HODGE REPRESENTATIONS: APPENDICES

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Abstract. Appendix to [HR20].

Conflicts of Interest. The authors declare none.

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Contents

A. Analysis of level two Hodge structures

This appendix contains proofs of the theorems in [HR20]. The general argument is outlined in §A.1. References to sections/equations/tables/propositions in this appendix all begin with letters ("A" or "B"); while references to the main document [HR20] begin with numbers (eg. Table 3.1 may be found in the main document). Theorems 4.1 and 4.3 of [HR20] are proved simultaneously in §§A.2–A.7; Theorem 4.4 of [HR20] is proved in §A.8; and Theorem 4.6 of [HR20] is proved in §A.9.

A.1. **Outline of the arguments.** The proofs of Theorems 4.1 and 4.3 proceed by considering each of the cases listed in [HR20, Table 3.1]. Given the pair $(\mathfrak{g}_\mathbb{C}, \mathtt{E})$ it suffices to determine when there exists an irreducible $\mathfrak{g}_\mathbb{C}$–representation U_μ of highest weight $\mu \in \mathfrak{h}^*$, and $c \in \mathbb{Q}$ satisfying the conditions of Theorem 3.1 for the specified $\mathbf{h}_\phi$. First note that U_μ has either two or three nontrivial $\mathtt{E}$–eigenvalues; equivalently (Remark 2.21),

$$(\mu + \mu^*)(\mathtt{E}) \ \in \ \{1, 2\}\,. \tag{A.1}$$

This gives us the following three possibilities (cf. §§2.5 and 2.7):

(a) If U_μ is real, then $c = 0$ (§2.7(a)) and it is necessary and sufficient that the $\mathtt{E}$–eigenspace decomposition (equivalently, the Hodge decomposition) of $V_\mathbb{C} = U_\mu$ be

$$V^{2,0} \oplus V^{1,1} \oplus V^{0,2} \ = \ U_1 \oplus U_0 \oplus U_{-1}\,,$$

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with $\dim U_{\pm 1} = h^{2,0}_\phi \in \{1,2\}$. In particular, $\mu(\mathtt{E}) = 1$.

(b) If U_μ is complex or quaternionic, so that $V_{\mathbb{C}} = U_\mu \oplus U_\mu^*$, and there are three nontrivial $\mathtt{E}$–eigenvalues, so that the $\mathtt{E}$–eigenspace decompositions are

$$\begin{array}{rcl} U_\mu & = & U_{\mu(\mathtt{E})} \oplus U_{\mu(\mathtt{E})-1} \oplus U_{\mu(\mathtt{E})-2} \\ U_\mu^* & = & U_{2-\mu(\mathtt{E})} \oplus U_{1-\mu(\mathtt{E})} \oplus U_{-\mu(\mathtt{E})} \,. \end{array}$$

Then we are looking for $c \in \mathbb{Q}$ so that

$$\begin{array}{c|ccc} V_{\mathbb{C}} & V^{2,0} & V^{1,1} & V^{0,2} \\ \hline U_\mu & U_{\mu(\mathtt{E})} & U_{\mu(\mathtt{E})-1} & U_{\mu(\mathtt{E})-2} \\ U_\mu^* & U_{2-\mu(\mathtt{E})} & U_{1-\mu(\mathtt{E})} & U_{-\mu(\mathtt{E})} \,. \end{array}$$

Equivalently, $\mu(\mathtt{E}) + c = 1$ and $2 - \mu(\mathtt{E}) - c = 1$. That is,

$$c \;=\; 1 - \mu(\mathtt{E})\,.$$

Note that each of the eigenspaces $U_{\pm\mu(\mathtt{E})}$ and $U_{\pm(\mu(\mathtt{E})-2)}$ must have dimension one, and we have $h^{2,0} = 2$. (In particular, this case will not appear in Theorem 4.1.)

(c) Suppose U_μ is complex, so that $V_{\mathbb{C}} = U_\mu \oplus U_\mu^*$, and there are two nontrivial $\mathtt{E}$–eigenvalues, so that the $\mathtt{E}$–eigenspace decompositions are

$$\begin{array}{rcl} U_\mu & = & U_{\mu(\mathtt{E})} \oplus U_{\mu(\mathtt{E})-1} \\ U_\mu^* & = & U^*_{1-\mu(\mathtt{E})} \oplus U^*_{-\mu(\mathtt{E})} \,. \end{array}$$

We are looking for $c \in \mathbb{Q}$ so that either

$$\begin{array}{c|ccc} V_{\mathbb{C}} & V^{2,0} & V^{1,1} & V^{0,2} \\ \hline U_\mu & U_{\mu(\mathtt{E})} & U_{\mu(\mathtt{E})-1} & \\ U_\mu^* & & U^*_{1-\mu(\mathtt{E})} & U^*_{-\mu(\mathtt{E})} \end{array}\,.$$

or

$$\begin{array}{c|ccc} V_{\mathbb{C}} & V^{2,0} & V^{1,1} & V^{0,2} \\ \hline U_\mu & & U_{\mu(\mathtt{E})} & U_{\mu(\mathtt{E})-1} \\ U_\mu^* & U^*_{1-\mu(\mathtt{E})} & U^*_{-\mu(\mathtt{E})} & \end{array}\,.$$

Equivalently, either

$$1 \;=\; \mu(\mathtt{E}) + c \quad \text{and} \quad \dim_{\mathