocity and pressure expression with regard to the permeability parameter K1, i.e. :

UX1→(UX10[X,Y1]+K1 UX11[X,Y1]+K1^2 UX12[X,Y1]

UY1→(UY10[X,Y1]+K1 UY11[X,Y1]+K1^2 UY12[X,Y1]

P1→(P10[X,Y1]+K1 P11[X,Y1]+K1^2 P12[X,Y1]

P2→(P20[X,Y2]+K1 P21[X,Y2]+K1^2 P22[X,Y2]

and substituting into the governing equations gives

In[68]:=
```
EquationList1 =
  Simplify[EquationList1 //. {UX1 → (UX10[#1, #2] + K1 UX11[#1, #2] + K1 ^ 2 UX12[#1, #2] &),
      UY1 → (UY10[#1, #2] + K1 UY11[#1, #2] + K1 ^ 2 UY12[#1, #2] &),
      P1 → (P10[#1, #2] + K1 P11[#1, #2] + K1 ^ 2 P12[#1, #2] &),
      P2 → (P20[#1, #2] + K1 P21[#1, #2] + K1 ^ 2 P22[#1, #2] &)}];
```

In[69]:= **TextbookEq[EquationList1]**

$$\frac{\partial UX10}{\partial X} + K1\,\frac{\partial UX11}{\partial X} + K1^2\,\frac{\partial UX12}{\partial X} + \frac{\partial UY10}{\partial Y1} + K1\,\frac{\partial UY11}{\partial Y1} + K1^2\,\frac{\partial UY12}{\partial Y1} = 0$$
$$\frac{\partial^2 UX10}{\partial Y1^2} + K1\,\frac{\partial^2 UX11}{\partial Y1^2} + \epsilon 1^2\left(\frac{\partial^2 UX10}{\partial X^2} + K1\left(\frac{\partial^2 UX11}{\partial X^2} + K1\,\frac{\partial^2 UX12}{\partial X^2}\right)\right) + K1^2\,\frac{\partial^2 UX12}{\partial Y1^2} = \frac{\partial P10}{\partial X} + K1\left(\frac{\partial P11}{\partial X} + K1\,\frac{\partial P12}{\partial X}\right)$$
$$\frac{\partial P10}{\partial Y1} + K1\left(\frac{\partial P11}{\partial Y1} + K1\,\frac{\partial P12}{\partial Y1}\right) = \epsilon 1^2\left(\frac{\partial^2 UY10}{\partial Y1^2} + K1\,\frac{\partial^2 UY11}{\partial Y1^2} + \epsilon 1^2\left(\frac{\partial^2 UY10}{\partial X^2} + K1\left(\frac{\partial^2 UY11}{\partial X^2} + K1\,\frac{\partial^2 UY12}{\partial X^2}\right)\right) + K1^2\,\frac{\partial^2 UY12}{\partial Y1^2}\right)$$
$$UX10 + K1\,(UX11 + K1\,UX12) = 0$$
$$UX10 + K1\,(UX11 + K1\,UX12) = 0$$
$$UY10 + K1\,(UY11 + K1\,UY12) = 0$$
$$K1\,(P10 + K1\,(P11 + K1\,P12)) + UY10 + K1\,UY11 + K1^2\,UY12 = K1\,(P20 + K1\,(P21 + K1\,P22))$$
$$Pin = P10 + K1\,(P11 + K1\,P12)$$
$$P10 + K1\,(P11 + K1\,P12) = 0$$

The leading order is given by

In[70]:=
```
EquationList1LeadingOrder = GetEquationListOrder[EquationList1, 0, ϵ1];
EquationList1LeadingOrder = GetEquationListOrder[EquationList1LeadingOrder, 0, K1];
```

In[71]:= **EquationList1LeadingOrder**

Out[71]=
```
{UY10^(0,1)[X, Y1] + UX10^(1,0)[X, Y1] == 0, UX10^(0,2)[X, Y1] - P10^(1,0)[X, Y1] == 0,
 P10^(0,1)[X, Y1] == 0, UX10[X, 1] == 0, UX10[X, 0] == 0, UY10[X, 1] == 0,
 UY10[X, 0] == 0, Pin - P10[0, Y1] == 0, P10[1, Y1] == 0}
```

In[72]:= **TextbookEq[EquationList1LeadingOrder]**

$$\frac{\partial UX10}{\partial X} + \frac{\partial UY10}{\partial Y1} = 0$$
$$-\frac{\partial P10}{\partial X} + \frac{\partial^2 UX10}{\partial Y1^2} = 0$$
$$\frac{\partial P10}{\partial Y1} = 0$$
$$UX10 = 0$$
$$UX10 = 0$$
$$UY10 = 0$$
$$UY10 = 0$$
$$Pin - P10 = 0$$
$$P10 = 0$$

Similarly for region 2:

UX2→(UX20[X,Y2]+K1 UX21[X,Y2]+K1^2 UX22[X,Y2]

UY2→(UY20[X,Y2] +K1 UY21[X,Y2] +K1^2 UY22[X,Y2]

P2→(P20[X,Y2] +K1 P21[X,Y2] +K1^2 P22[X,Y2]

In[73]:=
```
EquationList2 =
  Simplify[EquationList2 //. {UX2 → (UX20[#1, #2] + K1 UX21[#1, #2] + K1 ^ 2 UX22[#1, #2] &),
      UY2 → (UY20[#1, #2] + K1 UY21[#1, #2] + K1 ^ 2 UY22[#1, #2] &),
      P2 → (P20[#1, #2] + K1 P21[#1, #2] + K1 ^ 2 P22[#1, #2] &)}, AssumptionOnScaling];
```

In[74]:= **TextbookEq[EquationList2]**

$$\frac{\partial UX20}{\partial X}+K1\,\frac{\partial UX21}{\partial X}+K1^2\,\frac{\partial UX22}{\partial X}+\frac{\partial UY20}{\partial Y2}+K1\,\frac{\partial UY21}{\partial Y2}+K1^2\,\frac{\partial UY22}{\partial Y2}=0$$
$$\frac{\partial^2 UX20}{\partial Y2^2}+K1\,\frac{\partial^2 UX21}{\partial Y2^2}+\epsilon 2^2\left(\frac{\partial^2 UX20}{\partial X^2}+K1\left(\frac{\partial^2 UX21}{\partial X^2}+K1\,\frac{\partial^2 UX22}{\partial X^2}\right)\right)+K1^2\,\frac{\partial^2 UX22}{\partial Y2^2}=\frac{\partial P20}{\partial X}+K1\left(\frac{\partial P21}{\partial X}+K1\,\frac{\partial P22}{\partial X}\right)$$
$$\frac{\partial P20}{\partial Y2}+K1\left(\frac{\partial P21}{\partial Y2}+K1\,\frac{\partial P22}{\partial Y2}\right)=\epsilon 2^2\left(\frac{\partial^2 UY20}{\partial Y2^2}+K1\,\frac{\partial^2 UY21}{\partial Y2^2}+\epsilon 2^2\left(\frac{\partial^2 UY20}{\partial X^2}+K1\left(\frac{\partial^2 UY21}{\partial X^2}+K1\,\frac{\partial^2 UY22}{\partial X^2}\right)\right)+K1^2\,\frac{\partial^2 UY22}{\partial Y2^2}\right)$$
$$UX20+K1\,(UX21+K1\,UX22)=0$$
$$UX20+K1\,(UX21+K1\,UX22)=0$$
$$UY20+K1\,(UY21+K1\,UY22)=0$$
$$K2\,w\,\mu\,P1+UY20+K1\,(UY21+K1\,UY22)=K2\,w\,\mu\,(P20+K1\,(P21+K1\,P22))$$
$$\frac{\partial P20}{\partial 0}+K1\left(\frac{\partial P21}{\partial 0}+K1\,\frac{\partial P22}{\partial 0}\right)=0$$
$$P20+K1\,(P21+K1\,P22)=0$$

Note: GetEquationListOrder[EquationList2,1,ϵ2] gives the multiplier in EquationList2 of term having ϵ2 of the 1st order.

In[75]:= **EquationList2LeadingOrder = GetEquationListOrder[EquationList2, 1, ϵ2];**

In[76]:= **EquationList2LeadingOrder = GetEquationListOrder[EquationList2, 2, ϵ2];**

In[77]:= **EquationList2LeadingOrder = GetEquationListOrder[EquationList2, 0, ϵ2];**
EquationList2LeadingOrder = GetEquationListOrder[EquationList2LeadingOrder, 0, K1];

The leading order of region 2 is given by

In[79]:= **EquationList2LeadingOrder**

Out[79]= $\{UY20^{(0,1)}[X, Y2] + UX20^{(1,0)}[X, Y2] == 0,$
$UX20^{(0,2)}[X, Y2] - P20^{(1,0)}[X, Y2] == 0, P20^{(0,1)}[X, Y2] == 0, UX20[X, 0] == 0,$
$UX20[X, -1 - a1\,X - a2\,X^2 - a3\,X^3] == 0, UY20[X, -1 - a1\,X - a2\,X^2 - a3\,X^3] == 0,$
$K2\,w\,\mu\,P1[X, 0] - K2\,w\,\mu\,P20[X, 0] + UY20[X, 0] == 0, P20^{(1,0)}[0, Y2] == 0, P20[1, Y2] == 0\}$

In[80]:= **TextbookEq[EquationList2LeadingOrder]**

$$\frac{\partial UX20}{\partial X}+\frac{\partial UY20}{\partial Y2}=0$$
$$-\frac{\partial P20}{\partial X}+\frac{\partial^2 UX20}{\partial Y2^2}=0$$
$$\frac{\partial P20}{\partial Y2}=0$$
$$UX20=0$$
$$UX20=0$$
$$UY20=0$$
$$K2\,w\,\mu\,P1-K2\,w\,\mu\,P20+UY20=0$$
$$\frac{\partial P20}{\partial 0}=0$$
$$P20=0$$

2. Solving the leading order of region 1

The leading order for region 1 is given by:

In[81]:= **EquationList1LeadingOrder**

Out[81]= $\{UY10^{(0,1)}[X, Y1] + UX10^{(1,0)}[X, Y1] == 0, UX10^{(0,2)}[X, Y1] - P10^{(1,0)}[X, Y1] == 0, P10^{(0,1)}[X, Y1] == 0, UX10[X, 1] == 0, UX10[X, 0] == 0, UY10[X, 1] == 0, UY10[X, 0] == 0, Pin - P10[0, Y1] == 0, P10[1, Y1] == 0\}$

In[82]:= **TextbookEq[EquationList1LeadingOrder]**

$$\frac{\partial \mathrm{UX10}}{\partial X} + \frac{\partial \mathrm{UY10}}{\partial \mathrm{Y1}} = 0$$
$$-\frac{\partial \mathrm{P10}}{\partial X} + \frac{\partial^2 \mathrm{UX10}}{\partial \mathrm{Y1}^2} = 0$$
$$\frac{\partial \mathrm{P10}}{\partial \mathrm{Y1}} = 0$$
$$\mathrm{UX10} = 0$$
$$\mathrm{UX10} = 0$$
$$\mathrm{UY10} = 0$$
$$\mathrm{UY10} = 0$$
$$\mathrm{Pin} - \mathrm{P10} = 0$$
$$\mathrm{P10} = 0$$

Momentum in the Y direction gives

In[83]:= **EquationList1LeadingOrder[[3]]**

Out[83]= $P10^{(0,1)}[X, Y1] == 0$

we thus see that P10 is not a function of Y1, only X, and thus

In[84]:= **EquationList1LeadingOrder = Simplify[EquationList1LeadingOrder //. {P10 → (G10[#1] &)}]**

Out[84]= $\{UY10^{(0,1)}[X, Y1] + UX10^{(1,0)}[X, Y1] == 0, G10'[X] == UX10^{(0,2)}[X, Y1], True, UX10[X, 1] == 0, UX10[X, 0] == 0, UY10[X, 1] == 0, UY10[X, 0] == 0, Pin == G10[0], G10[1] == 0\}$

We now solve the X momentum conservation equation, given by

In[85]:= **EquationList1LeadingOrder[[2]]**

Out[85]= $G10'[X] == UX10^{(0,2)}[X, Y1]$

The solution for Ux along with the no-slip boundary condition is

In[86]:= **UX10Func[X_, Y1_] = Flatten[DSolve[{EquationList1LeadingOrder[[2]], EquationList1LeadingOrder[[4]], EquationList1LeadingOrder[[5]]}, UX10[*X*, Y1], {Y1}]][[1, 2]]**

Out[86]= $\frac{1}{2}\left(-Y1\, G10'[X] + Y1^2\, G10'[X]\right)$

Substituting the solution for Ux into the leading order equations

In[87]:= **EquationList1LeadingOrder = EquationList1LeadingOrder /. {UX10 → Function[{*X*, Y1}, UX10Func[*X*, Y1]]}**

Out[87]= $\{\frac{1}{2}\left(-Y1\, G10''[X] + Y1^2\, G10''[X]\right) + UY10^{(0,1)}[X, Y1] == 0, True, True, True, True, UY10[X, 1] == 0, UY10[X, 0] == 0, Pin == G10[0], G10[1] == 0\}$

We now solve the mass conservation equation, given by

In[88]:= **EquationList1LeadingOrder[[1]]**

Out[88]= $\frac{1}{2}\left(-Y1\,G10''[X]+Y1^2\,G10''[X]\right)+UY10^{(0,1)}[X,Y1]==0$

The expression for Uy is given when solving the mass conservation equation and by using the no penetration boundary condition at the top of region 1, i.e. UY10[X,1]==0:

In[89]:= **UY10Func[X_, Y1_] = Flatten[
DSolve[{EquationList1LeadingOrder[[1]], EquationList1LeadingOrder[[6]]}, UY10[*X*, Y1], {Y1}]][[1, 2]]**

Out[89]= $\frac{1}{12}\left(-G10''[X]+3\,Y1^2\,G10''[X]-2\,Y1^3\,G10''[X]\right)$

Substituting the expression for Uy10 into the equation list for the leading order gives:

In[90]:= **EquationList1LeadingOrder = EquationList1LeadingOrder /. {UY10 → Function[{*X*, Y1}, UY10Func[*X*, Y1]]}**

Out[90]= $\left\{\frac{1}{12}\left(6\,Y1\,G10''[X]-6\,Y1^2\,G10''[X]\right)+\frac{1}{2}\left(-Y1\,G10''[X]+Y1^2\,G10''[X]\right)==0,\right.$
$\left.\text{True, True, True, True, True, }-\frac{1}{12}G10''[X]==0,\ Pin==G10[0],\ G10[1]==0\right\}$

The pressure of the leading order for region 1 is a pressure driven channel flow:

In[91]:= **G10Func[X_] = Flatten[DSolve[{EquationList1LeadingOrder[[7]],
EquationList1LeadingOrder[[8]], EquationList1LeadingOrder[[9]]}, G10[*X*], {*X*}]][[1, 2]]**

Out[91]= Pin − Pin X

Substituting the solution for P1 into the leading order equations shows that all the equations are satisfied.

In[92]:= **EquationList1LeadingOrder = EquationList1LeadingOrder //. {G10 → Function[{*X*}, G10Func[*X*]]}**

Out[92]= {True, True, True, True, True, True, True, True, True}

As expected for a pressure driven channel flow, Uy in the leading order is zero

In[93]:= **UY10Func[X_, Y1_] = UY10Func[*X*, Y1] //. {G10 → Function[{*X*}, G10Func[*X*]]}**

Out[93]= 0

UY10, the leading order of the velocity in the direction perpendicular to the channel is zero since it is a pressure driven channel flow.

In[94]:= **UX10Func[X_, Y1_] = UX10Func[*X*, Y1] //. {G10 → Function[{*X*}, G10Func[*X*]]}**

Out[94]= $\frac{1}{2}\left(Pin\,Y1-Pin\,Y1^2\right)$

In summary, the pressure, Ux, uy:

In[95]:=
```
G10Func[X]
UX10Func[X, Y1]
UY10Func[X, Y1]
```

Out[95]= $\text{Pin} - \text{Pin}\,X$

Out[96]= $\frac{1}{2}\left(\text{Pin}\,Y1 - \text{Pin}\,Y1^2\right)$

Out[97]= 0

3. Solving the leading order of region 2

Polynomial h2 case:

The set of equations for the leading order of region 2:

In[98]:=
```
EquationList2LeadingOrder
```

Out[98]= $\{UY20^{(0,1)}[X, Y2] + UX20^{(1,0)}[X, Y2] == 0, UX20^{(0,2)}[X, Y2] - P20^{(1,0)}[X, Y2] == 0, P20^{(0,1)}[X, Y2] == 0, UX20[X, 0] == 0, UX20\left[X, -1 - a1\,X - a2\,X^2 - a3\,X^3\right] == 0, UY20\left[X, -1 - a1\,X - a2\,X^2 - a3\,X^3\right] == 0, K2\,w\,\mu\,P1[X, 0] - K2\,w\,\mu\,P20[X, 0] + UY20[X, 0] == 0, P20^{(1,0)}[0, Y2] == 0, P20[1, Y2] == 0\}$

In[99]:=
```
TextbookEq[EquationList2LeadingOrder]
```

$$\frac{\partial \text{UX20}}{\partial X} + \frac{\partial \text{UY20}}{\partial \text{Y2}} = 0$$
$$-\frac{\partial \text{P20}}{\partial X} + \frac{\partial^2 \text{UX20}}{\partial \text{Y2}^2} = 0$$
$$\frac{\partial \text{P20}}{\partial \text{Y2}} = 0$$
$$\text{UX20} = 0$$
$$\text{UX20} = 0$$
$$\text{UY20} = 0$$
$$\text{K2}\,w\,\mu\,\text{P1} - \text{K2}\,w\,\mu\,\text{P20} + \text{UY20} = 0$$
$$\frac{\partial \text{P20}}{\partial 0} = 0$$
$$\text{P20} = 0$$

Referring to the simplified case where the bottom wall of region 2 expands linearly downstream

In[100]:=
```
a2 = 0;
a3 = 0;
```

The simplified set of equations for the leading order of region 2 :

In[102]:=
```
EquationList2LeadingOrder
```

Out[102]= $\{UY20^{(0,1)}[X, Y2] + UX20^{(1,0)}[X, Y2] == 0, UX20^{(0,2)}[X, Y2] - P20^{(1,0)}[X, Y2] == 0, P20^{(0,1)}[X, Y2] == 0, UX20[X, 0] == 0, UX20[X, -1 - a1\,X] == 0, UY20[X, -1 - a1\,X] == 0, K2\,w\,\mu\,P1[X, 0] - K2\,w\,\mu\,P20[X, 0] + UY20[X, 0] == 0, P20^{(1,0)}[0, Y2] == 0, P20[1, Y2] == 0\}$

Assuming pC=pin, which is the most interesting limit

In[103]:= **EquationList2LeadingOrder = Flatten[Simplify[
{EquationList2LeadingOrder //. {P1 → Function[{X}, G10Func[X]]}}, AssumptionOnScaling]]**

Out[103]= {UY20^(0,1)[X, Y2] + UX20^(1,0)[X, Y2] == 0, UX20^(0,2)[X, Y2] == P20^(1,0)[X, Y2],
P20^(0,1)[X, Y2] == 0, UX20[X, 0] == 0, UX20[X, -1 - a1 X] == 0, UY20[X, -1 - a1 X] == 0,
UY20[X, 0] == K2 w μ (Pin (-1 + X) + P20[X, 0]), P20^(1,0)[0, Y2] == 0, P20[1, Y2] == 0}

In[104]:= **TextbookEq[EquationList2LeadingOrder]**

$$\frac{\partial \mathrm{UX20}}{\partial X} + \frac{\partial \mathrm{UY20}}{\partial \mathrm{Y2}} = 0$$
$$\frac{\partial^2 \mathrm{UX20}}{\partial \mathrm{Y2}^2} = \frac{\partial \mathrm{P20}}{\partial X}$$
$$\frac{\partial \mathrm{P20}}{\partial \mathrm{Y2}} = 0$$
$$\mathrm{UX20} = 0$$
$$\mathrm{UX20} = 0$$
$$\mathrm{UY20} = 0$$
$$\mathrm{UY20} = \mathrm{K2}\, w\, \mu\, (\mathrm{Pin}\,(-1 + X) + \mathrm{P20})$$
$$\frac{\partial \mathrm{P20}}{\partial 0} = 0$$
$$\mathrm{P20} = 0$$

Momentum in the Y direction gives

In[105]:= **TextbookEq[EquationList2LeadingOrder[[3]]]**

$$\frac{\partial \mathrm{P20}}{\partial \mathrm{Y2}} = 0$$

we thus see that P20 is not a function of Y2, and thus

In[106]:= **EquationList2LeadingOrder = Simplify[EquationList2LeadingOrder //. {P20 → (G20[#1] &)}]**

Out[106]= {UY20^(0,1)[X, Y2] + UX20^(1,0)[X, Y2] == 0, G20′[X] == UX20^(0,2)[X, Y2],
True, UX20[X, 0] == 0, UX20[X, -1 - a1 X] == 0, UY20[X, -1 - a1 X] == 0,
UY20[X, 0] == K2 w μ (Pin (-1 + X) + G20[X]), G20′[0] == 0, G20[1] == 0}

We now solve the X momentum conservation equation, given by

In[107]:= **TextbookEq[EquationList2LeadingOrder[[2]]]**

$$\frac{\partial \mathrm{G20}}{\partial X} = \frac{\partial^2 \mathrm{UX20}}{\partial \mathrm{Y2}^2}$$

Solving the equation for Ux while implementing the no slip boundary conditions at the top and bottom walls of region 2:

In[108]:= **TextbookEq[
UX20Func[X_, Y2_] = Flatten[DSolve[{EquationList2LeadingOrder[[2]], EquationList2LeadingOrder[[4]],
EquationList2LeadingOrder[[5]]}, UX20[X, Y2], {Y2}]][[1, 2]]]**

$$\frac{1}{2}\left(\mathrm{Y2}\,\frac{\partial \mathrm{G20}}{\partial X} + \mathrm{a1}\,X\,\mathrm{Y2}\,\frac{\partial \mathrm{G20}}{\partial X} + \mathrm{Y2}^2\,\frac{\partial \mathrm{G20}}{\partial X}\right)$$

In[109]:= **EquationList2LeadingOrder = EquationList2LeadingOrder /. {UX20 → Function[{X, Y2}, UX20Func[X, Y2]]}**

Out[109]= $\left\{\frac{1}{2}\left(\text{a1 Y2 G20}'[X]+\text{Y2 G20}''[X]+\text{a1 X Y2 G20}''[X]+\text{Y2}^2\text{ G20}''[X]\right)+\text{UY20}^{(0,1)}[X,\text{Y2}]==0,\text{True},\right.$
$\text{True},\text{True},\frac{1}{2}\left(\left(-1-\text{a1 X}\right)\text{G20}'[X]+\text{a1 X}\left(-1-\text{a1 X}\right)\text{G20}'[X]+\left(-1-\text{a1 X}\right)^2\text{G20}'[X]\right)==0,$
$\left.\text{UY20}[X,-1-\text{a1 X}]==0,\text{UY20}[X,0]==\text{K2 w}\,\mu\left(\text{Pin}\left(-1+X\right)+\text{G20}[X]\right),\text{G20}'[0]==0,\text{G20}[1]==0\right\}$

In[110]:= **EquationList2LeadingOrder;**

We now solve the mass conservation equation, given by

In[111]:= **TextbookEq[EquationList2LeadingOrder[[1]]]**

$$\frac{1}{2}\left(\text{a1 Y2}\frac{\partial \text{G20}}{\partial X}+\text{Y2}\frac{\partial^2 \text{G20}}{\partial X^2}+\text{a1}\,X\,\text{Y2}\frac{\partial^2 \text{G20}}{\partial X^2}+\text{Y2}^2\frac{\partial^2 \text{G20}}{\partial X^2}\right)+\frac{\partial \text{UY20}}{\partial \text{Y2}}=0$$

To obtain a solution for UY20

In[112]:= **TextbookEq[UY20Func[X_, Y2_] =
Flatten[DSolve[{EquationList2LeadingOrder[[1]], EquationList2LeadingOrder[[6]]}, UY20[X, Y2], {Y2}]][[
1, 2]]]**

$$\frac{1}{12}\left(3\,\text{a1}\frac{\partial \text{G20}}{\partial X}+6\,\text{a1}^2 X\frac{\partial \text{G20}}{\partial X}+3\,\text{a1}^3 X^2\frac{\partial \text{G20}}{\partial X}-3\,\text{a1 Y2}^2\frac{\partial \text{G20}}{\partial X}+\frac{\partial^2 \text{G20}}{\partial X^2}+3\,\text{a1}\,X\frac{\partial^2 \text{G20}}{\partial X^2}+3\,\text{a1}^2 X^2\frac{\partial^2 \text{G20}}{\partial X^2}+\text{a1}^3 X^3\frac{\partial^2 \text{G20}}{\partial X^2}-3\,\text{Y2}^2\frac{\partial^2 \text{G20}}{\partial X^2}-3\,\text{a1}\,X\,\text{Y2}^2\frac{\partial^2 \text{G20}}{\partial X^2}-2\,\text{Y2}^3\frac{\partial^2 \text{G20}}{\partial X^2}\right)$$

In[113]:= **EquationList2LeadingOrder = EquationList2LeadingOrder /. {UY20 → Function[{X, Y2}, UY20Func[X, Y2]]}**

Out[113]= $\left\{\frac{1}{12}\left(-6\,\text{a1 Y2 G20}'[X]-6\,\text{Y2 G20}''[X]-6\,\text{a1 X Y2 G20}''[X]-6\,\text{Y2}^2\text{ G20}''[X]\right)+\right.$
$\frac{1}{2}\left(\text{a1 Y2 G20}'[X]+\text{Y2 G20}''[X]+\text{a1 X Y2 G20}''[X]+\text{Y2}^2\text{ G20}''[X]\right)==0,\text{True},\text{True},$
$\text{True},\frac{1}{2}\left(\left(-1-\text{a1 X}\right)\text{G20}'[X]+\text{a1 X}\left(-1-\text{a1 X}\right)\text{G20}'[X]+\left(-1-\text{a1 X}\right)^2\text{G20}'[X]\right)==0,$
$\frac{1}{12}\left(3\,\text{a1 G20}'[X]+6\,\text{a1}^2\text{ X G20}'[X]+3\,\text{a1}^3\text{ X}^2\text{ G20}'[X]-3\,\text{a1}\left(-1-\text{a1 X}\right)^2\text{G20}'[X]+\right.$
$\text{G20}''[X]+3\,\text{a1 X G20}''[X]+3\,\text{a1}^2\text{ X}^2\text{ G20}''[X]+\text{a1}^3\text{ X}^3\text{ G20}''[X]-$
$\left.3\left(-1-\text{a1 X}\right)^2\text{G20}''[X]-3\,\text{a1 X}\left(-1-\text{a1 X}\right)^2\text{G20}''[X]-2\left(-1-\text{a1 X}\right)^3\text{G20}''[X]\right)==0,$
$\frac{1}{12}\left(3\,\text{a1 G20}'[X]+6\,\text{a1}^2\text{ X G20}'[X]+3\,\text{a1}^3\text{ X}^2\text{ G20}'[X]+\text{G20}''[X]+3\,\text{a1 X G20}''[X]+\right.$
$\left.\left.3\,\text{a1}^2\text{ X}^2\text{ G20}''[X]+\text{a1}^3\text{ X}^3\text{ G20}''[X]\right)==\text{K2 w}\,\mu\left(\text{Pin}\left(-1+X\right)+\text{G20}[X]\right),\text{G20}'[0]==0,\text{G20}[1]==0\right\}$

In[114]:= **TextbookEq[FullSimplify[EquationList2LeadingOrder[[7]]]]**

$$12\,\text{K2}\,w\,\mu\,(\text{Pin}\,(-1+X)+\text{G20})=(1+\text{a1}\,X)^2\left(3\,\text{a1}\frac{\partial \text{G20}}{\partial X}+(1+\text{a1}\,X)\frac{\partial^2 \text{G20}}{\partial X^2}\right)$$

In[115]:=
```
TextbookEq[EquationList2LeadingOrder[[7]]]
TextbookEq[EquationList2LeadingOrder[[8]]]
TextbookEq[EquationList2LeadingOrder[[9]]]
```

$$\frac{1}{12}\left(3\,\text{a1}\,\frac{\partial \text{G20}}{\partial X}+6\,\text{a1}^2 X\,\frac{\partial \text{G20}}{\partial X}+3\,\text{a1}^3 X^2\,\frac{\partial \text{G20}}{\partial X}+\frac{\partial^2 \text{G20}}{\partial X^2}+3\,\text{a1}\,X\,\frac{\partial^2 \text{G20}}{\partial X^2}+3\,\text{a1}^2 X^2\,\frac{\partial^2 \text{G20}}{\partial X^2}+\text{a1}^3 X^3\,\frac{\partial^2 \text{G20}}{\partial X^2}\right)= \text{K2}\,w\,\mu\,(\text{Pin}\,(-1+X)+\text{G20})$$

$$\frac{\partial \text{G20}}{\partial 0}=0$$

$$\text{G20}=0$$

G(x) Asymptotic expansion

Assuming a1<<1

Expression for G (x)~G0 + a1 G1

The assumptions on scaling, as mentioned before are:

In[118]:=
```
AssumptionOnScaling = {l > 0, h1 > 0, h2 > 0, pC1 > 0,
    uxC > 0, uyC > 0, ϵ1 > 0, ϵ2 > 0, μ > 0, K1 > 0, K2 > 0, a1 > 0, Pin > 0};
```

The equation representing the pressure for the leading order in region 2

In[119]:=
```
GOriginalEq =
  1/12 (3 a1 G20'[X] + 6 a1^2 X G20'[X] + 3 a1^3 X^2 G20'[X] + G20''[X] + 3 a1 X G20''[X] + 3 a1^2 X^2 G20''[X] +
     a1^3 X^3 G20''[X]) == K2 (Pin (-1 + X) + G20[X])
```

Out[119]=
$$\frac{1}{12}\left(3\,\text{a1}\,\text{G20}'[X]+6\,\text{a1}^2 X\,\text{G20}'[X]+3\,\text{a1}^3 X^2\,\text{G20}'[X]+\text{G20}''[X]+3\,\text{a1}\,X\,\text{G20}''[X]+3\,\text{a1}^2 X^2\,\text{G20}''[X]+\text{a1}^3 X^3\,\text{G20}''[X]\right)=\text{K2}\,(\text{Pin}\,(-1+X)+\text{G20}[X])$$

In[120]:=
```
GOriginalEqList = List[GOriginalEq];
```

In[121]:=
```
GAsymptoticExpList = Simplify[GOriginalEqList //. {G20 → (G0[#1] + a1 G1[#1] + a1 ^ 2 G2[#1]  &)}]
```

Out[121]=
$$\left\{\frac{1}{12}(1+\text{a1}\,X)^2\left(3\,\text{a1}\,\text{G0}'[X]+3\,\text{a1}^2\,\text{G1}'[X]+3\,\text{a1}^3\,\text{G2}'[X]+\text{G0}''[X]+\text{a1}\,X\,\text{G0}''[X]+\text{a1}\,\text{G1}''[X]+\text{a1}^2 X\,\text{G1}''[X]+\text{a1}^2\,\text{G2}''[X]+\text{a1}^3 X\,\text{G2}''[X]\right)=\text{K2}\,(\text{Pin}\,(-1+X)+\text{G0}[X]+\text{a1}\,\text{G1}[X]+\text{a1}^2\,\text{G2}[X])\right\}$$

Dividing the orders of the solution by orders of a1. The leading order, i.e. O(a1=0):

In[122]:=
```
GEqLeadingOrder = GetEquationListOrder[GAsymptoticExpList, 0, a1]
```

Out[122]=
$$\left\{-\text{K2}\,\text{Pin}\,(-1+X)-\text{K2}\,\text{G0}[X]+\frac{\text{G0}''[X]}{12}=0\right\}$$

In[123]:=
```
TextbookEq[FullSimplify[GEqLeadingOrder]]
```

$$12\,\text{K2}\,(\text{Pin}\,(-1+X)+\text{G0})=\frac{\partial^2 \text{G0}}{\partial X^2}$$

As expected, same as the case for a1=0:

$$\frac{G20''[X]}{12} == K2\ (Pin\ (-1+X) + G20[X])$$

In[124]:=
```
GEq1stOrder = GetEquationListOrder[GAsymptoticExpList, 1, a1]
```

Out[124]= $\left\{-K2\ G1[X] + \frac{G0'[X]}{4} + \frac{1}{4} X\ G0''[X] + \frac{G1''[X]}{12} == 0\right\}$

In[125]:=
```
TextbookEq[FullSimplify[GEq1stOrder]]
```

$$12\ K2\ G1 = 3\ \frac{\partial G0}{\partial X} + 3\ X\ \frac{\partial^2 G0}{\partial X^2} + \frac{\partial^2 G1}{\partial X^2}$$

Solving by orders.

First, leading order:

Boundary conditions:

In[126]:=
```
G0BC1 = D[G0[X], X] == 0; G0BC1 = G0BC1 //. X → 0;
G0BC2 = G0[1] == 0;
```

In[128]:=
```
FuncG0[X_] =
  Simplify[DSolve[{GEqLeadingOrder[[1]], G0BC1, G0BC2}, G0[X], {X}]][[1]][[1]][[2]]
```

Out[128]= $$\frac{Pin\left(6 - \frac{\sqrt{3}\ e^{-2\sqrt{3}\sqrt{K2}(-2+X)}}{\sqrt{K2}} + \frac{\sqrt{3}\ e^{2\sqrt{3}\sqrt{K2}X}}{\sqrt{K2}} - 6\ e^{4\sqrt{3}\sqrt{K2}}(-1+X) - 6X\right)}{6\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)}$$

In[129]:=
```
TextbookEq[FullSimplify[FuncG0[X]]]
```

$$\frac{Pin\left(6 - \frac{\sqrt{3}\ e^{-2\sqrt{3}\sqrt{K2}(-2+X)}}{\sqrt{K2}} + \frac{\sqrt{3}\ e^{2\sqrt{3}\sqrt{K2}X}}{\sqrt{K2}} - 6\ e^{4\sqrt{3}\sqrt{K2}}(-1+X) - 6X\right)}{6\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)}$$

Checking BC for G0:

In[130]:=
```
FuncG0[X] /. X → 1
D[FuncG0[X], X] //. X → 0
```

Out[130]= 0

Out[131]= 0

Next, solving the 1st order :

In[132]:=
```
G1BC1 = D[G1[X], X] == 0; G1BC1 = G1BC1 //. X → 0;
G1BC2 = G1[1] == 0;
```

Substituting G0 into the exp. for the 1st order:

In[134]:= **GEq1stOrder1 = GEq1stOrder //. G0 → Function[X, FuncG0[X]]**

Out[134]= $$\left\{\frac{\left(-6-6\,e^{4\sqrt{3}\sqrt{K2}}+6\,e^{-2\sqrt{3}\sqrt{K2}(-2+X)}+6\,e^{2\sqrt{3}\sqrt{K2}X}\right)\text{Pin}}{24\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)}+\frac{\left(-12\sqrt{3}\,e^{-2\sqrt{3}\sqrt{K2}(-2+X)}\sqrt{K2}+12\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}X}\sqrt{K2}\right)\text{Pin}\,X}{24\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)}-K2\,G1[X]+\frac{G1''[X]}{12}==0\right\}$$

In[135]:= **TextbookEq[FullSimplify[GEq1stOrder1]]**

$$\frac{1}{12}\left(\frac{3\left(-1-e^{4\sqrt{3}\sqrt{K2}}+e^{-2\sqrt{3}\sqrt{K2}(-2+X)}+e^{2\sqrt{3}\sqrt{K2}X}\right)\text{Pin}}{1+e^{4\sqrt{3}\sqrt{K2}}}+\frac{6\sqrt{3}\,e^{-2\sqrt{3}\sqrt{K2}(-2+X)}\left(-1+e^{4\sqrt{3}\sqrt{K2}(-1+X)}\right)\sqrt{K2}\,\text{Pin}\,X}{1+e^{4\sqrt{3}\sqrt{K2}}}+\frac{\partial^2 G1}{\partial X^2}-12\,K2\,G1\right)=0$$

In[136]:= **FuncG1[X_] =**
FullSimplify[DSolve[{GEq1stOrder1, G1BC1, G1BC2}, G1[X], {X}], AssumptionOnScaling][[
1]][[1]][[2]]

Out[136]= $$-\frac{1}{16\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^2 K2}e^{2\sqrt{3}\sqrt{K2}(2-3X)}\,\text{Pin}\left(8\,e^{6\sqrt{3}\sqrt{K2}X}-4\,e^{2\sqrt{3}\sqrt{K2}(-1+2X)}-4\,e^{2\sqrt{3}\sqrt{K2}(1+2X)}+4\,e^{2\sqrt{3}\sqrt{K2}(-2+3X)}+4\,e^{2\sqrt{3}\sqrt{K2}(2+3X)}-4\,e^{2\sqrt{3}\sqrt{K2}(-1+4X)}-4\,e^{2\sqrt{3}\sqrt{K2}(1+4X)}+e^{4\sqrt{3}\sqrt{K2}(1+X)}\left(-1-2\sqrt{3}\sqrt{K2}\,X+12\,K2\,X^2\right)+e^{4\sqrt{3}\sqrt{K2}(-1+2X)}\left(-1+2\sqrt{3}\sqrt{K2}\,X+12\,K2\,X^2\right)+e^{4\sqrt{3}\sqrt{K2}X}\left(1-2\sqrt{3}\sqrt{K2}\,X+12\,K2\left(-2+X^2\right)\right)+e^{8\sqrt{3}\sqrt{K2}X}\left(1+2\sqrt{3}\sqrt{K2}\,X+12\,K2\left(-2+X^2\right)\right)\right)$$

In[137]:= **TextbookEq[FullSimplify[FuncG1[X]]]**

$$-\frac{1}{16\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^2 K2}e^{2\sqrt{3}\sqrt{K2}(2-3X)}\,\text{Pin}\left(8\,e^{6\sqrt{3}\sqrt{K2}X}-4\,e^{2\sqrt{3}\sqrt{K2}(-1+2X)}-4\,e^{2\sqrt{3}\sqrt{K2}(1+2X)}+4\,e^{2\sqrt{3}\sqrt{K2}(-2+3X)}+4\,e^{2\sqrt{3}\sqrt{K2}(2+3X)}-4\,e^{2\sqrt{3}\sqrt{K2}(-1+4X)}-4\,e^{2\sqrt{3}\sqrt{K2}(1+4X)}+e^{4\sqrt{3}\sqrt{K2}(1+X)}\left(-1-2\sqrt{3}\sqrt{K2}\,X+12\,K2\,X^2\right)+e^{4\sqrt{3}\sqrt{K2}(-1+2X)}\left(-1+2\sqrt{3}\sqrt{K2}\,X+12\,K2\,X^2\right)+e^{4\sqrt{3}\sqrt{K2}X}\left(1-2\sqrt{3}\sqrt{K2}\,X+12\,K2\left(-2+X^2\right)\right)+e^{8\sqrt{3}\sqrt{K2}X}\left(1+2\sqrt{3}\sqrt{K2}\,X+12\,K2\left(-2+X^2\right)\right)\right)$$

Checking BC for G1 :

In[138]:= **Simplify[FuncG1[X] /. X → 1]**
Simplify[D[FuncG1[X], X] //. X → 0]

Out[138]= 0

Out[139]= 0

In[140]:= `FuncG01[X_] = Simplify[FuncG0[X] + a1 FuncG1[X]]`

Out[140]=

$$\begin{aligned}&\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^{2}}\\&Pin\left(8\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)\left(6-\frac{\sqrt{3}\,e^{-2\sqrt{3}\sqrt{K2}(-2+X)}}{\sqrt{K2}}+\frac{\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}X}}{\sqrt{K2}}-6e^{4\sqrt{3}\sqrt{K2}}(-1+X)-6X\right)-\right.\\&\quad\frac{1}{K2}3\,a1\,e^{2\sqrt{3}\sqrt{K2}(2-3X)}\\&\quad\left(8e^{6\sqrt{3}\sqrt{K2}X}-4e^{2\sqrt{3}\sqrt{K2}(-1+2X)}-4e^{2\sqrt{3}\sqrt{K2}(1+2X)}+4e^{2\sqrt{3}\sqrt{K2}(-2+3X)}+4e^{2\sqrt{3}\sqrt{K2}(2+3X)}-\right.\\&\quad 4e^{2\sqrt{3}\sqrt{K2}(-1+4X)}-4e^{2\sqrt{3}\sqrt{K2}(1+4X)}+e^{4\sqrt{3}\sqrt{K2}(1+X)}\left(-1-2\sqrt{3}\sqrt{K2}X+12K2X^{2}\right)+\\&\quad e^{4\sqrt{3}\sqrt{K2}(-1+2X)}\left(-1+2\sqrt{3}\sqrt{K2}X+12K2X^{2}\right)+e^{4\sqrt{3}\sqrt{K2}X}\left(1-2\sqrt{3}\sqrt{K2}X+12K2\left(-2+X^{2}\right)\right)+\\&\quad\left.\left. e^{8\sqrt{3}\sqrt{K2}X}\left(1+2\sqrt{3}\sqrt{K2}X+12K2\left(-2+X^{2}\right)\right)\right)\right)\end{aligned}$$

In[141]:= `FuncG201[X_] = FuncG01[X]`

Out[141]=

$$\begin{aligned}&\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^{2}}\\&Pin\left(8\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)\left(6-\frac{\sqrt{3}\,e^{-2\sqrt{3}\sqrt{K2}(-2+X)}}{\sqrt{K2}}+\frac{\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}X}}{\sqrt{K2}}-6e^{4\sqrt{3}\sqrt{K2}}(-1+X)-6X\right)-\right.\\&\quad\frac{1}{K2}3\,a1\,e^{2\sqrt{3}\sqrt{K2}(2-3X)}\\&\quad\left(8e^{6\sqrt{3}\sqrt{K2}X}-4e^{2\sqrt{3}\sqrt{K2}(-1+2X)}-4e^{2\sqrt{3}\sqrt{K2}(1+2X)}+4e^{2\sqrt{3}\sqrt{K2}(-2+3X)}+4e^{2\sqrt{3}\sqrt{K2}(2+3X)}-\right.\\&\quad 4e^{2\sqrt{3}\sqrt{K2}(-1+4X)}-4e^{2\sqrt{3}\sqrt{K2}(1+4X)}+e^{4\sqrt{3}\sqrt{K2}(1+X)}\left(-1-2\sqrt{3}\sqrt{K2}X+12K2X^{2}\right)+\\&\quad e^{4\sqrt{3}\sqrt{K2}(-1+2X)}\left(-1+2\sqrt{3}\sqrt{K2}X+12K2X^{2}\right)+e^{4\sqrt{3}\sqrt{K2}X}\left(1-2\sqrt{3}\sqrt{K2}X+12K2\left(-2+X^{2}\right)\right)+\\&\quad\left.\left. e^{8\sqrt{3}\sqrt{K2}X}\left(1+2\sqrt{3}\sqrt{K2}X+12K2\left(-2+X^{2}\right)\right)\right)\right)\end{aligned}$$

In[142]:= `TextbookEq[FuncG201[X]]`

$$\begin{aligned}&\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^{2}}\\&Pin\left(8\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)\left(6-\frac{\sqrt{3}\,e^{-2\sqrt{3}\sqrt{K2}(-2+X)}}{\sqrt{K2}}+\frac{\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}X}}{\sqrt{K2}}-6e^{4\sqrt{3}\sqrt{K2}}(-1+X)-6X\right)-\right.\\&\quad\frac{1}{K2}3\,a1\,e^{2\sqrt{3}\sqrt{K2}(2-3X)}\left(8e^{6\sqrt{3}\sqrt{K2}X}-4e^{2\sqrt{3}\sqrt{K2}(-1+2X)}-4e^{2\sqrt{3}\sqrt{K2}(1+2X)}+\right.\\&\quad 4e^{2\sqrt{3}\sqrt{K2}(-2+3X)}+4e^{2\sqrt{3}\sqrt{K2}(2+3X)}-4e^{2\sqrt{3}\sqrt{K2}(-1+4X)}-4e^{2\sqrt{3}\sqrt{K2}(1+4X)}+\\&\quad e^{4\sqrt{3}\sqrt{K2}(1+X)}\left(-1-2\sqrt{3}\sqrt{K2}X+12K2X^{2}\right)+e^{4\sqrt{3}\sqrt{K2}(-1+2X)}\left(-1+2\sqrt{3}\sqrt{K2}X+12K2X^{2}\right)+\\&\quad\left.\left. e^{4\sqrt{3}\sqrt{K2}X}\left(1-2\sqrt{3}\sqrt{K2}X+12K2\left(-2+X^{2}\right)\right)+e^{8\sqrt{3}\sqrt{K2}X}\left(1+2\sqrt{3}\sqrt{K2}X+12K2\left(-2+X^{2}\right)\right)\right)\right)\end{aligned}$$

Back to analysis - First order 2nd region

In[143]:= **EquationList2LeadingOrder1 = FullSimplify[EquationList2LeadingOrder]**

Out[143]= {True, True, True, True, True, True,
$12\,K2\,w\,\mu\,(Pin\,(-1+X)+G20[X]) == (1+a1\,X)^2\,(3\,a1\,G20'[X]+(1+a1\,X)\,G20''[X])$,
$G20'[0] == 0$, $G20[1] == 0$}

In[144]:= **G20EqLead = GetEquationListOrder[EquationList2LeadingOrder, 0, a1][[7]]**

Out[144]= $$-K2\,w\,\mu\,(Pin\,(-1+X)+G20[X])+\frac{G20''[X]}{12} == 0$$

In[145]:= **EquationList2LeadingOrder1a1 = GetEquationListOrder[EquationList2LeadingOrder, 1, a1][[7]]**

Out[145]= $$\frac{G20'[X]}{4}+\frac{1}{4}X\,G20''[X] == 0$$

In[146]:= **FuncG0[X]**

Out[146]= $$\frac{Pin\left(6-\frac{\sqrt{3}\,e^{-2\sqrt{3}\sqrt{K2}\,(-2+X)}}{\sqrt{K2}}+\frac{\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}\,X}}{\sqrt{K2}}-6\,e^{4\sqrt{3}\sqrt{K2}}\,(-1+X)-6\,X\right)}{6\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)}$$

In[147]:= **EquationList2LeadingOrdersub =**
Simplify[EquationList2LeadingOrder //. {G20 → Function[{*X*}, G20Func[*X*]]}]

Out[147]= {True, True, True, True, True, True,
$\frac{1}{12}(1+a1\,X)^2\,(3\,a1\,G20Func'[X]+(1+a1\,X)\,G20Func''[X]) == K2\,w\,\mu\,(Pin\,(-1+X)+G20Func[X])$,
$G20Func'[0] == 0$, $G20Func[1] == 0$}

In[148]:= **EquationList2LeadingOrdersub**

Out[148]= {True, True, True, True, True, True,
$\frac{1}{12}(1+a1\,X)^2\,(3\,a1\,G20Func'[X]+(1+a1\,X)\,G20Func''[X]) == K2\,w\,\mu\,(Pin\,(-1+X)+G20Func[X])$,
$G20Func'[0] == 0$, $G20Func[1] == 0$}

In[149]:= **UY20Func[*X*, Y2]**

Out[149]= $$\frac{1}{12}\left(3\,a1\,G20'[X]+6\,a1^2\,X\,G20'[X]+3\,a1^3\,X^2\,G20'[X]-3\,a1\,Y2^2\,G20'[X]+G20''[X]+3\,a1\,X\,G20''[X]+3\,a1^2\,X^2\,G20''[X]+a1^3\,X^3\,G20''[X]-3\,Y2^2\,G20''[X]-3\,a1\,X\,Y2^2\,G20''[X]-2\,Y2^3\,G20''[X]\right)$$

In[150]:=
```
UY20FuncSub[X_, Y2_] = Simplify[UY20Func[X, Y2] //. {G20 → Function[{X}, FuncG201[X]]}]
```

Out[150]=

$$-\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^2\sqrt{K2}}e^{-2\sqrt{3}\sqrt{K2}(-1+X)}\,\text{Pin}\,(1+a1\,X+Y2)$$
$$\Big(3\,a1^3\,X\,\Big(2\left(\sqrt{3}-2\sqrt{K2}\,X\right)+2\,e^{4\sqrt{3}\sqrt{K2}}\left(\sqrt{3}-2\sqrt{K2}\,X\right)-2\,e^{4\sqrt{3}\sqrt{K2}\,X}\left(\sqrt{3}+2\sqrt{K2}\,X\right)-$$
$$2\,e^{4\sqrt{3}\sqrt{K2}(1+X)}\left(\sqrt{3}+2\sqrt{K2}\,X\right)+4\,e^{6\sqrt{3}\sqrt{K2}}\sqrt{K2}\,X\left(3-4\sqrt{3}\sqrt{K2}\,X+3\,K2\,X^2\right)+$$
$$4\,e^{2\sqrt{3}\sqrt{K2}(-1+2X)}\sqrt{K2}\,X\left(3+4\sqrt{3}\sqrt{K2}\,X+3\,K2\,X^2\right)-$$
$$e^{2\sqrt{3}\sqrt{K2}}\left(\sqrt{3}-14\sqrt{K2}\,X-12\,K2^{3/2}X\left(-2+X^2\right)+4\sqrt{3}\,K2\left(-3+4X^2\right)\right)+$$
$$e^{2\sqrt{3}\sqrt{K2}(1+2X)}\left(\sqrt{3}+14\sqrt{K2}\,X+12\,K2^{3/2}X\left(-2+X^2\right)+4\sqrt{3}\,K2\left(-3+4X^2\right)\right)\Big)-$$
$$8\sqrt{3}\,e^{-2\sqrt{3}\sqrt{K2}}\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)\left(e^{4\sqrt{3}\sqrt{K2}}-e^{4\sqrt{3}\sqrt{K2}\,X}\right)K2\left(-1+Y2+2\,Y2^2\right)-$$
$$a1\sqrt{K2}\,\Big(12\,e^{2\sqrt{3}\sqrt{K2}(-1+X)}(-1+Y2)+24\,e^{2\sqrt{3}\sqrt{K2}(1+X)}(-1+Y2)+$$
$$12\,e^{2\sqrt{3}\sqrt{K2}(3+X)}(-1+Y2)-12\left(-1+Y2+2\,Y2^2\right)-12\,e^{4\sqrt{3}\sqrt{K2}}\left(-1+Y2+2\,Y2^2\right)-$$
$$12\,e^{4\sqrt{3}\sqrt{K2}\,X}\left(-1+Y2+2\,Y2^2\right)-12\,e^{4\sqrt{3}\sqrt{K2}(1+X)}\left(-1+Y2+2\,Y2^2\right)+e^{6\sqrt{3}\sqrt{K2}}$$
$$\left(3-3\,Y2+18\,Y2^2+36\,K2\,X^2\left(-1+Y2+2\,Y2^2\right)-2\sqrt{3}\sqrt{K2}\,X\left(-7+11\,Y2+30\,Y2^2\right)\right)+e^{2\sqrt{3}\sqrt{K2}}$$
$$\left(36\,K2\left(-2+X^2\right)\left(-1+Y2+2\,Y2^2\right)+3\left(-1+Y2+10\,Y2^2\right)-2\sqrt{3}\sqrt{K2}\,X\left(-7+11\,Y2+30\,Y2^2\right)\right)+$$
$$e^{2\sqrt{3}\sqrt{K2}(-1+2X)}\Big(3-3\,Y2+18\,Y2^2+36\,K2\,X^2\left(-1+Y2+2\,Y2^2\right)+$$
$$2\sqrt{3}\sqrt{K2}\,X\left(-7+11\,Y2+30\,Y2^2\right)\Big)+e^{2\sqrt{3}\sqrt{K2}(1+2X)}\Big(36\,K2\left(-2+X^2\right)\left(-1+Y2+2\,Y2^2\right)+$$
$$3\left(-1+Y2+10\,Y2^2\right)+2\sqrt{3}\sqrt{K2}\,X\left(-7+11\,Y2+30\,Y2^2\right)\Big)\Big)+$$
$$a1^2\,\Big(12\,e^{2\sqrt{3}\sqrt{K2}(-1+X)}\sqrt{K2}\,X+24\,e^{2\sqrt{3}\sqrt{K2}(1+X)}\sqrt{K2}\,X+12\,e^{2\sqrt{3}\sqrt{K2}(3+X)}\sqrt{K2}\,X+$$
$$e^{2\sqrt{3}\sqrt{K2}(1+2X)}\Big(2\sqrt{3}\,K2\left(X^2(35-24\,Y2)+18(-1+Y2)\right)+\sqrt{K2}\,X(45-42\,Y2)-$$
$$36\,K2^{3/2}X\left(-2+X^2\right)(-2+Y2)-3\sqrt{3}(-1+Y2)\Big)-6\,e^{4\sqrt{3}\sqrt{K2}}$$
$$\left(-2\sqrt{K2}\,X(-2+Y2)+\sqrt{3}(-1+Y2)\right)+6\,e^{4\sqrt{3}\sqrt{K2}\,X}\left(2\sqrt{K2}\,X(-2+Y2)+\sqrt{3}(-1+Y2)\right)+$$
$$6\,e^{4\sqrt{3}\sqrt{K2}(1+X)}\left(2\sqrt{K2}\,X(-2+Y2)+\sqrt{3}(-1+Y2)\right)+12\sqrt{K2}\,X(-2+Y2)-6\sqrt{3}(-1+Y2)-$$
$$e^{6\sqrt{3}\sqrt{K2}}\sqrt{K2}\,X\left(-33+2\sqrt{3}\sqrt{K2}\,X(35-24\,Y2)+36\,K2\,X^2(-2+Y2)+36\,Y2\right)-$$
$$e^{2\sqrt{3}\sqrt{K2}(-1+2X)}\sqrt{K2}\,X\left(-33+36\,K2\,X^2(-2+Y2)+36\,Y2+2\sqrt{3}\sqrt{K2}\,X(-35+24\,Y2)\right)+$$
$$e^{2\sqrt{3}\sqrt{K2}}\Big(\sqrt{K2}\,X(45-42\,Y2)-36\,K2^{3/2}X\left(-2+X^2\right)(-2+Y2)+$$
$$3\sqrt{3}(-1+Y2)+2\sqrt{3}\,K2\left(-18(-1+Y2)+X^2(-35+24\,Y2)\right)\Big)\Big)\Big)$$

In[151]:=
```
UX20FuncSub[X_, Y2_] = Simplify[UX20Func[X, Y2] //. {G20 → Function[{X}, FuncG201[X]]}]
```

Out[151]=

$$\frac{1}{8\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^2\sqrt{K2}}$$
$$e^{-2\sqrt{3}\sqrt{K2}\,X}\,\text{Pin}\,\Big(4\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)\left(-1+e^{2\sqrt{3}\sqrt{K2}\,X}\right)\left(-e^{4\sqrt{3}\sqrt{K2}}+e^{2\sqrt{3}\sqrt{K2}\,X}\right)\sqrt{K2}+$$
$$a1\,\Big(-2\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}}-2\sqrt{3}\,e^{6\sqrt{3}\sqrt{K2}}+2\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}(1+2X)}+$$
$$2\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}(3+2X)}+e^{4\sqrt{3}\sqrt{K2}\,X}\left(-9\sqrt{K2}\,X-6\sqrt{3}\,K2\,X^2\right)+$$
$$e^{8\sqrt{3}\sqrt{K2}}\left(-9\sqrt{K2}\,X+6\sqrt{3}\,K2\,X^2\right)+e^{4\sqrt{3}\sqrt{K2}}\left(\sqrt{3}-9\sqrt{K2}\,X+6\sqrt{3}\,K2\left(-2+X^2\right)\right)-$$
$$e^{4\sqrt{3}\sqrt{K2}(1+X)}\left(\sqrt{3}+9\sqrt{K2}\,X+6\sqrt{3}\,K2\left(-2+X^2\right)\right)\Big)\Big)\,Y2\,(1+a1\,X+Y2)$$

In summary, the pressure and velocities are given by:

In[152]:= **FuncG201[*X*]**
UY20FuncSub[*X*, Y2]
UX20FuncSub[*X*, Y2]

Out[152]=

$$\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^2}$$

$$\text{Pin}\left(8\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)\left(6-\frac{\sqrt{3}\,e^{-2\sqrt{3}\sqrt{K2}(-2+X)}}{\sqrt{K2}}+\frac{\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}X}}{\sqrt{K2}}-6\,e^{4\sqrt{3}\sqrt{K2}}\left(-1+X\right)-6X\right)-\right.$$

$$\frac{1}{K2}3\,a1\,e^{2\sqrt{3}\sqrt{K2}(2-3X)}$$

$$\left(8\,e^{6\sqrt{3}\sqrt{K2}X}-4\,e^{2\sqrt{3}\sqrt{K2}(-1+2X)}-4\,e^{2\sqrt{3}\sqrt{K2}(1+2X)}+4\,e^{2\sqrt{3}\sqrt{K2}(-2+3X)}+4\,e^{2\sqrt{3}\sqrt{K2}(2+3X)}-\right.$$

$$4\,e^{2\sqrt{3}\sqrt{K2}(-1+4X)}-4\,e^{2\sqrt{3}\sqrt{K2}(1+4X)}+e^{4\sqrt{3}\sqrt{K2}(1+X)}\left(-1-2\sqrt{3}\sqrt{K2}X+12\,K2\,X^2\right)+$$

$$e^{4\sqrt{3}\sqrt{K2}(-1+2X)}\left(-1+2\sqrt{3}\sqrt{K2}X+12\,K2\,X^2\right)+e^{4\sqrt{3}\sqrt{K2}X}\left(1-2\sqrt{3}\sqrt{K2}X+12\,K2\left(-2+X^2\right)\right)+$$

$$\left.\left.e^{8\sqrt{3}\sqrt{K2}X}\left(1+2\sqrt{3}\sqrt{K2}X+12\,K2\left(-2+X^2\right)\right)\right)\right)$$

Out[153]=

$$
\begin{aligned}
&-\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^2\sqrt{K2}}\,e^{-2\sqrt{3}\sqrt{K2}(-1+X)}\,\text{Pin}\,(1+a1\,X+Y2)\\
&\Big(3\,a1^3\,X\,\Big(2\,(\sqrt{3}-2\sqrt{K2}\,X)+2\,e^{4\sqrt{3}\sqrt{K2}}\,(\sqrt{3}-2\sqrt{K2}\,X)-2\,e^{4\sqrt{3}\sqrt{K2}\,X}\,(\sqrt{3}+2\sqrt{K2}\,X)-\\
&\quad 2\,e^{4\sqrt{3}\sqrt{K2}(1+X)}\,(\sqrt{3}+2\sqrt{K2}\,X)+4\,e^{6\sqrt{3}\sqrt{K2}}\sqrt{K2}\,X\,(3-4\sqrt{3}\sqrt{K2}\,X+3\,K2\,X^2)+\\
&\quad 4\,e^{2\sqrt{3}\sqrt{K2}(-1+2X)}\sqrt{K2}\,X\,(3+4\sqrt{3}\sqrt{K2}\,X+3\,K2\,X^2)-\\
&\quad e^{2\sqrt{3}\sqrt{K2}}\,(\sqrt{3}-14\sqrt{K2}\,X-12\,K2^{3/2}\,X\,(-2+X^2)+4\sqrt{3}\,K2\,(-3+4\,X^2))+\\
&\quad e^{2\sqrt{3}\sqrt{K2}(1+2X)}\,(\sqrt{3}+14\sqrt{K2}\,X+12\,K2^{3/2}\,X\,(-2+X^2)+4\sqrt{3}\,K2\,(-3+4\,X^2))\Big)\Big)-\\
&\;8\sqrt{3}\,e^{-2\sqrt{3}\sqrt{K2}}\,(1+e^{4\sqrt{3}\sqrt{K2}})\,(e^{4\sqrt{3}\sqrt{K2}}-e^{4\sqrt{3}\sqrt{K2}\,X})\,K2\,(-1+Y2+2\,Y2^2)-\\
&\;a1\sqrt{K2}\,\Big(12\,e^{2\sqrt{3}\sqrt{K2}(-1+X)}\,(-1+Y2)+24\,e^{2\sqrt{3}\sqrt{K2}(1+X)}\,(-1+Y2)+\\
&\quad 12\,e^{2\sqrt{3}\sqrt{K2}(3+X)}\,(-1+Y2)-12\,(-1+Y2+2\,Y2^2)-12\,e^{4\sqrt{3}\sqrt{K2}}\,(-1+Y2+2\,Y2^2)-\\
&\quad 12\,e^{4\sqrt{3}\sqrt{K2}\,X}\,(-1+Y2+2\,Y2^2)-12\,e^{4\sqrt{3}\sqrt{K2}(1+X)}\,(-1+Y2+2\,Y2^2)+e^{6\sqrt{3}\sqrt{K2}}\\
&\quad\;(3-3\,Y2+18\,Y2^2+36\,K2\,X^2\,(-1+Y2+2\,Y2^2)-2\sqrt{3}\sqrt{K2}\,X\,(-7+11\,Y2+30\,Y2^2))+e^{2\sqrt{3}\sqrt{K2}}\\
&\quad\;(36\,K2\,(-2+X^2)\,(-1+Y2+2\,Y2^2)+3\,(-1+Y2+10\,Y2^2)-2\sqrt{3}\sqrt{K2}\,X\,(-7+11\,Y2+30\,Y2^2))+\\
&\quad e^{2\sqrt{3}\sqrt{K2}(-1+2X)}\,\Big(3-3\,Y2+18\,Y2^2+36\,K2\,X^2\,(-1+Y2+2\,Y2^2)+\\
&\qquad 2\sqrt{3}\sqrt{K2}\,X\,(-7+11\,Y2+30\,Y2^2)\Big)+e^{2\sqrt{3}\sqrt{K2}(1+2X)}\,\Big(36\,K2\,(-2+X^2)\,(-1+Y2+2\,Y2^2)+\\
&\qquad 3\,(-1+Y2+10\,Y2^2)+2\sqrt{3}\sqrt{K2}\,X\,(-7+11\,Y2+30\,Y2^2)\Big)\Big)+\\
&\;a1^2\,\Big(12\,e^{2\sqrt{3}\sqrt{K2}(-1+X)}\sqrt{K2}\,X+24\,e^{2\sqrt{3}\sqrt{K2}(1+X)}\sqrt{K2}\,X+12\,e^{2\sqrt{3}\sqrt{K2}(3+X)}\sqrt{K2}\,X+\\
&\quad e^{2\sqrt{3}\sqrt{K2}(1+2X)}\,\Big(2\sqrt{3}\,K2\,(X^2\,(35-24\,Y2)+18\,(-1+Y2))+\sqrt{K2}\,X\,(45-42\,Y2)-\\
&\qquad 36\,K2^{3/2}\,X\,(-2+X^2)\,(-2+Y2)-3\sqrt{3}\,(-1+Y2)\Big)-6\,e^{4\sqrt{3}\sqrt{K2}}\\
&\quad\;(-2\sqrt{K2}\,X\,(-2+Y2)+\sqrt{3}\,(-1+Y2))+6\,e^{4\sqrt{3}\sqrt{K2}\,X}\,(2\sqrt{K2}\,X\,(-2+Y2)+\sqrt{3}\,(-1+Y2))+\\
&\quad 6\,e^{4\sqrt{3}\sqrt{K2}(1+X)}\,(2\sqrt{K2}\,X\,(-2+Y2)+\sqrt{3}\,(-1+Y2))+12\sqrt{K2}\,X\,(-2+Y2)-6\sqrt{3}\,(-1+Y2)-\\
&\quad e^{6\sqrt{3}\sqrt{K2}}\sqrt{K2}\,X\,(-33+2\sqrt{3}\sqrt{K2}\,X\,(35-24\,Y2)+36\,K2\,X^2\,(-2+Y2)+36\,Y2)-\\
&\quad e^{2\sqrt{3}\sqrt{K2}(-1+2X)}\sqrt{K2}\,X\,(-33+36\,K2\,X^2\,(-2+Y2)+36\,Y2+2\sqrt{3}\sqrt{K2}\,X\,(-35+24\,Y2))+\\
&\quad e^{2\sqrt{3}\sqrt{K2}}\,\Big(\sqrt{K2}\,X\,(45-42\,Y2)-36\,K2^{3/2}\,X\,(-2+X^2)\,(-2+Y2)+\\
&\qquad 3\sqrt{3}\,(-1+Y2)+2\sqrt{3}\,K2\,(-18\,(-1+Y2)+X^2\,(-35+24\,Y2))\Big)\Big)\Big)
\end{aligned}
$$

Out[154]=

$$
\begin{aligned}
&\frac{1}{8\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^2\sqrt{K2}}\\
&e^{-2\sqrt{3}\sqrt{K2}\,X}\,\text{Pin}\,\Big(4\,(1+e^{4\sqrt{3}\sqrt{K2}})\,(-1+e^{2\sqrt{3}\sqrt{K2}\,X})\,(-e^{4\sqrt{3}\sqrt{K2}}+e^{2\sqrt{3}\sqrt{K2}\,X})\sqrt{K2}+\\
&\quad a1\,\Big(-2\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}}-2\sqrt{3}\,e^{6\sqrt{3}\sqrt{K2}}+2\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}(1+2X)}+\\
&\qquad 2\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}(3+2X)}+e^{4\sqrt{3}\sqrt{K2}\,X}\,(-9\sqrt{K2}\,X-6\sqrt{3}\,K2\,X^2)+\\
&\qquad e^{8\sqrt{3}\sqrt{K2}}\,(-9\sqrt{K2}\,X+6\sqrt{3}\,K2\,X^2)+e^{4\sqrt{3}\sqrt{K2}}\,(\sqrt{3}-9\sqrt{K2}\,X+6\sqrt{3}\,K2\,(-2+X^2))-\\
&\qquad e^{4\sqrt{3}\sqrt{K2}(1+X)}\,(\sqrt{3}+9\sqrt{K2}\,X+6\sqrt{3}\,K2\,(-2+X^2))\Big)\Big)\,Y2\,(1+a1\,X+Y2)
\end{aligned}
$$

4. Solving the first order region 1

In[155]:= `TextbookEq[EquationList1]`

$$\frac{\partial UX10}{\partial X} + K1\,\frac{\partial UX11}{\partial X} + K1^2\,\frac{\partial UX12}{\partial X} + \frac{\partial UY10}{\partial Y1} + K1\,\frac{\partial UY11}{\partial Y1} + K1^2\,\frac{\partial UY12}{\partial Y1} = 0$$
$$\frac{\partial^2 UX10}{\partial Y1^2} + K1\,\frac{\partial^2 UX11}{\partial Y1^2} + \epsilon1^2\left(\frac{\partial^2 UX10}{\partial X^2} + K1\left(\frac{\partial^2 UX11}{\partial X^2} + K1\,\frac{\partial^2 UX12}{\partial X^2}\right)\right) + K1^2\,\frac{\partial^2 UX12}{\partial Y1^2} = \frac{\partial P10}{\partial X} + K1\left(\frac{\partial P11}{\partial X} + K1\,\frac{\partial P12}{\partial X}\right)$$
$$\frac{\partial P10}{\partial Y1} + K1\left(\frac{\partial P11}{\partial Y1} + K1\,\frac{\partial P12}{\partial Y1}\right) = \epsilon1^2\left(\frac{\partial^2 UY10}{\partial Y1^2} + K1\,\frac{\partial^2 UY11}{\partial Y1^2} + \epsilon1^2\left(\frac{\partial^2 UY10}{\partial X^2} + K1\left(\frac{\partial^2 UY11}{\partial X^2} + K1\,\frac{\partial^2 UY12}{\partial X^2}\right)\right) + K1^2\,\frac{\partial^2 UY12}{\partial Y1^2}\right)$$
$$UX10 + K1\,(UX11 + K1\,UX12) = 0$$
$$UX10 + K1\,(UX11 + K1\,UX12) = 0$$
$$UY10 + K1\,(UY11 + K1\,UY12) = 0$$
$$K1\,(P10 + K1\,(P11 + K1\,P12)) + UY10 + K1\,UY11 + K1^2\,UY12 = K1\,(P20 + K1\,(P21 + K1\,P22))$$
$$Pin = P10 + K1\,(P11 + K1\,P12)$$
$$P10 + K1\,(P11 + K1\,P12) = 0$$

In[156]:= `TextbookEq[EquationList1FirstOrder = GetEquationListOrder[EquationList1, 1, K1]]`

$$\frac{\partial UX11}{\partial X} + \frac{\partial UY11}{\partial Y1} = 0$$
$$-\frac{\partial P11}{\partial X} + \epsilon1^2\,\frac{\partial^2 UX11}{\partial X^2} + \frac{\partial^2 UX11}{\partial Y1^2} = 0$$
$$\frac{\partial P11}{\partial Y1} - \epsilon1^4\,\frac{\partial^2 UY11}{\partial X^2} - \epsilon1^2\,\frac{\partial^2 UY11}{\partial Y1^2} = 0$$
$$UX11 = 0$$
$$UX11 = 0$$
$$UY11 = 0$$
$$P10 - P20 + UY11 = 0$$
$$-P11 = 0$$
$$P11 = 0$$

In[157]:= `TextbookEq[EquationList1FirstOrder = GetEquationListOrder[EquationList1FirstOrder, 0, ϵ1]]`

$$\frac{\partial UX11}{\partial X} + \frac{\partial UY11}{\partial Y1} = 0$$
$$-\frac{\partial P11}{\partial X} + \frac{\partial^2 UX11}{\partial Y1^2} = 0$$
$$\frac{\partial P11}{\partial Y1} = 0$$
$$UX11 = 0$$
$$UX11 = 0$$
$$UY11 = 0$$
$$P10 - P20 + UY11 = 0$$
$$-P11 = 0$$
$$P11 = 0$$

In[158]:= `TextbookEq[EquationList1FirstOrder]`

$$\frac{\partial UX11}{\partial X} + \frac{\partial UY11}{\partial Y1} = 0$$
$$-\frac{\partial P11}{\partial X} + \frac{\partial^2 UX11}{\partial Y1^2} = 0$$
$$\frac{\partial P11}{\partial Y1} = 0$$
$$UX11 = 0$$
$$UX11 = 0$$
$$UY11 = 0$$
$$P10 - P20 + UY11 = 0$$
$$-P11 = 0$$
$$P11 = 0$$

In[159]:= `TextbookEq[EquationList1FirstOrder[[3]]]`

$$\frac{\partial \mathrm{P11}}{\partial \mathrm{Y1}} = 0$$

we thus see that P11 is not a function of Y1, and thus

In[160]:= **TextbookEq[EquationList1FirstOrder = Simplify[EquationList1FirstOrder //. {P11 → (G11[#1] &)}]]**

$$\frac{\partial \mathrm{UX11}}{\partial X} + \frac{\partial \mathrm{UY11}}{\partial \mathrm{Y1}} = 0$$
$$\frac{\partial \mathrm{G11}}{\partial X} = \frac{\partial^2 \mathrm{UX11}}{\partial \mathrm{Y1}^2}$$
True
$$\mathrm{UX11} = 0$$
$$\mathrm{UX11} = 0$$
$$\mathrm{UY11} = 0$$
$$\mathrm{P10} + \mathrm{UY11} = \mathrm{P20}$$
$$\mathrm{G11} = 0$$
$$\mathrm{G11} = 0$$

Substituting P20 into the permeability boundary condition

In[161]:= **EquationList1FirstOrder = Simplify[EquationList1FirstOrder //. {P20 → Function[{X}, FuncG01[X]]}]**

Out[161]= $\Big\{\mathrm{UY11}^{(0,1)}[X, \mathrm{Y1}] + \mathrm{UX11}^{(1,0)}[X, \mathrm{Y1}] == 0,\ \mathrm{G11}'[X] == \mathrm{UX11}^{(0,2)}[X, \mathrm{Y1}],\ \mathrm{True},\ \mathrm{UX11}[X, 1] == 0,$
$\mathrm{UX11}[X, 0] == 0,\ \mathrm{UY11}[X, 1] == 0,\ \mathrm{P10}[X, 0] + \mathrm{UY11}[X, 0] == \frac{1}{48\left(1 + e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\right)^2}$
$\mathrm{Pin}\Big(8\left(1 + e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\right)\Big(6 - \frac{\sqrt{3}\, e^{-2\sqrt{3}\sqrt{\mathrm{K2}}(-2+X)}}{\sqrt{\mathrm{K2}}} + \frac{\sqrt{3}\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}X}}{\sqrt{\mathrm{K2}}} - 6\, e^{4\sqrt{3}\sqrt{\mathrm{K2}}}(-1 + X) - 6X\Big) -$
$\frac{1}{\mathrm{K2}} 3\, \mathrm{a1}\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}(2-3X)}\Big(8\, e^{6\sqrt{3}\sqrt{\mathrm{K2}}X} - 4\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}(-1+2X)} - 4\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}(1+2X)} +$
$4\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}(-2+3X)} + 4\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}(2+3X)} - 4\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}(-1+4X)} - 4\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}(1+4X)} +$
$e^{4\sqrt{3}\sqrt{\mathrm{K2}}(1+X)}\left(-1 - 2\sqrt{3}\sqrt{\mathrm{K2}}\,X + 12\,\mathrm{K2}\,X^2\right) + e^{4\sqrt{3}\sqrt{\mathrm{K2}}(-1+2X)}\left(-1 + 2\sqrt{3}\sqrt{\mathrm{K2}}\,X + 12\,\mathrm{K2}\,X^2\right) +$
$e^{4\sqrt{3}\sqrt{\mathrm{K2}}X}\left(1 - 2\sqrt{3}\sqrt{\mathrm{K2}}\,X + 12\,\mathrm{K2}\left(-2 + X^2\right)\right) +$
$e^{8\sqrt{3}\sqrt{\mathrm{K2}}X}\left(1 + 2\sqrt{3}\sqrt{\mathrm{K2}}\,X + 12\,\mathrm{K2}\left(-2 + X^2\right)\right)\Big)\Big),\ \mathrm{G11}[0] == 0,\ \mathrm{G11}[1] == 0\Big\}$

In[162]:= **EquationList1FirstOrder = Simplify[EquationList1FirstOrder //. {P10 → Function[{X}, G10Func[X]]}]**

Out[162]= $\Big\{\mathrm{UY11}^{(0,1)}[X, \mathrm{Y1}] + \mathrm{UX11}^{(1,0)}[X, \mathrm{Y1}] == 0,\ \mathrm{G11}'[X] == \mathrm{UX11}^{(0,2)}[X, \mathrm{Y1}],$
$\mathrm{True},\ \mathrm{UX11}[X, 1] == 0,\ \mathrm{UX11}[X, 0] == 0,\ \mathrm{UY11}[X, 1] == 0,$
$8\sqrt{3}\, e^{-2\sqrt{3}\sqrt{\mathrm{K2}}X}\left(-e^{4\sqrt{3}\sqrt{\mathrm{K2}}} + e^{4\sqrt{3}\sqrt{\mathrm{K2}}X}\right)\mathrm{Pin} == \frac{1}{\left(1 + e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\right)\sqrt{\mathrm{K2}}}$
$3\Big(\mathrm{a1}\, e^{-2\sqrt{3}\sqrt{\mathrm{K2}}X}\,\mathrm{Pin}\Big(-4\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}} - 4\, e^{6\sqrt{3}\sqrt{\mathrm{K2}}} + 4\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}X} + 8\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}(2+X)} + 4\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}(4+X)} -$
$4\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}(1+2X)} - 4\, e^{2\sqrt{3}\sqrt{\mathrm{K2}}(3+2X)} + e^{8\sqrt{3}\sqrt{\mathrm{K2}}}\left(-1 - 2\sqrt{3}\sqrt{\mathrm{K2}}\,X + 12\,\mathrm{K2}\,X^2\right) +$
$e^{4\sqrt{3}\sqrt{\mathrm{K2}}X}\left(-1 + 2\sqrt{3}\sqrt{\mathrm{K2}}\,X + 12\,\mathrm{K2}\,X^2\right) + e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\left(1 - 2\sqrt{3}\sqrt{\mathrm{K2}}\,X + 12\,\mathrm{K2}\left(-2 + X^2\right)\right) +$
$e^{4\sqrt{3}\sqrt{\mathrm{K2}}(1+X)}\left(1 + 2\sqrt{3}\sqrt{\mathrm{K2}}\,X + 12\,\mathrm{K2}\left(-2 + X^2\right)\right)\Big) +$
$16\left(1 + e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\right)^2 \mathrm{K2}\,\mathrm{UY11}[X, 0]\Big),\ \mathrm{G11}[0] == 0,\ \mathrm{G11}[1] == 0\Big\}$

In[163]:= **TextbookEq[EquationList1FirstOrder[[2]]]**

$$\frac{\partial G11}{\partial X} = \frac{\partial^2 UX11}{\partial Y1^2}$$

The solution along with the no-slip boundary condition is

In[164]:= **TextbookEq[UX11Func[X_, Y1_] =
Flatten[DSolve[{EquationList1FirstOrder[[2]], EquationList1FirstOrder[[4]], EquationList1FirstOrder[[5]]}, UX11[X, Y1], {Y1}]][[1, 2]]]**

$$\frac{1}{2}\left(-Y1\,\frac{\partial G11}{\partial X} + Y1^2\,\frac{\partial G11}{\partial X}\right)$$

Substituting into the leading order equations

In[165]:= **EquationList1FirstOrder = EquationList1FirstOrder /. {UX11 → Function[{X, Y1}, UX11Func[X, Y1]]}**

Out[165]=
$$\Big\{\frac{1}{2}\left(-Y1\,G11''[X] + Y1^2\,G11''[X]\right) + UY11^{(0,1)}[X, Y1] == 0,\ \text{True},\ \text{True},\ \text{True},\ \text{True},$$
$$UY11[X, 1] == 0,\ 8\sqrt{3}\,e^{-2\sqrt{3}\sqrt{K2}\,X}\left(-e^{4\sqrt{3}\sqrt{K2}} + e^{4\sqrt{3}\sqrt{K2}\,X}\right) Pin == \frac{1}{\left(1 + e^{4\sqrt{3}\sqrt{K2}}\right)\sqrt{K2}}$$
$$3\Big(a1\,e^{-2\sqrt{3}\sqrt{K2}\,X}\,Pin\Big(-4\,e^{2\sqrt{3}\sqrt{K2}} - 4\,e^{6\sqrt{3}\sqrt{K2}} + 4\,e^{2\sqrt{3}\sqrt{K2}\,X} + 8\,e^{2\sqrt{3}\sqrt{K2}\,(2+X)} + 4\,e^{2\sqrt{3}\sqrt{K2}\,(4+X)} -$$
$$4\,e^{2\sqrt{3}\sqrt{K2}\,(1+2X)} - 4\,e^{2\sqrt{3}\sqrt{K2}\,(3+2X)} + e^{8\sqrt{3}\sqrt{K2}}\left(-1 - 2\sqrt{3}\sqrt{K2}\,X + 12\,K2\,X^2\right) +$$
$$e^{4\sqrt{3}\sqrt{K2}\,X}\left(-1 + 2\sqrt{3}\sqrt{K2}\,X + 12\,K2\,X^2\right) + e^{4\sqrt{3}\sqrt{K2}}\left(1 - 2\sqrt{3}\sqrt{K2}\,X + 12\,K2\left(-2 + X^2\right)\right) +$$
$$e^{4\sqrt{3}\sqrt{K2}\,(1+X)}\left(1 + 2\sqrt{3}\sqrt{K2}\,X + 12\,K2\left(-2 + X^2\right)\right)\Big) +$$
$$16\left(1 + e^{4\sqrt{3}\sqrt{K2}}\right)^2 K2\,UY11[X, 0]\Big),\ G11[0] == 0,\ G11[1] == 0\Big\}$$

We now solve the mass conservation equation, given by

In[166]:= **TextbookEq[EquationList1FirstOrder[[1]]]**

$$\frac{1}{2}\left(-Y1\,\frac{\partial^2 G11}{\partial X^2} + Y1^2\,\frac{\partial^2 G11}{\partial X^2}\right) + \frac{\partial UY11}{\partial Y1} = 0$$

In[167]:= **TextbookEq[UY11Func[X_, Y1_] =
Flatten[DSolve[{EquationList1FirstOrder[[1]], EquationList1FirstOrder[[6]]}, UY11[X, Y1], {Y1}]][[1, 2]]]**

$$\frac{1}{12}\left(-\frac{\partial^2 G11}{\partial X^2} + 3\,Y1^2\,\frac{\partial^2 G11}{\partial X^2} - 2\,Y1^3\,\frac{\partial^2 G11}{\partial X^2}\right)$$

In[168]:= **EquationList1FirstOrder = Simplify[EquationList1FirstOrder /. {UY11 → Function[{X, Y1}, UY11Func[X, Y1]]}]**

Out[168]=

$$\Big\{\text{True, True, True, True, True, True, } 8\sqrt{3}\,e^{-2\sqrt{3}\sqrt{K2}\,X}\left(-e^{4\sqrt{3}\sqrt{K2}}+e^{4\sqrt{3}\sqrt{K2}\,X}\right)\text{Pin}+4\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)\sqrt{K2}\,\text{G11}''[X] == \frac{1}{\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)\sqrt{K2}} 3\,\text{a1}\,e^{-2\sqrt{3}\sqrt{K2}\,X}\,\text{Pin}\Big(-4\,e^{2\sqrt{3}\sqrt{K2}}-4\,e^{6\sqrt{3}\sqrt{K2}}+4\,e^{2\sqrt{3}\sqrt{K2}\,X}+8\,e^{2\sqrt{3}\sqrt{K2}\,(2+X)}+4\,e^{2\sqrt{3}\sqrt{K2}\,(4+X)}-4\,e^{2\sqrt{3}\sqrt{K2}\,(1+2X)}-4\,e^{2\sqrt{3}\sqrt{K2}\,(3+2X)}+e^{8\sqrt{3}\sqrt{K2}}\left(-1-2\sqrt{3}\sqrt{K2}\,X+12\,K2\,X^2\right)+e^{4\sqrt{3}\sqrt{K2}\,X}\left(-1+2\sqrt{3}\sqrt{K2}\,X+12\,K2\,X^2\right)+e^{4\sqrt{3}\sqrt{K2}}\left(1-2\sqrt{3}\sqrt{K2}\,X+12\,K2\left(-2+X^2\right)\right)+e^{4\sqrt{3}\sqrt{K2}\,(1+X)}\left(1+2\sqrt{3}\sqrt{K2}\,X+12\,K2\left(-2+X^2\right)\right)\Big),\ \text{G11}[0]==0,\ \text{G11}[1]==0\Big\}$$

In[169]:=

```
(*EquationList1FirstOrder=
 Simplify[EquationList1FirstOrder//.{P10→Function[{X},G10Func[X]]}]*)
```

Now solving the ODE for G11 with the existing BC

In[170]:= **TextbookEq[EquationList1FirstOrder[[7]]]**
TextbookEq[EquationList1FirstOrder[[8]]]
TextbookEq[EquationList1FirstOrder[[9]]]

$$8\sqrt{3}\,e^{-2\sqrt{3}\sqrt{K2}\,X}\left(-e^{4\sqrt{3}\sqrt{K2}}+e^{4\sqrt{3}\sqrt{K2}\,X}\right)\text{Pin}+4\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)\sqrt{K2}\,\frac{\partial^2 \text{G11}}{\partial X^2}= \frac{1}{\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)\sqrt{K2}}3\,\text{a1}\,e^{-2\sqrt{3}\sqrt{K2}\,X}\,\text{Pin}\left(-4\,e^{2\sqrt{3}\sqrt{K2}}-4\,e^{6\sqrt{3}\sqrt{K2}}+4\,e^{2\sqrt{3}\sqrt{K2}\,X}+8\,e^{2\sqrt{3}\sqrt{K2}\,(2+X)}+4\,e^{2\sqrt{3}\sqrt{K2}\,(4+X)}-4\,e^{2\sqrt{3}\sqrt{K2}\,(1+2X)}-4\,e^{2\sqrt{3}\sqrt{K2}\,(3+2X)}+e^{8\sqrt{3}\sqrt{K2}}\left(-1-2\sqrt{3}\sqrt{K2}\,X+12\,K2\,X^2\right)+e^{4\sqrt{3}\sqrt{K2}\,X}\left(-1+2\sqrt{3}\sqrt{K2}\,X+12\,K2\,X^2\right)+e^{4\sqrt{3}\sqrt{K2}}\left(1-2\sqrt{3}\sqrt{K2}\,X+12\,K2\left(-2+X^2\right)\right)+e^{4\sqrt{3}\sqrt{K2}\,(1+X)}\left(1+2\sqrt{3}\sqrt{K2}\,X+12\,K2\left(-2+X^2\right)\right)\right)$$

$$\text{G11}=0$$

$$\text{G11}=0$$

In[173]:= **G11Func[X_] = Simplify[DSolve[{EquationList1FirstOrder[[7]], EquationList1FirstOrder[[8]], EquationList1FirstOrder[[9]]}, G11[*X*], {*X*}], AssumptionOnScaling][[1]][[1]][[2]]**

Out[173]=

$$\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^2 K2^2} e^{-2\sqrt{3}\sqrt{K2}X}\,\text{Pin}\Big(8\sqrt{3}\sqrt{K2}\Big(e^{4\sqrt{3}\sqrt{K2}}+e^{8\sqrt{3}\sqrt{K2}}-e^{4\sqrt{3}\sqrt{K2}X}-e^{4\sqrt{3}\sqrt{K2}(1+X)}-e^{2\sqrt{3}\sqrt{K2}X}(-1+X)+e^{2\sqrt{3}\sqrt{K2}(4+X)}(-1+X)\Big)+3\,a1\Big(-4e^{2\sqrt{3}\sqrt{K2}}-4e^{6\sqrt{3}\sqrt{K2}}-4e^{2\sqrt{3}\sqrt{K2}(1+2X)}-4e^{2\sqrt{3}\sqrt{K2}(3+2X)}+e^{2\sqrt{3}\sqrt{K2}(1+X)}(8-16X)+e^{2\sqrt{3}\sqrt{K2}(3+X)}(8-16X)+2e^{2\sqrt{3}\sqrt{K2}(2+X)}\left(-5+24K2(-1+X)^2+9X\right)+3e^{8\sqrt{3}\sqrt{K2}}\left(1+2\sqrt{3}\sqrt{K2}X+4K2X^2\right)+e^{4\sqrt{3}\sqrt{K2}X}\left(3-6\sqrt{3}\sqrt{K2}X+12K2X^2\right)+e^{2\sqrt{3}\sqrt{K2}X}\left(-3+(7-24K2)X+24K2X^2\right)+e^{2\sqrt{3}\sqrt{K2}(4+X)}\left(-3+(7-24K2)X+24K2X^2\right)+e^{4\sqrt{3}\sqrt{K2}(1+X)}\left(5-6\sqrt{3}\sqrt{K2}X+12K2(-2+X^2)\right)+e^{4\sqrt{3}\sqrt{K2}}\left(5+6\sqrt{3}\sqrt{K2}X+12K2(-2+X^2)\right)\Big)\Big)$$

In[174]:= **TextbookEq[UY11Func[X, Y1]]**

$$\frac{1}{12}\left(-\frac{\partial^2 G11}{\partial X^2}+3\,Y1^2\,\frac{\partial^2 G11}{\partial X^2}-2\,Y1^3\,\frac{\partial^2 G11}{\partial X^2}\right)$$

In[175]:= **UY11Func[X_, Y1_] = Simplify[UY11Func[*X*, Y1] //. {G11 → Function[{*X*}, G11Func[*X*]]}]**

Out[175]=

$$\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^2 K2} e^{-2\sqrt{3}\sqrt{K2}X}\,\text{Pin}\Big(-8\sqrt{3}\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)\left(e^{4\sqrt{3}\sqrt{K2}}-e^{4\sqrt{3}\sqrt{K2}X}\right)\sqrt{K2}-3\,a1\Big(-4e^{2\sqrt{3}\sqrt{K2}}-4e^{6\sqrt{3}\sqrt{K2}}+4e^{2\sqrt{3}\sqrt{K2}X}+8e^{2\sqrt{3}\sqrt{K2}(2+X)}+4e^{2\sqrt{3}\sqrt{K2}(4+X)}-4e^{2\sqrt{3}\sqrt{K2}(1+2X)}-4e^{2\sqrt{3}\sqrt{K2}(3+2X)}+e^{8\sqrt{3}\sqrt{K2}}\left(-1-2\sqrt{3}\sqrt{K2}X+12K2X^2\right)+e^{4\sqrt{3}\sqrt{K2}X}\left(-1+2\sqrt{3}\sqrt{K2}X+12K2X^2\right)+e^{4\sqrt{3}\sqrt{K2}}\left(1-2\sqrt{3}\sqrt{K2}X+12K2(-2+X^2)\right)+e^{4\sqrt{3}\sqrt{K2}(1+X)}\left(1+2\sqrt{3}\sqrt{K2}X+12K2(-2+X^2)\right)\Big)\Big)(-1+Y1)^2(1+2Y1)$$

In[176]:= **TextbookEq[UX11Func[*X*, Y1]]**

$$\frac{1}{2}\left(-Y1\,\frac{\partial G11}{\partial X}+Y1^2\,\frac{\partial G11}{\partial X}\right)$$

In[177]:= **UX11Func[X_, Y1_] = Simplify[UX11Func[*X*, Y1] //. {G11 → Function[{*X*}, G11Func[*X*]]}]**

Out[177]=

$$\frac{1}{96\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)^2 K2^2} e^{-2\sqrt{3}\sqrt{K2}\,X}\,\mathrm{Pin}\,\Big(-8\left(1+e^{4\sqrt{3}\sqrt{K2}}\right)\left(\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}\,X}-\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}\,(2+X)}+6\,e^{4\sqrt{3}\sqrt{K2}}\sqrt{K2}+6\,e^{4\sqrt{3}\sqrt{K2}\,X}\sqrt{K2}\right)\sqrt{K2}+3\,a1\Big(-16\,e^{2\sqrt{3}\sqrt{K2}\,(1+X)}-16\,e^{2\sqrt{3}\sqrt{K2}\,(3+X)}+8\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}}\sqrt{K2}+8\sqrt{3}\,e^{6\sqrt{3}\sqrt{K2}}\sqrt{K2}-8\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}\,(1+2X)}\sqrt{K2}-8\sqrt{3}\,e^{2\sqrt{3}\sqrt{K2}\,(3+2X)}\sqrt{K2}+6\,e^{2\sqrt{3}\sqrt{K2}\,(2+X)}\left(3+16\,K2\,(-1+X)\right)+12\,e^{4\sqrt{3}\sqrt{K2}\,X}\,K2\,X\left(-1+2\sqrt{3}\sqrt{K2}\,X\right)-12\,e^{8\sqrt{3}\sqrt{K2}}\,K2\,X\left(1+2\sqrt{3}\sqrt{K2}\,X\right)+e^{2\sqrt{3}\sqrt{K2}\,X}\left(7+24\,K2\,(-1+2X)\right)+e^{2\sqrt{3}\sqrt{K2}\,(4+X)}\left(7+24\,K2\,(-1+2X)\right)+4\,e^{4\sqrt{3}\sqrt{K2}\,(1+X)}\sqrt{K2}\left(\sqrt{3}-3\sqrt{K2}\,X+6\sqrt{3}\,K2\left(-2+X^2\right)\right)-4\,e^{4\sqrt{3}\sqrt{K2}}\sqrt{K2}\left(\sqrt{3}+3\sqrt{K2}\,X+6\sqrt{3}\,K2\left(-2+X^2\right)\right)\Big)\Big)\,(-1+Y1)\,Y1$$

Summary:

In[178]:=

```
TextbookEq[G11Func[X]]
TextbookEq[UX11Func[X, Y1]]
TextbookEq[UY11Func[X, Y1]]
```

$$
\begin{aligned}
&\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\right)^2\mathrm{K2}^2}e^{-2\sqrt{3}\sqrt{\mathrm{K2}}X}\mathrm{Pin}\Big(8\sqrt{3}\sqrt{\mathrm{K2}}\\
&\Big(e^{4\sqrt{3}\sqrt{\mathrm{K2}}}+e^{8\sqrt{3}\sqrt{\mathrm{K2}}}-e^{4\sqrt{3}\sqrt{\mathrm{K2}}X}-e^{4\sqrt{3}\sqrt{\mathrm{K2}}(1+X)}-e^{2\sqrt{3}\sqrt{\mathrm{K2}}X}(-1+X)+e^{2\sqrt{3}\sqrt{\mathrm{K2}}(4+X)}(-1+X)\Big)+\\
&3\,\mathrm{a1}\Big(-4e^{2\sqrt{3}\sqrt{\mathrm{K2}}}-4e^{6\sqrt{3}\sqrt{\mathrm{K2}}}-4e^{2\sqrt{3}\sqrt{\mathrm{K2}}(1+2X)}-4e^{2\sqrt{3}\sqrt{\mathrm{K2}}(3+2X)}+e^{2\sqrt{3}\sqrt{\mathrm{K2}}(1+X)}(8-16X)+\\
&e^{2\sqrt{3}\sqrt{\mathrm{K2}}(3+X)}(8-16X)+2e^{2\sqrt{3}\sqrt{\mathrm{K2}}(2+X)}\left(-5+24\,\mathrm{K2}(-1+X)^2+9X\right)+\\
&3e^{8\sqrt{3}\sqrt{\mathrm{K2}}}\left(1+2\sqrt{3}\sqrt{\mathrm{K2}}X+4\,\mathrm{K2}X^2\right)+e^{4\sqrt{3}\sqrt{\mathrm{K2}}X}\left(3-6\sqrt{3}\sqrt{\mathrm{K2}}X+12\,\mathrm{K2}X^2\right)+\\
&e^{2\sqrt{3}\sqrt{\mathrm{K2}}X}\left(-3+(7-24\,\mathrm{K2})X+24\,\mathrm{K2}X^2\right)+e^{2\sqrt{3}\sqrt{\mathrm{K2}}(4+X)}\left(-3+(7-24\,\mathrm{K2})X+24\,\mathrm{K2}X^2\right)+\\
&e^{4\sqrt{3}\sqrt{\mathrm{K2}}(1+X)}\left(5-6\sqrt{3}\sqrt{\mathrm{K2}}X+12\,\mathrm{K2}\left(-2+X^2\right)\right)+e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\left(5+6\sqrt{3}\sqrt{\mathrm{K2}}X+12\,\mathrm{K2}\left(-2+X^2\right)\right)\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{96\left(1+e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\right)^2\mathrm{K2}^2}e^{-2\sqrt{3}\sqrt{\mathrm{K2}}X}\mathrm{Pin}\\
&\Big(-8\left(1+e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\right)\left(\sqrt{3}e^{2\sqrt{3}\sqrt{\mathrm{K2}}X}-\sqrt{3}e^{2\sqrt{3}\sqrt{\mathrm{K2}}(2+X)}+6e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\sqrt{\mathrm{K2}}+6e^{4\sqrt{3}\sqrt{\mathrm{K2}}X}\sqrt{\mathrm{K2}}\right)\sqrt{\mathrm{K2}}+\\
&3\,\mathrm{a1}\Big(-16e^{2\sqrt{3}\sqrt{\mathrm{K2}}(1+X)}-16e^{2\sqrt{3}\sqrt{\mathrm{K2}}(3+X)}+8\sqrt{3}e^{2\sqrt{3}\sqrt{\mathrm{K2}}}\sqrt{\mathrm{K2}}+\\
&8\sqrt{3}e^{6\sqrt{3}\sqrt{\mathrm{K2}}}\sqrt{\mathrm{K2}}-8\sqrt{3}e^{2\sqrt{3}\sqrt{\mathrm{K2}}(1+2X)}\sqrt{\mathrm{K2}}-8\sqrt{3}e^{2\sqrt{3}\sqrt{\mathrm{K2}}(3+2X)}\sqrt{\mathrm{K2}}+\\
&6e^{2\sqrt{3}\sqrt{\mathrm{K2}}(2+X)}(3+16\,\mathrm{K2}(-1+X))+12e^{4\sqrt{3}\sqrt{\mathrm{K2}}X}\mathrm{K2}X\left(-1+2\sqrt{3}\sqrt{\mathrm{K2}}X\right)-\\
&12e^{8\sqrt{3}\sqrt{\mathrm{K2}}}\mathrm{K2}X\left(1+2\sqrt{3}\sqrt{\mathrm{K2}}X\right)+e^{2\sqrt{3}\sqrt{\mathrm{K2}}X}(7+24\,\mathrm{K2}(-1+2X))+\\
&e^{2\sqrt{3}\sqrt{\mathrm{K2}}(4+X)}(7+24\,\mathrm{K2}(-1+2X))+4e^{4\sqrt{3}\sqrt{\mathrm{K2}}(1+X)}\sqrt{\mathrm{K2}}\left(\sqrt{3}-3\sqrt{\mathrm{K2}}X+6\sqrt{3}\,\mathrm{K2}\left(-2+X^2\right)\right)-\\
&4e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\sqrt{\mathrm{K2}}\left(\sqrt{3}+3\sqrt{\mathrm{K2}}X+6\sqrt{3}\,\mathrm{K2}\left(-2+X^2\right)\right)\Big)\Big)(-1+\mathrm{Y1})\,\mathrm{Y1}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\right)^2\mathrm{K2}}e^{-2\sqrt{3}\sqrt{\mathrm{K2}}X}\mathrm{Pin}\Big(-8\sqrt{3}\left(1+e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\right)\left(e^{4\sqrt{3}\sqrt{\mathrm{K2}}}-e^{4\sqrt{3}\sqrt{\mathrm{K2}}X}\right)\sqrt{\mathrm{K2}}-\\
&3\,\mathrm{a1}\Big(-4e^{2\sqrt{3}\sqrt{\mathrm{K2}}}-4e^{6\sqrt{3}\sqrt{\mathrm{K2}}}+4e^{2\sqrt{3}\sqrt{\mathrm{K2}}X}+8e^{2\sqrt{3}\sqrt{\mathrm{K2}}(2+X)}+4e^{2\sqrt{3}\sqrt{\mathrm{K2}}(4+X)}-\\
&4e^{2\sqrt{3}\sqrt{\mathrm{K2}}(1+2X)}-4e^{2\sqrt{3}\sqrt{\mathrm{K2}}(3+2X)}+e^{8\sqrt{3}\sqrt{\mathrm{K2}}}\left(-1-2\sqrt{3}\sqrt{\mathrm{K2}}X+12\,\mathrm{K2}X^2\right)+\\
&e^{4\sqrt{3}\sqrt{\mathrm{K2}}X}\left(-1+2\sqrt{3}\sqrt{\mathrm{K2}}X+12\,\mathrm{K2}X^2\right)+e^{4\sqrt{3}\sqrt{\mathrm{K2}}}\left(1-2\sqrt{3}\sqrt{\mathrm{K2}}X+12\,\mathrm{K2}\left(-2+X^2\right)\right)+\\
&e^{4\sqrt{3}\sqrt{\mathrm{K2}}(1+X)}\left(1+2\sqrt{3}\sqrt{\mathrm{K2}}X+12\,\mathrm{K2}\left(-2+X^2\right)\right)\Big)\Big)(-1+\mathrm{Y1})^2(1+2\,\mathrm{Y1})
\end{aligned}
$$

5. Results

Non-dimensional pressure and velocity functions

K1, K2 to K & ratio of h1/h2

In[181]:=

```
ShallowRelation1 = ϵ1 == h1 / l;
ShallowRelation2 = ϵ2 == h2 / l;
```

In[183]:=
```
FirstScalingRelation1 = uyC/ux1C == ϵ1;
FirstScalingRelation2 = uyC/ux2C == ϵ2;
```

In[185]:=
```
Kparameter1 = K1 == k μ / (l ϵ1^3);
Kparameter2 = K2 == k μ / (l ϵ2^3);
```

In[187]:=
```
AssumptionOnScaling = {l > 0, h1 > 0, h2 > 0, pC1 > 0, uxC > 0,
    uyC > 0, ϵ1 > 0, ϵ2 > 0, μ > 0, K1 > 0, K2 > 0, a1 > 0, Pin > 0, R > 0};
```

Substituting K1 and K2 (Only where K1 or K2 exist):

Region 1 :

Pressure

In[188]:=
```
G10Func[X];
G11Func[X];
G11FuncR[X] = Simplify[G11Func[X] //. K2 → {Kp * R}];
```

UY

In[191]:=
```
UX10Func[X, Y1];
UX11Func[X, Y1];
UX11FuncR[X, Y1] = Simplify[UX11Func[X, Y1] //. K2 → {Kp * R}];
```

UY

In[194]:=
```
UY10Func[X, Y1];
UY11Func[X, Y1];
UY11FuncR[X, Y1] = Simplify[UY11Func[X, Y1] //. K2 → {Kp * R}];
```

Region 1

Pressure - definitions

In[197]:=
```
G10Func[X];
G11FuncR[X];
GFuncAsy[X_, Kp_, R_, Pin_, a1_] = G10Func[X] + Kp G11FuncR[X]
GFuncAsyCorr[X_, Kp_, R_, Pin_, a1_] = Kp G11FuncR[X];
```

Out[199]=

$$\Big\{\mathrm{Pin} - \mathrm{Pin}\,X + \frac{1}{48\left(1 + e^{4\sqrt{3}\sqrt{\mathrm{Kp}\,R}}\right)^2 \mathrm{Kp}\,R^2} e^{-2\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,X}\,\mathrm{Pin}\Big(8\sqrt{3}\sqrt{\mathrm{Kp}\,R}\Big(e^{4\sqrt{3}\sqrt{\mathrm{Kp}\,R}} + e^{8\sqrt{3}\sqrt{\mathrm{Kp}\,R}} - e^{4\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,X} - e^{4\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,(1+X)} - e^{2\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,X}(-1+X) + e^{2\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,(4+X)}(-1+X)\Big) + 3\,\mathrm{a1}\Big(-4\,e^{2\sqrt{3}\sqrt{\mathrm{Kp}\,R}} - 4\,e^{6\sqrt{3}\sqrt{\mathrm{Kp}\,R}} - 4\,e^{2\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,(1+2X)} - 4\,e^{2\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,(3+2X)} + e^{2\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,(1+X)}(8-16X) + e^{2\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,(3+X)}(8-16X) + 2\,e^{2\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,(2+X)}\left(-5 + 24\,\mathrm{Kp}\,R\,(-1+X)^2 + 9X\right) + 3\,e^{8\sqrt{3}\sqrt{\mathrm{Kp}\,R}}\left(1 + 2\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,X + 4\,\mathrm{Kp}\,R\,X^2\right) + e^{4\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,X}\left(3 - 6\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,X + 12\,\mathrm{Kp}\,R\,X^2\right) + e^{2\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,X}\left(-3 + (7 - 24\,\mathrm{Kp}\,R)\,X + 24\,\mathrm{Kp}\,R\,X^2\right) + e^{2\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,(4+X)}\left(-3 + (7 - 24\,\mathrm{Kp}\,R)\,X + 24\,\mathrm{Kp}\,R\,X^2\right) + e^{4\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,(1+X)}\left(5 - 6\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,X + 12\,\mathrm{Kp}\,R\,(-2+X^2)\right) + e^{4\sqrt{3}\sqrt{\mathrm{Kp}\,R}}\left(5 + 6\sqrt{3}\sqrt{\mathrm{Kp}\,R}\,X + 12\,\mathrm{Kp}\,R\,(-2+X^2)\right)\Big)\Big)\Big\}$$

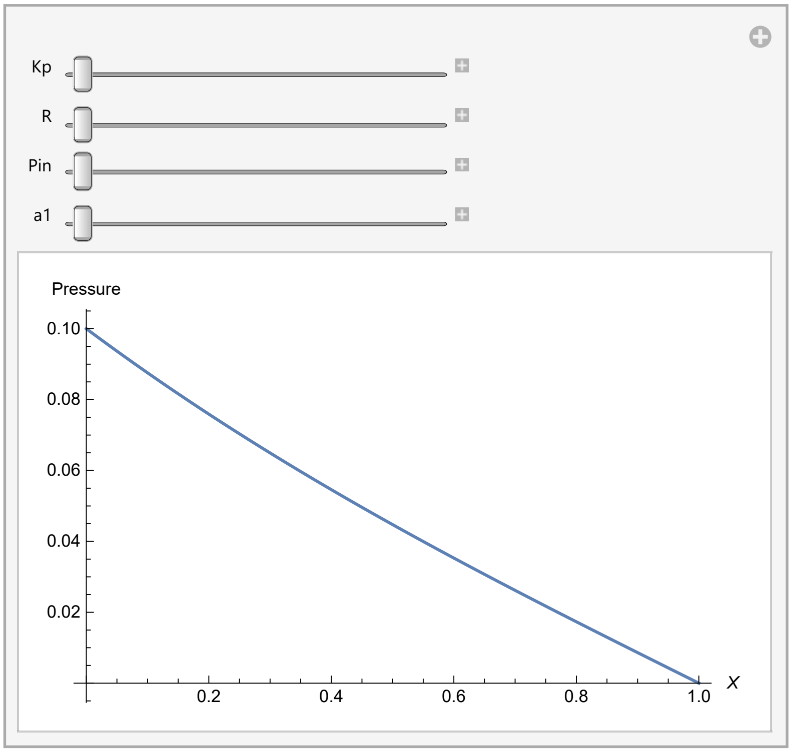
In[201]:=
```
GFuncAsy[0.5, 0.50, 0.3, 0.1, 1, 0.2];
GFuncAsyCorr[0.5, 0.50, 0.3, 0.1, 1, 0.2];
```

Pressure - total

The leading order is simple pressure driven channel flow.

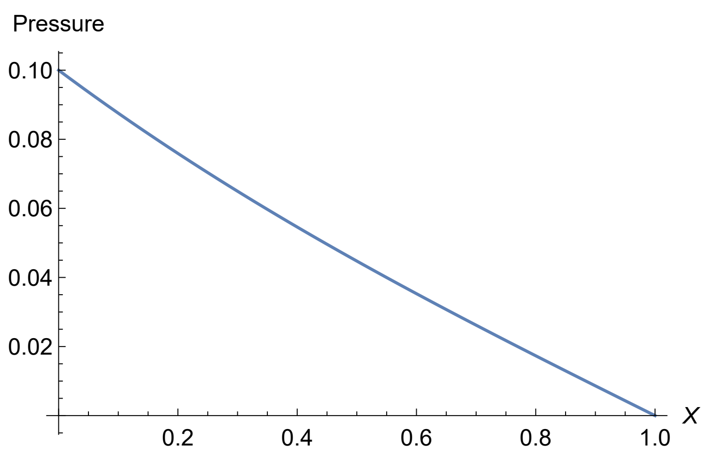
In[203]:=
```
Manipulate[Plot[GFuncAsy[X, Kp, R, Pin, a1], {X, 0, 1}, AxesLabel → {X, Pressure}],
 {Kp, 0.1, 1}, {R, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.1 , 1}]
```

Out[203]=



In[204]:=
```
P1Order0Fig =
 DynamicModule[{a1 = 0.1`, Kp = 0.1`, Pin = 0.1`, R = 1}, Plot[GFuncAsy[X, Kp, R, Pin, a1],
   {X, 0, 1}, AxesLabel → {X, Pressure}, LabelStyle → (FontSize → 12)]]
```

Out[204]=

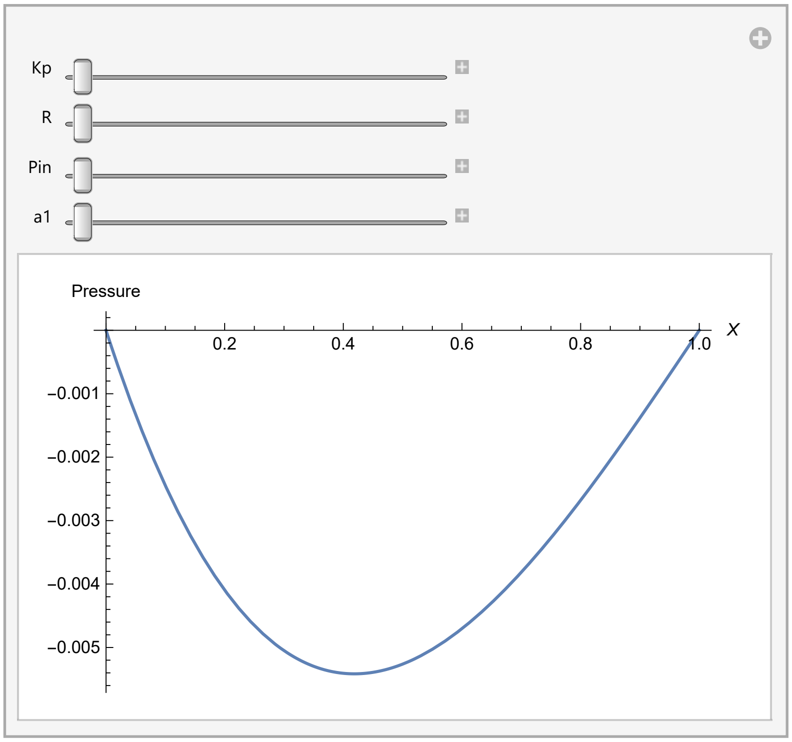


Pressure - asymptotic correction

In[205]:=
```
GFuncAsyCorr[X, Kp, R, Pin, a1];
```

In[206]:=
```
Manipulate[Plot[GFuncAsyCorr[X, Kp, R, Pin, a1], {X, 0, 1}, AxesLabel → {X, Pressure}],
 {Kp, 0.1, 1}, {R, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.1 , 1}]
```

Out[206]=



Ux1 - definitions

In[207]:=
```
UX10Func[X, Y1];
UX11Func[X, Y1];
UXAsy[X_, Y1_, Kp_, R_, Pin_, a1_] = UX10Func[X, Y1] + Kp UX11FuncR[X, Y1];
UXAsyCorr[X_, Y1_, Kp_, R_, Pin_, a1_] = Kp * UX11FuncR[X, Y1];
```

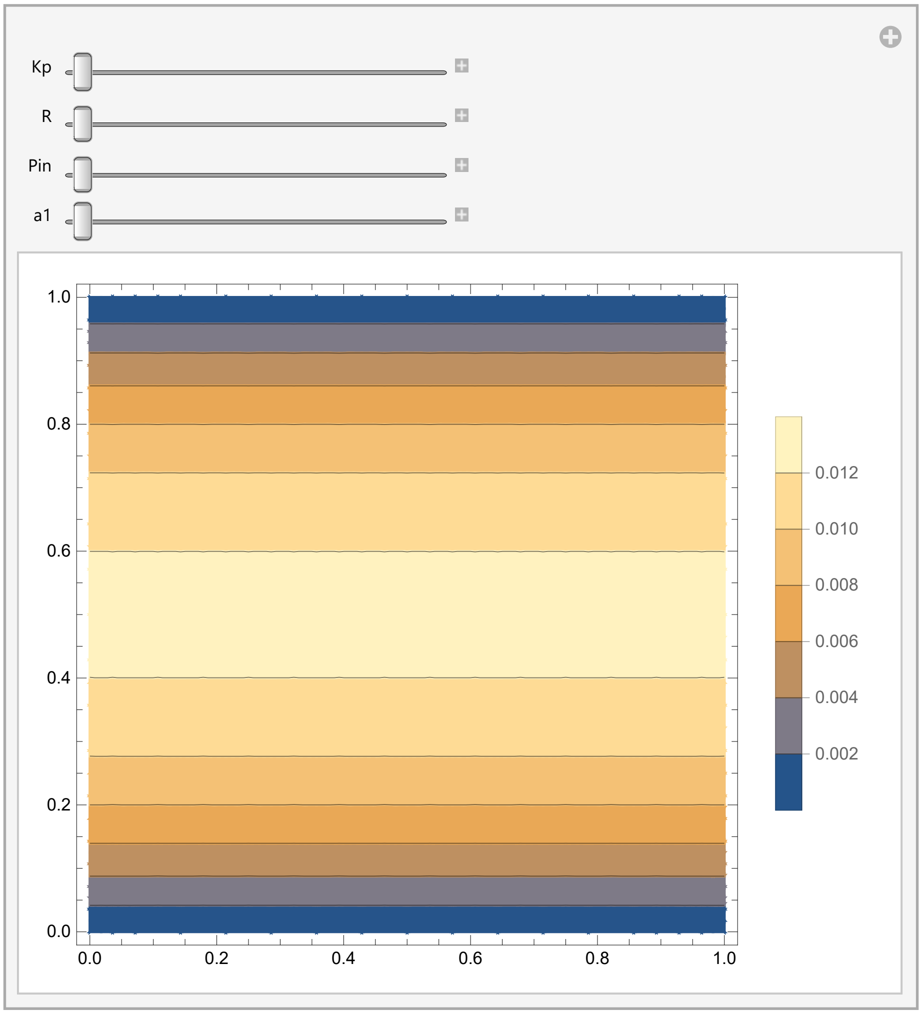
In[211]:=
```
UXAsy[0.5, 0.50, 0.3, 0.1, 1, 0.2];
UXAsyCorr[0.5, 0.50, 0.3, 0.1, 1, 0.2];
```

In[213]:=
```
UX10Func[X_, Y1_, Kp_, R_, Pin_, a1_] = UX10Func[X, Y1];
```

Ux1 - leading order

In[214]:=
```
Manipulate[ContourPlot[UX10Func[X, Y1, Kp, R, Pin, a1],
   {X, 0, 1}, {Y1, 0, 1}, AxesLabel → {X, Y1, UX}, PlotLegends → Automatic],
 {Kp, 0.1, 1}, {R, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.1 , 1}]
```

Out[214]=

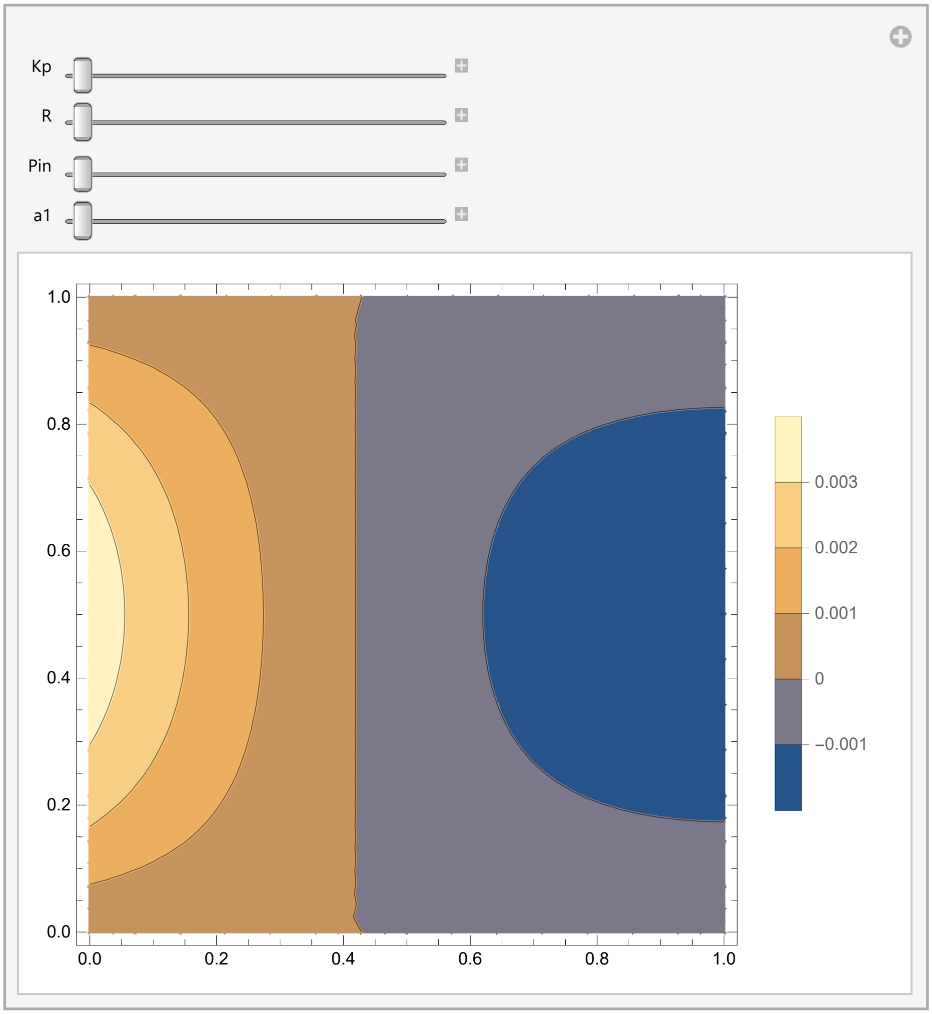


Ux1 - asymptotic correction

In[215]:=
```
UXAsyCorr[X, Y1, Kp, R, Pin,  a1];
```

In[216]:=
```
Manipulate[ContourPlot[UXAsyCorr[X, Y1, Kp, R, Pin, a1],
  {X, 0, 1}, {Y1, 0, 1}, AxesLabel → {X, Y1, UX}, PlotLegends → Automatic],
 {Kp, 0.1, 1}, {R, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.1 , 1}]
```

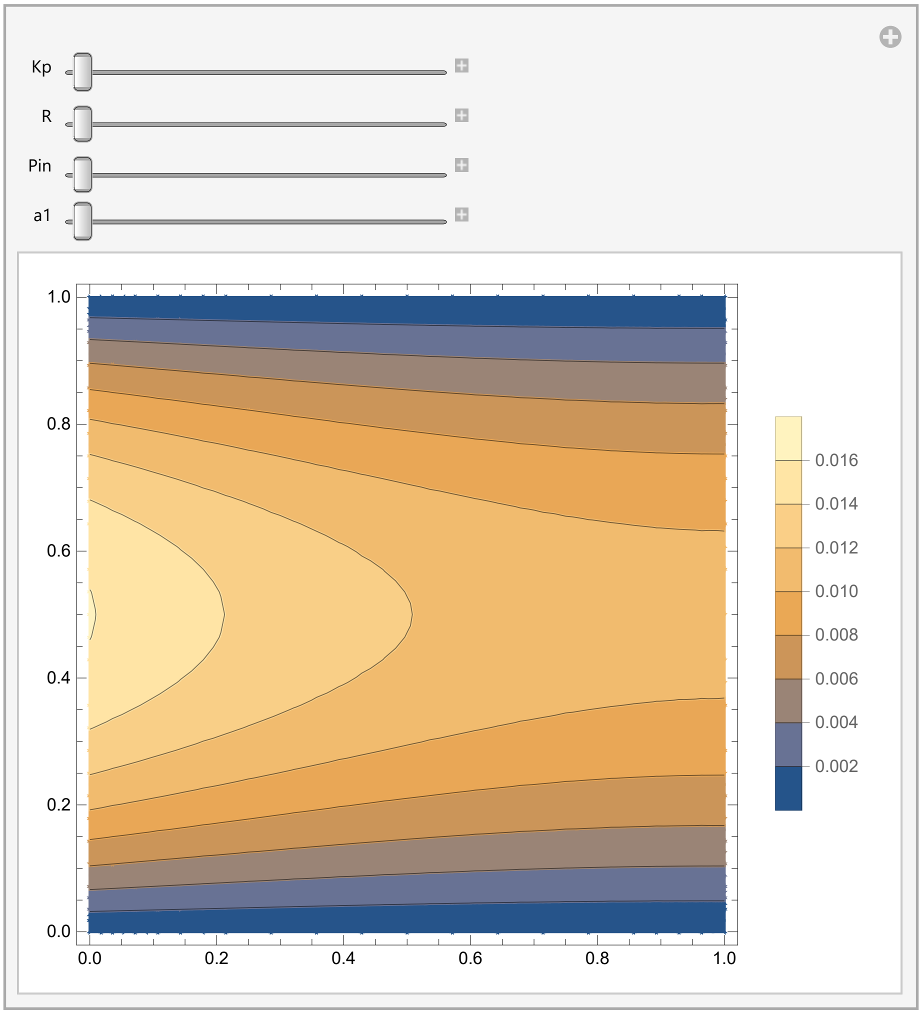
Out[216]=



Ux1 - total flow

In[217]:=
```
Manipulate[ContourPlot[UXAsy[X, Y1, Kp, R, Pin, a1],
   {X, 0, 1}, {Y1, 0, 1}, AxesLabel → {X, Y1, UX}, PlotLegends → Automatic],
 {Kp, 0.1, 1}, {R, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.1 , 1}]
```

Out[217]=



Uy1

In[218]:=
```
UY10Func[X, Y1];
UY11Func[X, Y1];
UYAsy[X_, Y1_, Kp_, R_, Pin_, a1_] = UY10Func[X, Y1] + Kp * UY11FuncR[X, Y1];
UYAsyCorr[X_, Y1_, Kp_, R_, Pin_, a1_] = Kp * UY11Func[X, Y1];
```

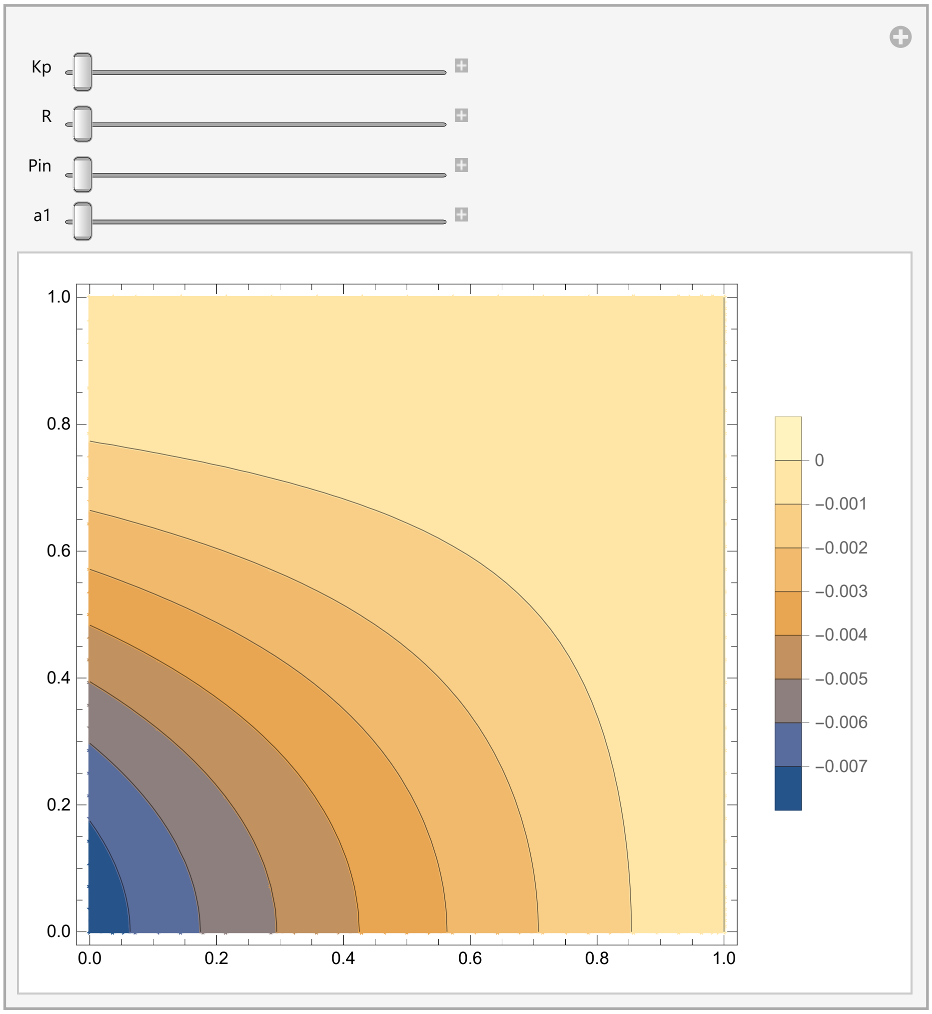
Since the leading order is essentially zero, only the asymptotic exp. matters in this case.

In[222]:=
```
UYAsy[0.5, 0.50, 0.5, 0.5, 0.5, 0.5];
UYAsyCorr[0.5, 0.50, 0.5, 0.5, 0.5, 0.5];
```

Uy1 - total flow

In[224]:=
```
Manipulate[ContourPlot[UYAsy[X, Y1, Kp, R, Pin, a1],
   {X, 0, 1}, {Y1, 0, 1}, AxesLabel → {X, Y1, UX}, PlotLegends → Automatic],
  {Kp, 0.1, 1}, {R, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.1 , 1}]
```

Out[224]=



UX and UY stream map - total flow fields

In[225]:=
```
UXAsy[X, Y1, Kp, R, Pin, a1];
UYAsy[X, Y1, Kp, R, Pin, a1];
```

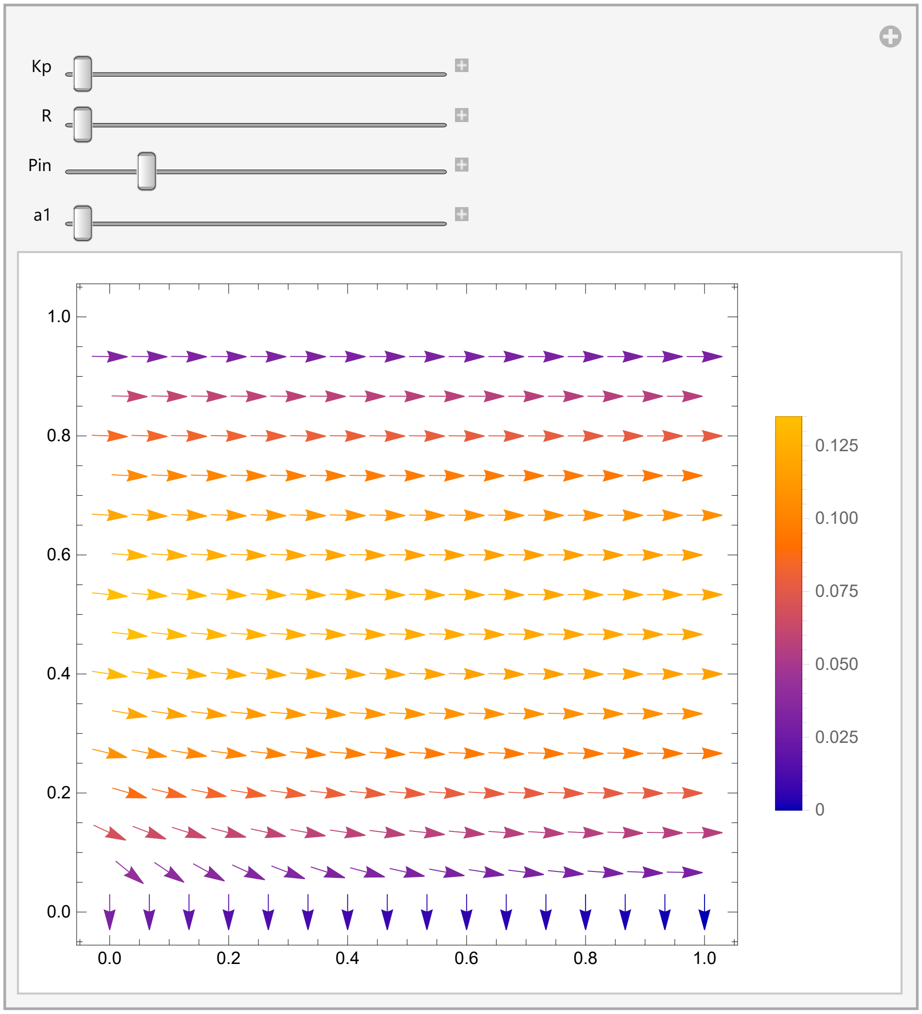
In[227]:=
```
UXAsy[1, 0.5, 0.1, 10, 1, 0.1];
UYAsy[0.5, 0.5, 0.1, 10, 1, 0.1];
```

In[229]:=
```
VectorPlot[{X, Y1}, {X, 0, 1}, {Y1, 0, 1}, PlotLegends -> Automatic];
```

In[230]:=
```
UXAsy[X, Y1, Kp, R, Pin, a1] * 0;
```

In[231]:=
```
Manipulate[
 VectorPlot[{UXAsy[X, Y1, Kp, R, Pin, a1][[1]], UYAsy[X, Y1, Kp, R, Pin, a1][[1]]},
  {X, 0, 1}, {Y1, 0, 1}, PlotLegends -> Automatic],
 {Kp, 0.1, 1}, {R, 10, 1000}, {{Pin, 1}, 0.1, 5}, {a1, 0.1, 1}]
```

Out[231]=

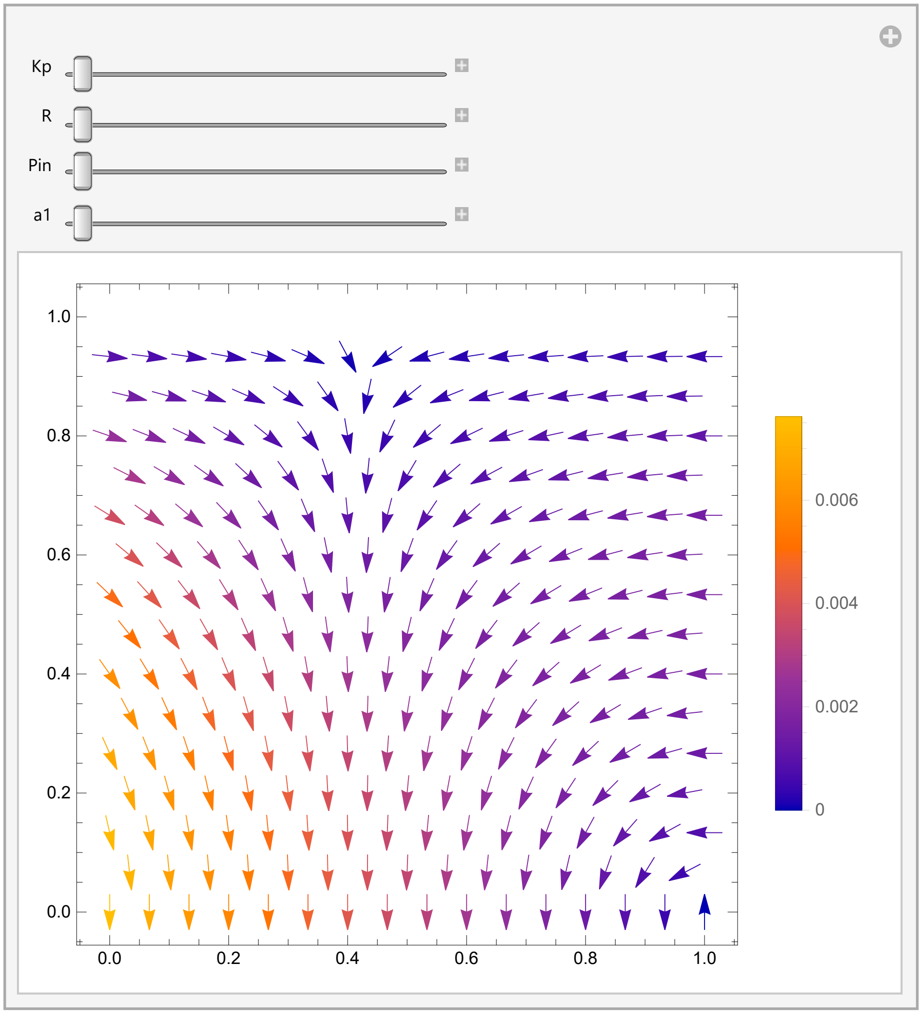


UX and UY stream map - asymptotic correction

In[232]:= **UXAsyCorr[*X*, Y1, Kp, *R*, Pin, a1];**
UYAsyCorr[*X*, Y1, Kp, *R*, Pin, a1];

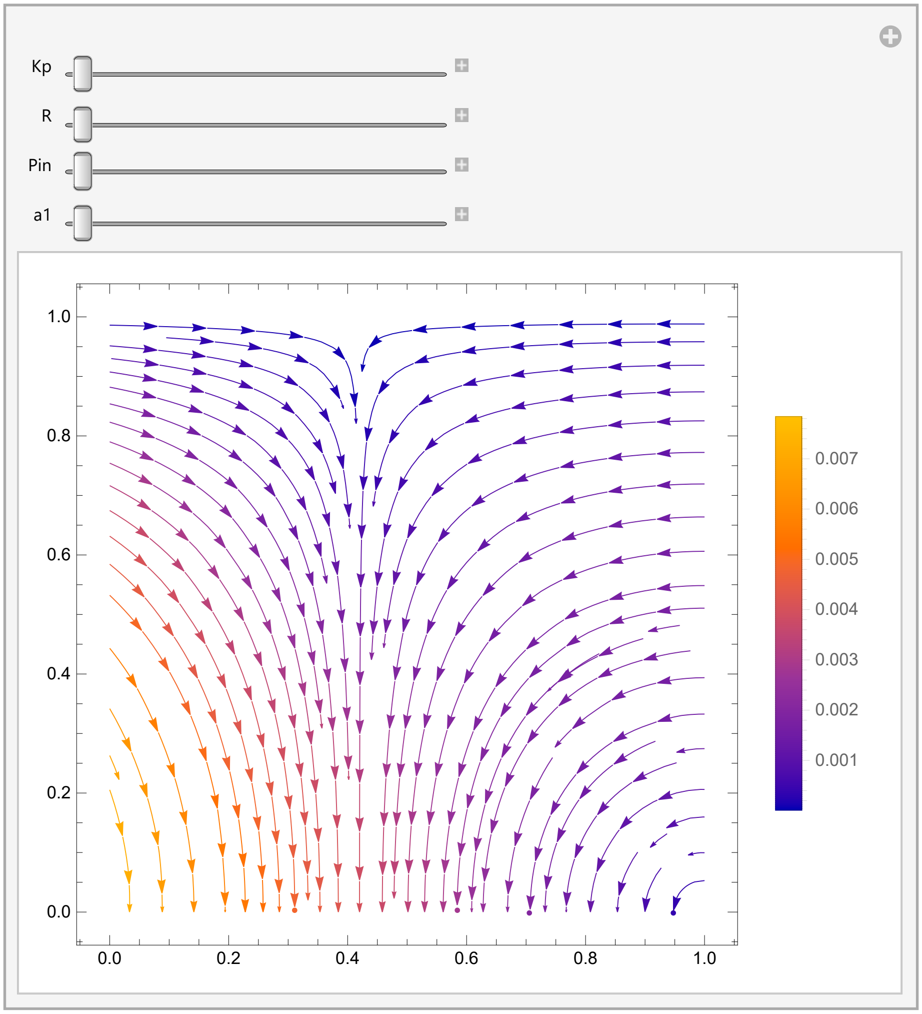
In[234]:=
```
Manipulate[
 VectorPlot[{UXAsyCorr[X, Y1, Kp, R, Pin, a1][[1]], UYAsyCorr[X, Y1, Kp, R, Pin, a1][[1]]},
  {X, 0, 1}, {Y1, 0, 1}, PlotLegends → Automatic],
 {Kp, 0.1, 1}, {R, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.1 , 1}]
```

Out[234]=



In[235]:=
```
Manipulate[
 StreamPlot[{UXAsyCorr[X, Y1, Kp, R, Pin, a1][[1]], UYAsyCorr[X, Y1, Kp, R, Pin, a1][[1]]},
  {X, 0, 1}, {Y1, 0, 1}, StreamColorFunction → Automatic, PlotLegends -> Automatic],
 {Kp, 0.1, 1}, {R, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.1 , 1}]
```

Out[235]=



Region 2

Pressure 2

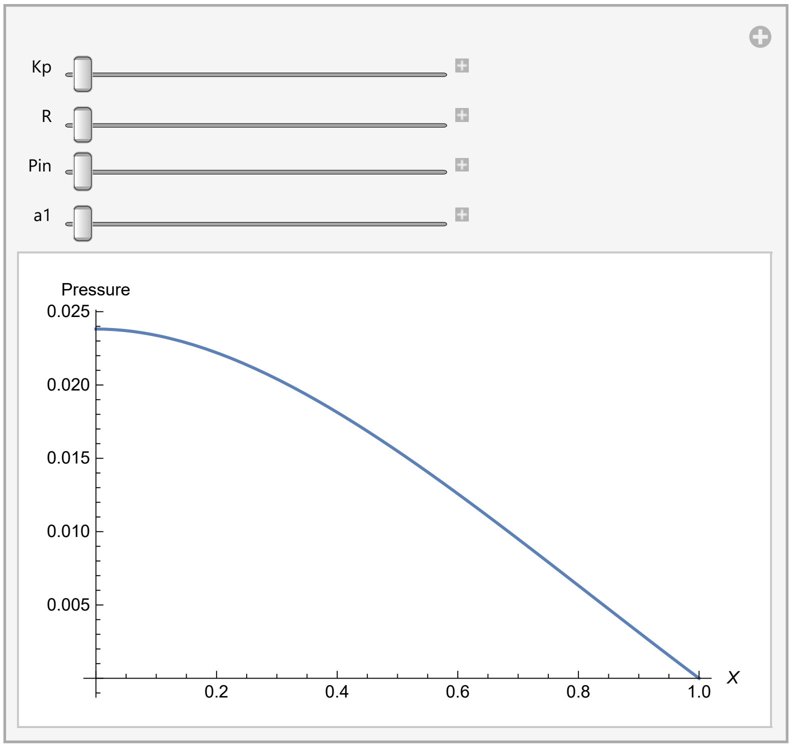
In[236]:=
```
FuncG201[X];
FuncG201R[X] = Simplify[FuncG201[X] //. K2 → {Kp * R}];
G2FuncAsy[X_, Y2_, Kp_, R_, Pin_, a1_] = FuncG201R[X];
```

In[239]:= G2 is not *a* function of Y2 or *K*

Out[239]= a function G2 is K not of or Y2

In[240]:=
```
Manipulate[Plot[G2FuncAsy[X, Y2, Kp, R, Pin, a1], {X, 0, 1}, AxesLabel → {X, Pressure}],
 {Kp, 0.1, 1}, {R, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.1 , 1}]
```

Out[240]=

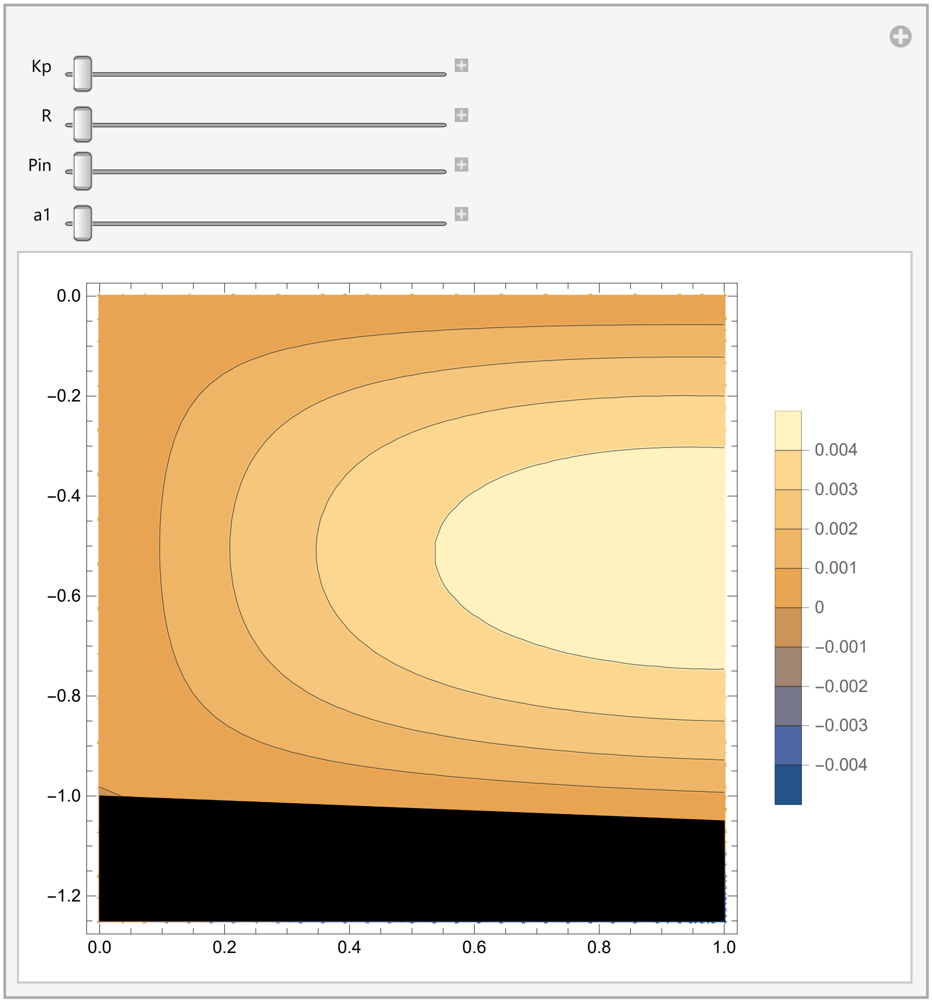


UX2

In[241]:=
```
UX20FuncSub[X, Y2];
UX20FuncSubR[X, Y2] = Simplify[UX20FuncSub[X, Y2] //. K2 → {Kp∗R}];
UX20Asy[X_, Y2_, Kp_, R_, Pin_, a1_] = UX20FuncSubR[X, Y2];
```

In[244]:=
```
Manipulate[Show[{ContourPlot[UX20Asy[X, Y2, Kp, R, Pin, a1],
    {X, 0, 1}, {Y2, -1.25, 0}, AxesLabel → {X, Y2, UX}, PlotLegends → Automatic],
   Graphics[Polygon[{{0, -1}, {1, -1 - a1}, {1, -1 - 5 a1}, {0, -1 - 5 a1}}]]}],
 {Kp, 0.1, 1}, {R, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.05 , 1}]
```

Out[244]=

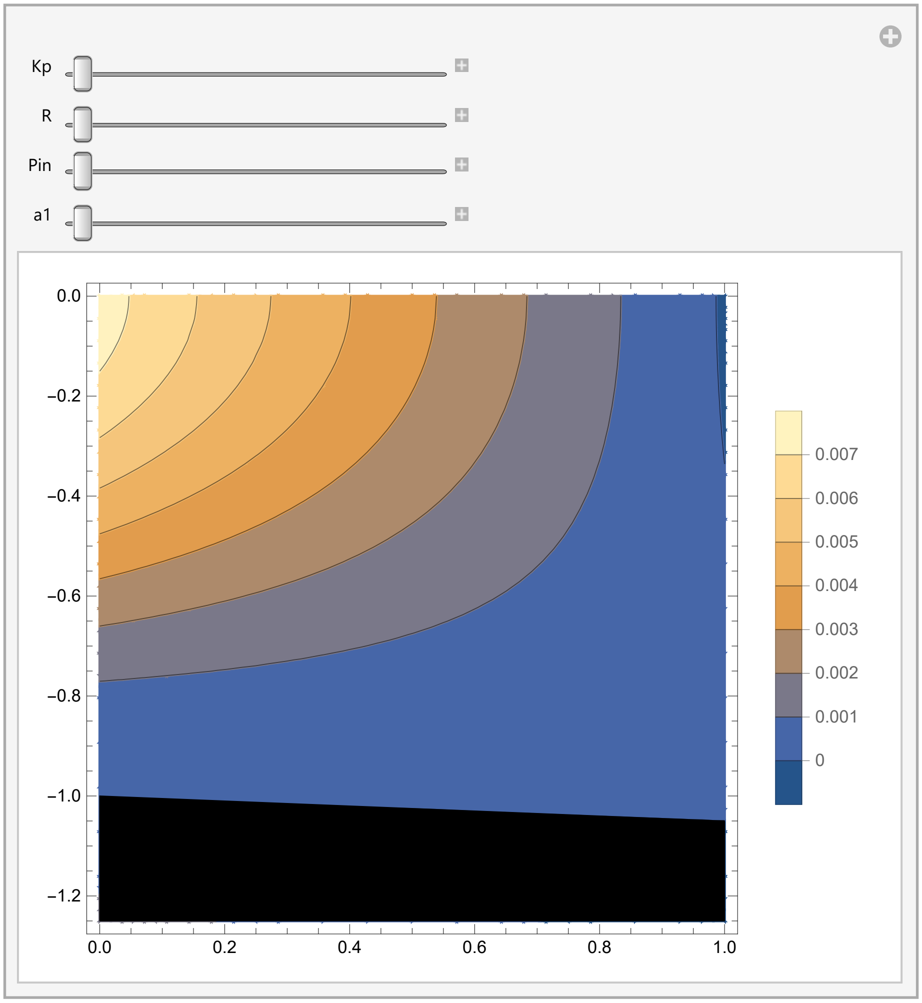


UY2

In[245]:=
```
UY20FuncSub[X, Y2];
UY20FuncSubR[X, Y2] = Simplify[UY20FuncSub[X, Y2] //. K2 → {Kp*R}];
UY20Asy[X_, Y2_, Kp_, R_, Pin_, a1_] = UY20FuncSubR[X, Y2];
```

In[248]:=
```
Manipulate[Show[{ContourPlot[-UY20Asy[X, Y2, Kp, R, Pin, a1],
    {X, 0, 1}, {Y2, -1.25, 0}, AxesLabel → {X, Y2, UY}, PlotLegends → Automatic],
   Graphics[Polygon[{{0, -1}, {1, -1 - a1}, {1, -1 - 5 a1}, {0, -1 - 5 a1}}]]}],
 {Kp, 0.1, 1}, {R, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.05 , 1}]
```

Out[248]=



UX2, UY2 - total flow field

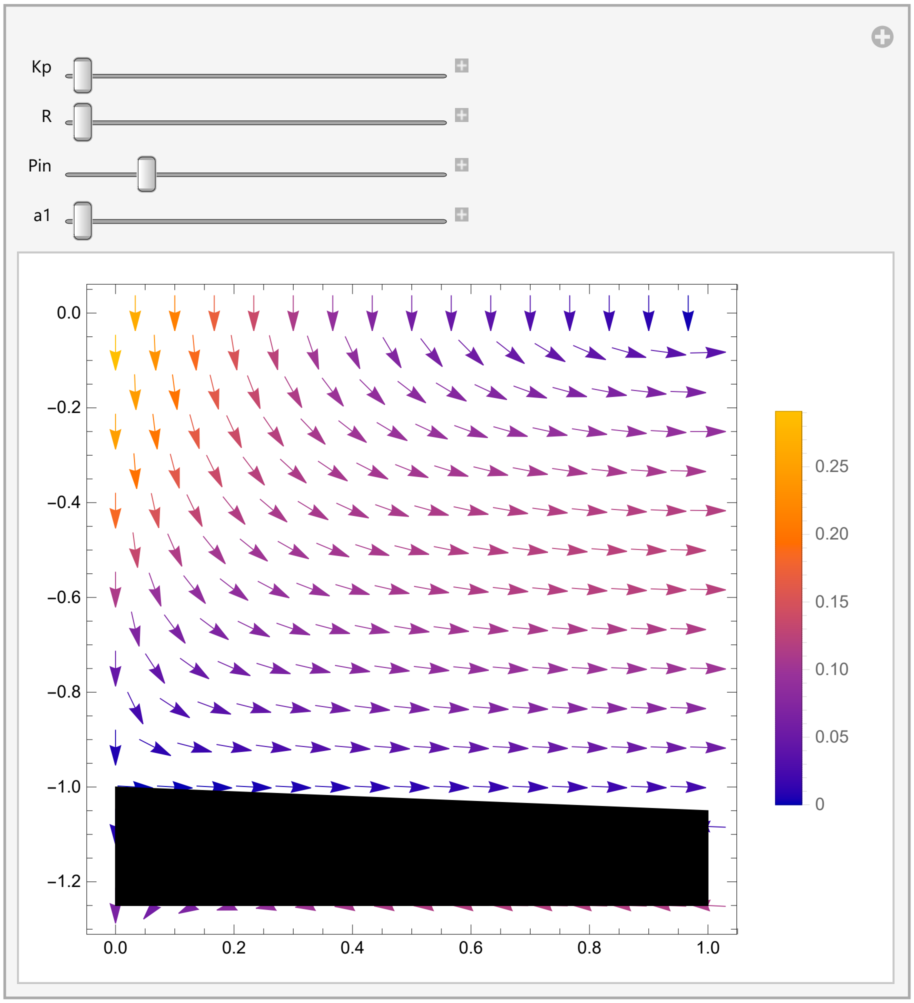
In[249]:=
```
UX20Asy[X, Y2, Kp, R, Pin, a1][[1]];
```

In[250]:=
```
UY20Asy[X, Y2, Kp, R, Pin, a1][[1]];
```

In[251]:=

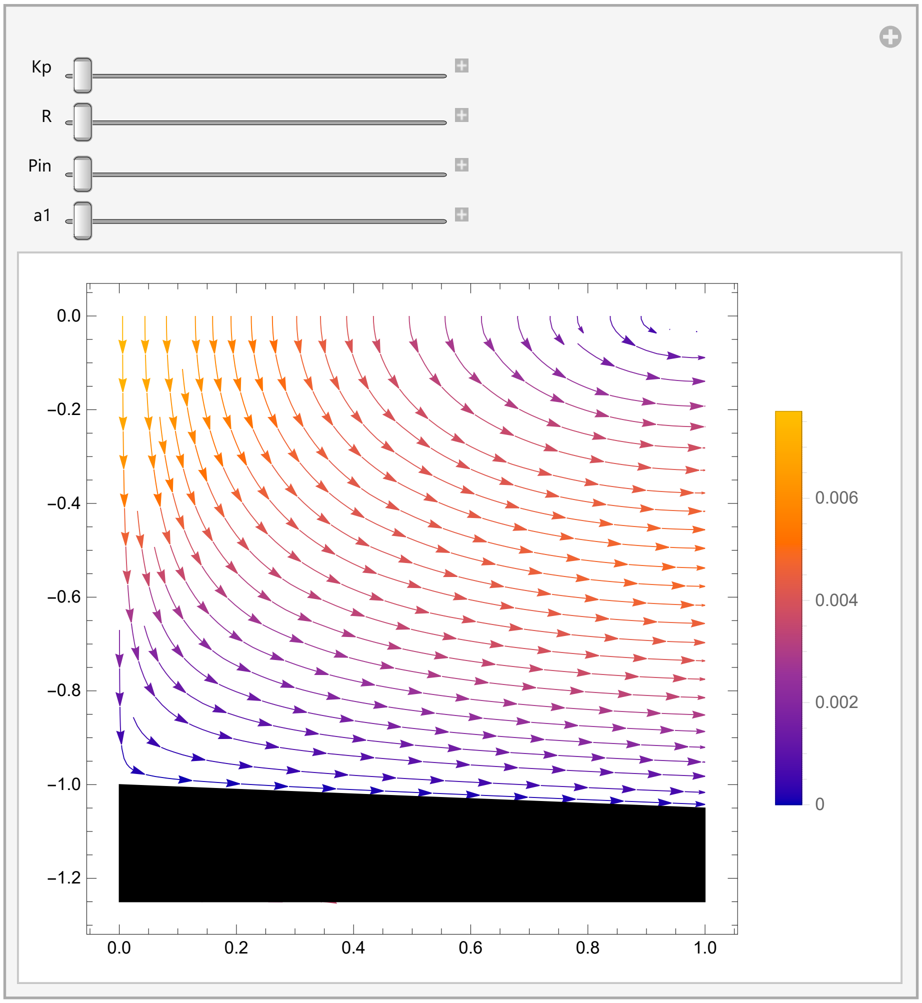
```
Manipulate[Show[
  {VectorPlot[{UX20Asy[X, Y2, Kp, R, Pin, a1][[1]], UY20Asy[X, Y2, Kp, R, Pin, a1][[1]]},
    {X, 0, 1}, {Y2, -1.25, 0}, PlotLegends -> Automatic],
   Graphics[Polygon[{{0, -1}, {1, -1 - a1}, {1, -1 - 5 a1}, {0, -1 - 5 a1}}]]}],
 {Kp, 0.1, 1}, {R, 10, 1000}, {{Pin, 1}, 0.1, 5}, {a1, 0.05 , 1}]
```

Out[251]=



In[252]:=
```
p1 = Manipulate[Show[
    {StreamPlot[{UX20Asy[X, Y2, Kp, R, Pin, a1][[1]], UY20Asy[X, Y2, Kp, R, Pin, a1][[1]]},
      {X, 0, 1}, {Y2, -1.25, 0}, StreamColorFunction → Automatic, PlotLegends -> Automatic],
     Graphics[Polygon[{{0, -1}, {1, -1 - a1}, {1, -1 - 5 a1}, {0, -1 - 5 a1}}]]}],
  {Kp, 0.1, 1}, {R, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.05 , 1}]
```

Out[252]=

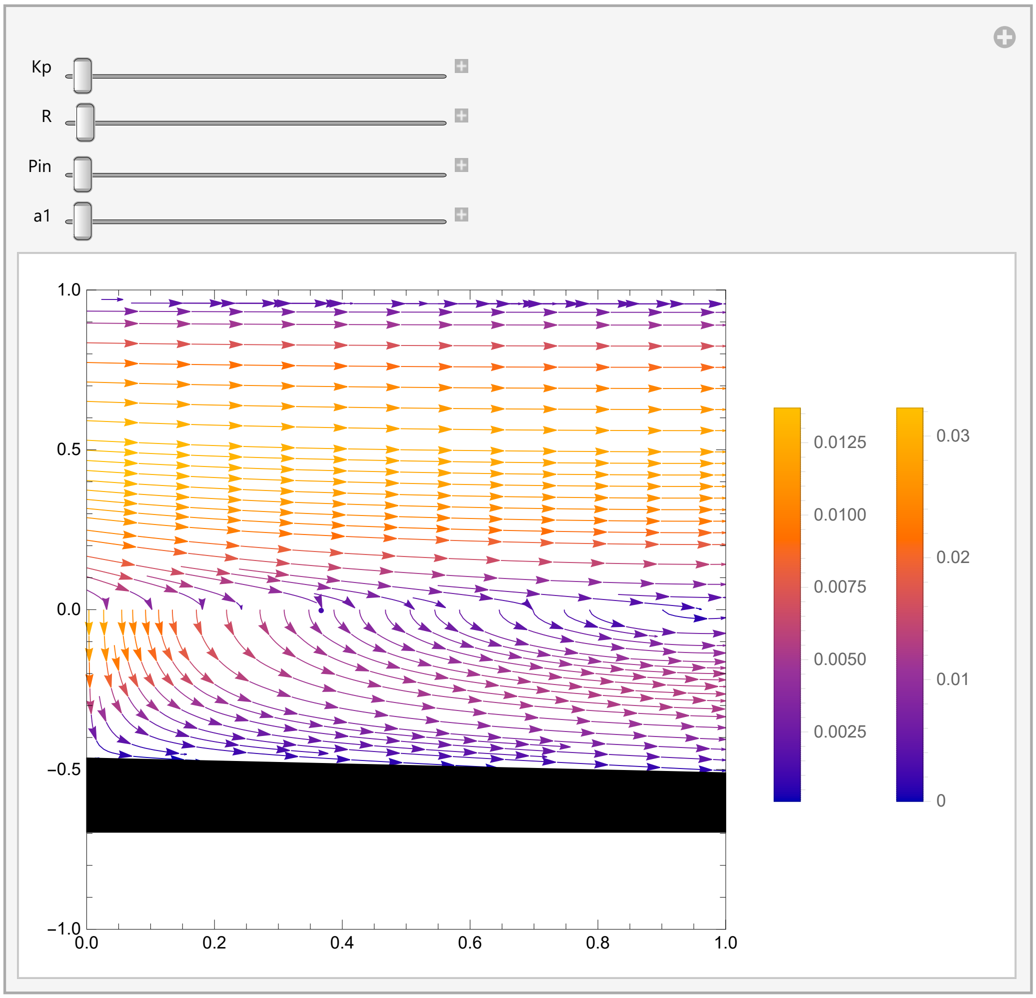


Total flow field

In[253]:=
```
H2Border[X_, a1_] = -(1 + a1 * X);
H2Border[X, a1];
H2Border[0.5, 0.5];
```

In[256]:=
```
Manipulate[
 Show[{StreamPlot[{UXAsy[X, Y1, Kp, R, Pin, a1][[1]], UYAsy[X, Y1, Kp, R, Pin, a1][[1]]},
    {X, 0, 1}, {Y1, 0, 1}, StreamColorFunction → Automatic,
    PlotLegends -> Automatic, PlotRange → {{0, 1}, {-1, 1}}],
   StreamPlot[{UX20Asy[X, Y2 / (R^(-1/3)), Kp, R, Pin, a1][[1]],
     UY20Asy[X, Y2 / (R^(-1/3)), Kp, R, Pin, a1][[1]]}, {X, 0, 1},
    {Y2, -(R^(-1/3)) (1 + a1), 0}, StreamColorFunction → Automatic,
    PlotLegends -> Automatic, PlotRange → All],
   Graphics[Polygon[{{0, -1 (R^(-1/3))}, {1, (R^(-1/3)) (-1 - a1)},
      {1, (R^(-1/3)) (-1 - 5 a1)}, {0, (R^(-1/3)) (-1 - 5 a1)}}]]}],
 {Kp, 0.1, 1}, {{R, 10}, 1 , 1000}, {Pin, 0.1 , 5}, {a1, 0.1 , 1}]
```

Out[256]=



6. Dimensional velocity and pressure functions

6.1 Top Region - Leading Order

P10

In[257]:=
```
G10Func[X];
G10FuncDim[X] =
  Simplify[G10Func[X] //. {UX1 → (ux1/ux1C [X[#1], Y1[#2]] &), UY1 → (uy1/uyC [X[#1], Y1[#2]] &),
      P1 → (p1/pC1 [X[#1], Y1[#2]] &), P2 → (p2/pC1 [X[#1], 0] &),
      Pin → ( pin/pC1 ), X → (x/ l), Y1 → (y/h1)}, AssumptionOnScaling];
```

In[259]:=
```
TextbookEq[G10Func[X]]
TextbookEq[G10FuncDim[X]]
```

$$\text{Pin} - \text{Pin}\,X$$

$$\frac{\text{pin}\,(l - x)}{l\,\text{pC1}}$$

U10

In[261]:=
```
UX10Func[X, Y1]
UX10FuncDim[X, Y1] = Simplify[
   UX10Func[X, Y1] //. {UX1 → (ux1/ux1C [X[#1], Y1[#2]] &), UY1 → (uy1/uyC [X[#1], Y1[#2]] &),
      P1 → (p1/pC1 [X[#1], Y1[#2]] &), P2 → (p2/pC1 [X[#1], 0] &),
      Pin → ( pin/pC1 ), X → (x/ l), Y1 → (y/h1)}, AssumptionOnScaling];
```

Out[261]= $\frac{1}{2}\left(\text{Pin Y1} - \text{Pin Y1}^2\right)$

In[263]:=
```
TextbookEq[UX10Func[X, Y1]]
TextbookEq[UX10FuncDim[X, Y1]]
```

$$\frac{1}{2}\left(\text{Pin Y1} - \text{Pin Y1}^2\right)$$

$$\frac{\text{pin}\,(\text{h1} - y)\,y}{2\,\text{h1}^2\,\text{pC1}}$$

V10

In[265]:=
```
UY10Func[X, Y1]
UY10FuncDim[X, Y1] =
  Simplify[UY10Func[X, Y1] //. {UX1 → (ux1/ux1C [X[#1], Y1[#2]] &) , UY1 → (uy1/uyC [X[#1], Y1[#2]] &),
        P1 → (p1/pC1 [X[#1], Y1[#2]] &), P2 → (p2/pC1 [X[#1], 0] &),
        Pin → ( pin/pC1 ), X → (x/ l), Y1 → (y/h1)}, AssumptionOnScaling]
```

Out[265]= 0

Out[266]= 0

6.2 Top Region - First Order

P11

In[267]:=
```
G11Func[X];
```

In[268]:=
```
G11FuncDim[X] =
   Simplify[G11Func[X] //. {UX1 → (ux1/ux1C [X[#1], Y1[#2]] &) , UY1 → (uy1/uyC [X[#1], Y1[#2]] &),
        P1 → (p1/pC1 [X[#1], Y1[#2]] &), P2 → (p2/pC1 [X[#1], 0] &), Pin → ( pin/pC1 ),
        X → (x/ l), Y1 → (y/h1), K2 → (K (h1/h2))^3}, AssumptionOnScaling];
```

In[269]:= **TextbookEq[G11FuncDim[*X*]]**

$$
\begin{aligned}
&\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\right)^2 \mathrm{h1}^6 K^6 l^2\,\mathrm{pC1}}\\
&e^{-\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}\,\mathrm{h2}^3\,\mathrm{pin}\left(8\sqrt{3}\,\mathrm{h1}\,\mathrm{h2}\sqrt{\mathrm{h1}\,\mathrm{h2}\,K^3}\,l\left(e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\,l+e^{8\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\,l-e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}\,l-\right.\right.\\
&\left.e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,(l+x)}{l}}\,l+e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}\,(l-x)+e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(4+\frac{x}{l}\right)}(-l+x)\right)+\\
&3\,\mathrm{a1}\left(-4\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\,\mathrm{h2}^3 l^2-4\,e^{6\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\,\mathrm{h2}^3 l^2-4\,e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,(l+2x)}{l}}\,\mathrm{h2}^3 l^2-4\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(3+\frac{2x}{l}\right)}\,\mathrm{h2}^3 l^2+\right.\\
&8\,e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,(l+x)}{l}}\,\mathrm{h2}^3\,l\,(l-2x)+8\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(3+\frac{x}{l}\right)}\,\mathrm{h2}^3\,l\,(l-2x)+\\
&3\,e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}\left(\mathrm{h2}^3 l^2-2\sqrt{3}\,\mathrm{h1}\,\mathrm{h2}\sqrt{\mathrm{h1}\,\mathrm{h2}\,K^3}\,l\,x+4\,\mathrm{h1}^3 K^3 x^2\right)+\\
&3\,e^{8\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\left(\mathrm{h2}^3 l^2+2\sqrt{3}\,\mathrm{h1}\,\mathrm{h2}\sqrt{\mathrm{h1}\,\mathrm{h2}\,K^3}\,l\,x+4\,\mathrm{h1}^3 K^3 x^2\right)+\\
&e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}\left(24\,\mathrm{h1}^3 K^3 x\,(-l+x)+\mathrm{h2}^3\,l\,(-3\,l+7x)\right)+e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(4+\frac{x}{l}\right)}\\
&\left(24\,\mathrm{h1}^3 K^3 x\,(-l+x)+\mathrm{h2}^3\,l\,(-3\,l+7x)\right)+2\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(2+\frac{x}{l}\right)}\left(24\,\mathrm{h1}^3 K^3 (l-x)^2+\mathrm{h2}^3\,l\,(-5\,l+9x)\right)+\\
&e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,(l+x)}{l}}\left(5\,\mathrm{h2}^3 l^2-6\sqrt{3}\,\mathrm{h1}\,\mathrm{h2}\sqrt{\mathrm{h1}\,\mathrm{h2}\,K^3}\,l\,x+12\,\mathrm{h1}^3 K^3\left(-2\,l^2+x^2\right)\right)+\\
&\left.\left.e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\left(5\,\mathrm{h2}^3 l^2+6\sqrt{3}\,\mathrm{h1}\,\mathrm{h2}\sqrt{\mathrm{h1}\,\mathrm{h2}\,K^3}\,l\,x+12\,\mathrm{h1}^3 K^3\left(-2\,l^2+x^2\right)\right)\right)\right)
\end{aligned}
$$

U11

In[270]:= **UX11Func[*X*, Y1];**

In[271]:= **UX11FuncDim[*X*, Y1] = Simplify[**

```
UX11FuncDim[X, Y1] = Simplify[
    UX11Func[X, Y1] //. {UX1 → (ux1/ux1C [X[#1], Y1[#2]] &) , UY1 → (uy1/uyC [X[#1], Y1[#2]] &),
      P1 → (p1/pC1 [X[#1], Y1[#2]] &), P2 → (p2/pC1 [X[#1], 0] &), Pin → ( pin/pC1 ),
      X → (x/ l), Y1 → (y/h1), K2 → (K (h1/h2)) ^ 3}, AssumptionOnScaling];
```

In[272]:= **TextbookEq[UX11FuncDim[X, Y1]]**

$$
\begin{aligned}
&\frac{1}{96\left(1+e^{4\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}}\right)^2 h1^7 K^6\,\mathrm{pC1}}\\
&e^{-\frac{2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\,x}{l}}\,h2^3\,\mathrm{pin}\Bigg(-8\left(1+e^{4\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}}\right)h1\Bigg(6\,e^{4\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}}\,h1^2K^3+6\,e^{\frac{4\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\,x}{l}}\,h1^2K^3+\\
&\sqrt{3}\,e^{\frac{2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\,x}{l}}\,h2\sqrt{h1\,h2\,K^3}-\sqrt{3}\,e^{2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\left(2+\frac{x}{l}\right)}h2\sqrt{h1\,h2\,K^3}\Bigg)+\frac{1}{l^2}\\
&3\,a1\Bigg(-16\,e^{\frac{2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\,(l+x)}{l}}\,h2^3l^2-16\,e^{2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\left(3+\frac{x}{l}\right)}h2^3l^2+8\sqrt{3}\,e^{2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}}\,h1\,h2\sqrt{h1\,h2\,K^3}\,l^2+\\
&8\sqrt{3}\,e^{6\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}}\,h1\,h2\sqrt{h1\,h2\,K^3}\,l^2-8\sqrt{3}\,e^{\frac{2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\,(l+2x)}{l}}\,h1\,h2\sqrt{h1\,h2\,K^3}\,l^2-\\
&8\sqrt{3}\,e^{2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\left(3+\frac{2x}{l}\right)}h1\,h2\sqrt{h1\,h2\,K^3}\,l^2+e^{\frac{2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\,x}{l}}\,l\left(7\,h2^3\,l-24\,h1^3K^3(l-2x)\right)+\\
&e^{2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\left(4+\frac{x}{l}\right)}\,l\left(7\,h2^3\,l-24\,h1^3K^3(l-2x)\right)-12\,e^{\frac{4\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\,x}{l}}\,h1^3K^3x\left(l-2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\,x\right)-\\
&12\,e^{8\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}}\,h1^3K^3x\left(l+2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\,x\right)+6\,e^{2\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\left(2+\frac{x}{l}\right)}\,l\left(3\,h2^3\,l+16\,h1^3K^3(-l+x)\right)+\\
&4\,e^{4\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}}\,h1\left(-\sqrt{3}\,h2\sqrt{h1\,h2\,K^3}\,l^2-3\,h1^2K^3\,l\,x+6\sqrt{3}\,h1^{7/2}\left(\frac{K^3}{h2}\right)^{3/2}\left(2\,l^2-x^2\right)\right)-\\
&4\,e^{\frac{4\sqrt{3}\sqrt{\frac{h1^3K^3}{h2^3}}\,(l+x)}{l}}\,h1\left(-\sqrt{3}\,h2\sqrt{h1\,h2\,K^3}\,l^2+3\,h1^2K^3\,l\,x+6\sqrt{3}\,h1^{7/2}\left(\frac{K^3}{h2}\right)^{3/2}\left(2\,l^2-x^2\right)\right)\Bigg)\Bigg)\,y\left(-1+\frac{y}{h1}\right)
\end{aligned}
$$

V11

In[273]:= **UY11Func[X, Y1];**

In[274]:= **UY11FuncDim[X, Y1] = Simplify[**
UY11Func[X, Y1] //. {UX1 → (ux1/ux1C [X[#1], Y1[#2]] &) , UY1 → (uy1/uyC [X[#1], Y1[#2]] &),
P1 → (p1/pC1 [X[#1], Y1[#2]] &), P2 → (p2/pC1 [X[#1], 0] &), Pin → (pin/pC1),
X → (x/ l), Y1 → (y/h1), K2 → (K (h1/h2)) ^ 3}, AssumptionOnScaling];

In[275]:= **TextbookEq[UY11FuncDim[*X*, Y1]]**

$$\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}\right)^2 h1^3 K^3\, pC1}$$

$$e^{-\frac{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}}\, h2^3\, pin\left(-8\sqrt{3}\left(1+e^{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}\right)\left(e^{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}-e^{\frac{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}}\right)\sqrt{\frac{h1^3 K^3}{h2^3}}-3\, a1\right.$$

$$\left(-4\, e^{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}-4\, e^{6\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}+4\, e^{\frac{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}}-4\, e^{\frac{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,(l+2x)}{l}}+8\, e^{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\left(2+\frac{x}{l}\right)}+4\, e^{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\left(4+\frac{x}{l}\right)}-\right.$$

$$4\, e^{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\left(3+\frac{2x}{l}\right)}+e^{8\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}\left(-1-\frac{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}+\frac{12\, h1^3 K^3 x^2}{h2^3\, l^2}\right)+e^{\frac{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}}$$

$$\left(-1+\frac{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}+\frac{12\, h1^3 K^3 x^2}{h2^3\, l^2}\right)+e^{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}\left(1-\frac{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}+\frac{12\, h1^3 K^3\left(-2+\frac{x^2}{l^2}\right)}{h2^3}\right)+$$

$$\left.\left.e^{\frac{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,(l+x)}{l}}\left(1+\frac{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}+\frac{12\, h1^3 K^3\left(-2+\frac{x^2}{l^2}\right)}{h2^3}\right)\right)\right)\left(-1+\frac{y}{h1}\right)^2\left(1+\frac{2y}{h1}\right)$$

6.3 Top Region - Total Flow field

P1

In[276]:= **P1[X_, Y1_] = G10FuncDim[*X*] + *K* G11FuncDim[*X*];**
TextbookEq[P1[*X*, Y1]]

$$
\begin{aligned}
&\frac{\mathrm{pin}\,(l-x)}{l\,\mathrm{pC1}}+\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\right)^2 \mathrm{h1}^6 K^5 l^2\,\mathrm{pC1}}\\
&\quad e^{-\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}\,\mathrm{h2}^3\,\mathrm{pin}\biggl(8\sqrt{3}\,\mathrm{h1}\,\mathrm{h2}\sqrt{\mathrm{h1}\,\mathrm{h2}\,K^3}\;l\biggl(e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\,l+e^{8\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\,l-\\
&\quad e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}\,l-e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,(l+x)}{l}}\,l+e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}\,(l-x)+e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(4+\frac{x}{l}\right)}\,(-l+x)\biggr)+\\
&3\,\mathrm{a1}\biggl(-4\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\,\mathrm{h2}^3 l^2-4\,e^{6\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\,\mathrm{h2}^3 l^2-4\,e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,(l+2x)}{l}}\,\mathrm{h2}^3 l^2-4\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(3+\frac{2x}{l}\right)}\,\mathrm{h2}^3 l^2+\\
&\quad 8\,e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,(l+x)}{l}}\,\mathrm{h2}^3\,l\,(l-2x)+8\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(3+\frac{x}{l}\right)}\,\mathrm{h2}^3\,l\,(l-2x)+\\
&\quad 3\,e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}\left(\mathrm{h2}^3 l^2-2\sqrt{3}\,\mathrm{h1}\,\mathrm{h2}\sqrt{\mathrm{h1}\,\mathrm{h2}\,K^3}\;l\,x+4\,\mathrm{h1}^3 K^3 x^2\right)+\\
&\quad 3\,e^{8\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\left(\mathrm{h2}^3 l^2+2\sqrt{3}\,\mathrm{h1}\,\mathrm{h2}\sqrt{\mathrm{h1}\,\mathrm{h2}\,K^3}\;l\,x+4\,\mathrm{h1}^3 K^3 x^2\right)+\\
&\quad e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}\left(24\,\mathrm{h1}^3 K^3 x\,(-l+x)+\mathrm{h2}^3\,l\,(-3\,l+7\,x)\right)+e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(4+\frac{x}{l}\right)}\\
&\left(24\,\mathrm{h1}^3 K^3 x\,(-l+x)+\mathrm{h2}^3\,l\,(-3\,l+7\,x)\right)+2\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(2+\frac{x}{l}\right)}\left(24\,\mathrm{h1}^3 K^3 (l-x)^2+\mathrm{h2}^3\,l\,(-5\,l+9\,x)\right)+\\
&\quad e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,(l+x)}{l}}\left(5\,\mathrm{h2}^3 l^2-6\sqrt{3}\,\mathrm{h1}\,\mathrm{h2}\sqrt{\mathrm{h1}\,\mathrm{h2}\,K^3}\;l\,x+12\,\mathrm{h1}^3 K^3\left(-2\,l^2+x^2\right)\right)+\\
&\quad e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\left(5\,\mathrm{h2}^3 l^2+6\sqrt{3}\,\mathrm{h1}\,\mathrm{h2}\sqrt{\mathrm{h1}\,\mathrm{h2}\,K^3}\;l\,x+12\,\mathrm{h1}^3 K^3\left(-2\,l^2+x^2\right)\right)\biggr)\biggr)
\end{aligned}
$$

U1

In[278]:= **U1[X_, Y1_] = UX10FuncDim[*X*, Y1] + *K* UX11FuncDim[*X*, Y1];**
TextbookEq[U1[*X*, Y1]]

$$
\begin{aligned}
&\frac{\text{pin}\,(\text{h1}-y)\,y}{2\,\text{h1}^2\,\text{pC1}}+\frac{1}{96\left(1+e^{4\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}}\right)^2\text{h1}^7K^5\,\text{pC1}}\\
&e^{-\frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\,x}{l}}\,\text{h2}^3\,\text{pin}\left(-8\left(1+e^{4\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}}\right)\text{h1}\left(6\,e^{4\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}}\,\text{h1}^2K^3+6\,e^{\frac{4\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\,x}{l}}\,\text{h1}^2K^3+\right.\right.\\
&\left.\sqrt{3}\,e^{\frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\,x}{l}}\,\text{h2}\sqrt{\text{h1}\,\text{h2}\,K^3}-\sqrt{3}\,e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\left(2+\frac{x}{l}\right)}\,\text{h2}\sqrt{\text{h1}\,\text{h2}\,K^3}\right)+\frac{1}{l^2}\\
&3\,\text{a1}\left(-16\,e^{\frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\,(l+x)}{l}}\,\text{h2}^3l^2-16\,e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\left(3+\frac{x}{l}\right)}\,\text{h2}^3l^2+8\sqrt{3}\,e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}}\,\text{h1}\,\text{h2}\sqrt{\text{h1}\,\text{h2}\,K^3}\,l^2+\right.\\
&8\sqrt{3}\,e^{6\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}}\,\text{h1}\,\text{h2}\sqrt{\text{h1}\,\text{h2}\,K^3}\,l^2-8\sqrt{3}\,e^{\frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\,(l+2x)}{l}}\,\text{h1}\,\text{h2}\sqrt{\text{h1}\,\text{h2}\,K^3}\,l^2-\\
&8\sqrt{3}\,e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\left(3+\frac{2x}{l}\right)}\,\text{h1}\,\text{h2}\sqrt{\text{h1}\,\text{h2}\,K^3}\,l^2+e^{\frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\,x}{l}}\,l\left(7\,\text{h2}^3l-24\,\text{h1}^3K^3(l-2x)\right)+\\
&e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\left(4+\frac{x}{l}\right)}\,l\left(7\,\text{h2}^3l-24\,\text{h1}^3K^3(l-2x)\right)-12\,e^{\frac{4\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\,x}{l}}\,\text{h1}^3K^3x\left(l-2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\,x\right)-\\
&12\,e^{8\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}}\,\text{h1}^3K^3x\left(l+2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\,x\right)+6\,e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\left(2+\frac{x}{l}\right)}\,l\left(3\,\text{h2}^3l+16\,\text{h1}^3K^3(-l+x)\right)+\\
&4\,e^{4\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}}\,\text{h1}\left(-\sqrt{3}\,\text{h2}\sqrt{\text{h1}\,\text{h2}\,K^3}\,l^2-3\,\text{h1}^2K^3\,l\,x+6\sqrt{3}\,\text{h1}^{7/2}\left(\frac{K^3}{\text{h2}}\right)^{3/2}\left(2\,l^2-x^2\right)\right)-\\
&\left.\left.4\,e^{\frac{4\sqrt{3}\sqrt{\frac{\text{h1}^3K^3}{\text{h2}^3}}\,(l+x)}{l}}\,\text{h1}\left(-\sqrt{3}\,\text{h2}\sqrt{\text{h1}\,\text{h2}\,K^3}\,l^2+3\,\text{h1}^2K^3\,l\,x+6\sqrt{3}\,\text{h1}^{7/2}\left(\frac{K^3}{\text{h2}}\right)^{3/2}\left(2\,l^2-x^2\right)\right)\right)\right)y\left(-1+\frac{y}{\text{h1}}\right)
\end{aligned}
$$

V1

In[280]:=
```
V1[X_, Y1_] = UY10FuncDim[X, Y1] + K UY11FuncDim[X, Y1];
TextbookEq[V1[X, Y1]]
```

$$\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\right)^2 \mathrm{h1}^3 K^2\,\mathrm{pC1}}$$

$$e^{-\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}\,\mathrm{h2}^3\,\mathrm{pin}\left(-8\sqrt{3}\left(1+e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\right)\left(e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}-e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}\right)\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}-3\,\mathrm{a1}\right.$$

$$\left(-4e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}-4e^{6\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}+4e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}-4e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,(l+2x)}{l}}+8e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(2+\frac{x}{l}\right)}+4e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(4+\frac{x}{l}\right)}-\right.$$

$$4e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\left(3+\frac{2x}{l}\right)}+e^{8\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\left(-1-\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}+\frac{12\,\mathrm{h1}^3 K^3 x^2}{\mathrm{h2}^3\,l^2}\right)+e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}}$$

$$\left(-1+\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}+\frac{12\,\mathrm{h1}^3 K^3 x^2}{\mathrm{h2}^3\,l^2}\right)+e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}}\left(1-\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}+\frac{12\,\mathrm{h1}^3 K^3\left(-2+\frac{x^2}{l^2}\right)}{\mathrm{h2}^3}\right)+$$

$$\left.\left.e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,(l+x)}{l}}\left(1+\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3 K^3}{\mathrm{h2}^3}}\,x}{l}+\frac{12\,\mathrm{h1}^3 K^3\left(-2+\frac{x^2}{l^2}\right)}{\mathrm{h2}^3}\right)\right)\right)\left(-1+\frac{y}{\mathrm{h1}}\right)^2\left(1+\frac{2y}{\mathrm{h1}}\right)$$

6.4 Bottom Region - Leading Order (Total Flow field)

P2

In[282]:=
```
FuncG201[X];
```

In[283]:=
```
FuncG201Dim[X] = Simplify[FuncG201[X] //. {UX2 → (ux2/ux2C [X[#1]] &),
    UY2 → (uy2/uyC [X[#1]] &), P2 → (p2/pC2 [X[#1]] &), P1 → (p1/pC2 [X[#1]] &),
    Pin → (pin/pC2), X → (x/l), Y2 → (y/h2), K2 → (K (h1/h2))^3}, AssumptionOnScaling];
```

In[284]:= **TextbookEq[FuncG201Dim[*X*]]**

$$\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}}\right)^2\mathrm{pC2}}\mathrm{pin}\left(8\left(1+e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}}\right)\left(6+\frac{\sqrt{3}\,e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\,x}{l}}}{\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}}-\frac{\sqrt{3}\,e^{-2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\left(-2+\frac{x}{l}\right)}}{\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}}+\frac{6\,e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}}\,(l-x)}{l}-\frac{6x}{l}\right)-\right.$$
$$\frac{1}{\mathrm{h1}^3K^3}3\,\mathrm{a1}\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\left(2-\frac{3x}{l}\right)}\mathrm{h2}^3\left(8\,e^{\frac{6\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\,x}{l}}-4\,e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\,(l+2x)}{l}}-4\,e^{\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\,(l+4x)}{l}}-\right.$$
$$4\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\left(-1+\frac{2x}{l}\right)}+4\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\left(-2+\frac{3x}{l}\right)}+4\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\left(2+\frac{3x}{l}\right)}-4\,e^{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\left(-1+\frac{4x}{l}\right)}+$$
$$e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\,(l+x)}{l}}\left(-1-\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\,x}{l}+\frac{12\,\mathrm{h1}^3K^3x^2}{\mathrm{h2}^3l^2}\right)+e^{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\left(-1+\frac{2x}{l}\right)}$$
$$\left(-1+\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\,x}{l}+\frac{12\,\mathrm{h1}^3K^3x^2}{\mathrm{h2}^3l^2}\right)+e^{\frac{4\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\,x}{l}}\left(1-\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\,x}{l}+\frac{12\,\mathrm{h1}^3K^3\left(-2+\frac{x^2}{l^2}\right)}{\mathrm{h2}^3}\right)+$$
$$\left.\left.e^{\frac{8\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\,x}{l}}\left(1+\frac{2\sqrt{3}\sqrt{\frac{\mathrm{h1}^3K^3}{\mathrm{h2}^3}}\,x}{l}+\frac{12\,\mathrm{h1}^3K^3\left(-2+\frac{x^2}{l^2}\right)}{\mathrm{h2}^3}\right)\right)\right)$$

U2

In[285]:= **UX20FuncSub[*X*, Y2];**

In[286]:=
```
UX20FuncSubDim[X, Y2] = Simplify[
    UX20FuncSub[X, Y2] //. {UX2 → (ux2/ux2C [X[#1], Y2[#2]] &), UY2 → (uy2/uyC [X[#1], Y2[#2]] &),
      P2 → (p2/pC2 [X[#1], Y2[#2]] &), P1 → (p1/pC2 [X[#1], 0] &), Pin → (pin/pC2),
      X → (x/l), Y2 → (y/h2), K2 → (K (h1/h2))^3}, AssumptionOnScaling];
```

In[287]:= **TextbookEq[UX20FuncSubDim[X, Y2]]**

$$-\frac{1}{8\left(1+e^{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}\right)^2 h1^{3/2}\, h2^{7/2}\sqrt{K^3}\, l^3\, pC2}$$

$$e^{-\frac{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}}\, pin\left(4\left(1+e^{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}\right)\left(e^{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}-e^{\frac{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}}\right)\left(-1+e^{\frac{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}}\right) h1\, h2\sqrt{h1\, h2\, K^3}\, l^2+\right.$$
$$a1\left(2\sqrt{3}\, e^{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}\, h2^3\, l^2+2\sqrt{3}\, e^{6\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}\, h2^3\, l^2-2\sqrt{3}\, e^{\frac{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}(l+2x)}{l}}\, h2^3\, l^2-\right.$$
$$2\sqrt{3}\, e^{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\left(3+\frac{2x}{l}\right)} h2^3\, l^2+3\, e^{8\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}\, h1\, x\left(3\, h2\sqrt{h1\, h2\, K^3}\, l-2\sqrt{3}\, h1^2 K^3 x\right)+$$
$$3\, e^{\frac{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}}\, h1\, x\left(3\, h2\sqrt{h1\, h2\, K^3}\, l+2\sqrt{3}\, h1^2 K^3 x\right)+$$
$$e^{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}\left(-\sqrt{3}\, h2^3\, l^2+9\, h1\, h2\sqrt{h1\, h2\, K^3}\, l\, x+6\sqrt{3}\, h1^3 K^3\left(2\, l^2-x^2\right)\right)+$$
$$\left.\left.e^{\frac{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}(l+x)}{l}}\left(\sqrt{3}\, h2^3\, l^2+9\, h1\, h2\sqrt{h1\, h2\, K^3}\, l\, x+6\sqrt{3}\, h1^3 K^3\left(-2\, l^2+x^2\right)\right)\right)\right) y\,(h2\,(l+a1\, x)+l\, y)$$

V2

In[288]:= **UY20FuncSub[X, Y2];**
UY20FuncSubDim[X, Y2] = Simplify[
UY20FuncSub[X, Y2] //. {UX2 → (ux2/ux2C [X[#1], Y2[#2]] &), UY2 → (uy2/uyC [X[#1], Y2[#2]] &),
P2 → (p2/pC2 [X[#1], Y2[#2]] &), P1 → (p1/pC2 [X[#1], 0] &), Pin → (pin/pC2),
X → (x/l), Y2 → (y/h2), K2 → (K (h1/h2))^3}, AssumptionOnScaling];

In[290]:= **TextbookEq[UY20FuncSubDim[X, Y2]]**

$$-\frac{1}{48\left(1+e^{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}\right)^2\sqrt{\frac{h1^3 K^3}{h2^3}}\, pC2}$$

$$e^{\frac{2\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}(l-x)}{l}}\, pin\left(1+\frac{a1\, x}{l}+\frac{y}{h2}\right)\left(\frac{1}{l}3\, a1^3\, x\left(2\left(\sqrt{3}-\frac{2\sqrt{\frac{h1^3 K^3}{h2^3}}\, x}{l}\right)+2\, e^{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}}\left(\sqrt{3}-\frac{2\sqrt{\frac{h1^3 K^3}{h2^3}}\, x}{l}\right)-\right.\right.$$
$$2\, e^{\frac{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}\,x}{l}}\left(\sqrt{3}+\frac{2\sqrt{\frac{h1^3 K^3}{h2^3}}\, x}{l}\right)-2\, e^{\frac{4\sqrt{3}\sqrt{\frac{h1^3 K^3}{h2^3}}(l+x)}{l}}\left(\sqrt{3}+\frac{2\sqrt{\frac{h1^3 K^3}{h2^3}}\, x}{l}\right)+$$

$$\frac{4\, e^{6\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}}\,\text{h1}^{3/2}\sqrt{K^3}\,x\left(3\,\text{h2}^3\,l^2-4\sqrt{3}\,\text{h1}\,\text{h2}\sqrt{\text{h1}\,\text{h2}\,K^3}\,l\,x+3\,\text{h1}^3 K^3 x^2\right)}{\text{h2}^{9/2}\,l^3}+$$

$$\frac{4\, e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\left(-1+\frac{2x}{l}\right)}\,\text{h1}^{3/2}\sqrt{K^3}\,x\left(3\,\text{h2}^3\,l^2+4\sqrt{3}\,\text{h1}\,\text{h2}\sqrt{\text{h1}\,\text{h2}\,K^3}\,l\,x+3\,\text{h1}^3 K^3 x^2\right)}{\text{h2}^{9/2}\,l^3}-$$

$$e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}}\left(\sqrt{3}-\frac{14\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\,x}{l}-\frac{12\left(\frac{\text{h1}^3 K^3}{\text{h2}^3}\right)^{3/2} x\left(-2\,l^2+x^2\right)}{l^3}+\frac{4\sqrt{3}\,\text{h1}^3 K^3\left(-3+\frac{4x^2}{l^2}\right)}{\text{h2}^3}\right)+$$

$$e^{\frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}(l+2x)}{l}}\left(\sqrt{3}+\frac{14\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\,x}{l}+\frac{12\left(\frac{\text{h1}^3 K^3}{\text{h2}^3}\right)^{3/2} x\left(-2\,l^2+x^2\right)}{l^3}+\frac{4\sqrt{3}\,\text{h1}^3 K^3\left(-3+\frac{4x^2}{l^2}\right)}{\text{h2}^3}\right)\Bigg)\Bigg)+$$

$$\frac{8\sqrt{3}\,e^{-2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}}\left(1+e^{4\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}}\right)\left(e^{4\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}}-e^{\frac{4\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\,x}{l}}\right)\text{h1}^3 K^3\left(\text{h2}^2-\text{h2}\,y-2\,y^2\right)}{\text{h2}^5}+\frac{1}{\text{h2}^4\,l^3}$$

$$\text{a1}^2\Bigg(24\,e^{\frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}(l+x)}{l}}\,\text{h1}^{3/2}\,\text{h2}^{5/2}\sqrt{K^3}\,l^2 x+12\,e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\left(-1+\frac{x}{l}\right)}\,\text{h1}^{3/2}\,\text{h2}^{5/2}\sqrt{K^3}\,l^2 x+12\,e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\left(3+\frac{x}{l}\right)}$$

$$\text{h1}^{3/2}\,\text{h2}^{5/2}\sqrt{K^3}\,l^2 x+6\,e^{4\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}}\,\text{h2}^3\,l^2\left(\sqrt{3}\,l\,(\text{h2}-y)-2\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\,x\,(2\,\text{h2}-y)\right)-$$

$$6\,e^{\frac{4\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\,x}{l}}\,\text{h2}^3\,l^2\left(\sqrt{3}\,l\,(\text{h2}-y)+2\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\,x\,(2\,\text{h2}-y)\right)-$$

$$6\,e^{\frac{4\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}(l+x)}{l}}\,\text{h2}^3\,l^2\left(\sqrt{3}\,l\,(\text{h2}-y)+2\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\,x\,(2\,\text{h2}-y)\right)+$$

$$e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}}\Bigg(-2\sqrt{3}\,\text{h1}^3 K^3\,l\left(x^2\,(35\,\text{h2}-24\,y)-18\,l^2\,(\text{h2}-y)\right)+3\,(\text{h1}\,\text{h2})^{3/2}\sqrt{K^3}\,l^2 x\,(15\,\text{h2}-14\,y)-$$

$$3\sqrt{3}\,\text{h2}^3\,l^3\,(\text{h2}-y)+36\,\text{h2}^3\left(\frac{\text{h1}^3 K^3}{\text{h2}^3}\right)^{3/2} x\left(-2\,l^2+x^2\right)(2\,\text{h2}-y)\Bigg)+$$

$$e^{\frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}(l+2x)}{l}}\Bigg(2\sqrt{3}\,\text{h1}^3 K^3\,l\left(x^2\,(35\,\text{h2}-24\,y)-18\,l^2\,(\text{h2}-y)\right)+3\,(\text{h1}\,\text{h2})^{3/2}\sqrt{K^3}\,l^2 x\,(15\,\text{h2}-14\,y)+$$

$$3\sqrt{3}\,\text{h2}^3\,l^3\,(\text{h2}-y)+36\,\text{h2}^3\left(\frac{\text{h1}^3 K^3}{\text{h2}^3}\right)^{3/2} x\left(-2\,l^2+x^2\right)(2\,\text{h2}-y)\Bigg)+$$

$$6\sqrt{3}\,\text{h2}^3\,l^3\,(\text{h2}-y)-12\,(\text{h1}\,\text{h2})^{3/2}\sqrt{K^3}\,l^2 x\,(2\,\text{h2}-y)+e^{6\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\,x\Big(33\,\text{h2}^4\,l^2-$$

$$2\sqrt{3}\,(\text{h1}\,\text{h2})^{3/2}\sqrt{K^3}\,l\,x\,(35\,\text{h2}-24\,y)+36\,\text{h1}^3 K^3 x^2\,(2\,\text{h2}-y)-36\,\text{h2}^3\,l^2\,y\Big)+e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\left(-1+\frac{2x}{l}\right)}$$

$$\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\, x\left(33\,\text{h2}^4\, l^2 + 2\,\sqrt{3}\,(\text{h1}\,\text{h2})^{3/2}\,\sqrt{K^3}\, l\, x\,(35\,\text{h2} - 24\,y) + 36\,\text{h1}^3 K^3 x^2\,(2\,\text{h2} - y) - 36\,\text{h2}^3\, l^2\, y\right)\Bigg) -$$

$$\text{a1}\,\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\left(24\, e^{\frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}(l+x)}{l}}\left(-1 + \frac{y}{\text{h2}}\right) + 12\, e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\left(-1+\frac{x}{l}\right)}\left(-1 + \frac{y}{\text{h2}}\right) +\right.$$

$$12\, e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\left(3+\frac{x}{l}\right)}\left(-1 + \frac{y}{\text{h2}}\right) + \frac{12\left(\text{h2}^2 - \text{h2}\, y - 2\, y^2\right)}{\text{h2}^2} + \frac{12\, e^{4\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}}\left(\text{h2}^2 - \text{h2}\, y - 2\, y^2\right)}{\text{h2}^2} +$$

$$\frac{12\, e^{\frac{4\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\, x}{l}}\left(\text{h2}^2 - \text{h2}\, y - 2\, y^2\right)}{\text{h2}^2} + \frac{12\, e^{\frac{4\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}(l+x)}{l}}\left(\text{h2}^2 - \text{h2}\, y - 2\, y^2\right)}{\text{h2}^2} + e^{6\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}}$$

$$\left(3 - \frac{3\,y}{\text{h2}} + \frac{18\,y^2}{\text{h2}^2} - \frac{36\,\text{h1}^3 K^3 x^2\left(\text{h2}^2 - \text{h2}\, y - 2\, y^2\right)}{\text{h2}^5\, l^2} - \frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\, x\left(-7 + \frac{11\,y}{\text{h2}} + \frac{30\,y^2}{\text{h2}^2}\right)}{l}\right) + e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}}$$

$$\left(-3 + \frac{3\,y}{\text{h2}} + \frac{30\,y^2}{\text{h2}^2} + \frac{36\,\text{h1}^3 K^3\left(-2 + \frac{x^2}{l^2}\right)\left(-1 + \frac{y}{\text{h2}} + \frac{2\,y^2}{\text{h2}^2}\right)}{\text{h2}^3} - \frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\, x\left(-7 + \frac{11\,y}{\text{h2}} + \frac{30\,y^2}{\text{h2}^2}\right)}{l}\right) +$$

$$e^{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\left(-1+\frac{2\,x}{l}\right)}\left(3 - \frac{3\,y}{\text{h2}} + \frac{18\,y^2}{\text{h2}^2} - \frac{36\,\text{h1}^3 K^3 x^2\left(\text{h2}^2 - \text{h2}\, y - 2\, y^2\right)}{\text{h2}^5\, l^2} +\right.$$

$$\left.\frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\, x\left(-7 + \frac{11\,y}{\text{h2}} + \frac{30\,y^2}{\text{h2}^2}\right)}{l}\right) + e^{\frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}(l+2\,x)}{l}}$$

$$\left.\left(-3 + \frac{3\,y}{\text{h2}} + \frac{30\,y^2}{\text{h2}^2} + \frac{36\,\text{h1}^3 K^3\left(-2 + \frac{x^2}{l^2}\right)\left(-1 + \frac{y}{\text{h2}} + \frac{2\,y^2}{\text{h2}^2}\right)}{\text{h2}^3} + \frac{2\sqrt{3}\sqrt{\frac{\text{h1}^3 K^3}{\text{h2}^3}}\, x\left(-7 + \frac{11\,y}{\text{h2}} + \frac{30\,y^2}{\text{h2}^2}\right)}{l}\right)\right)\Bigg)$$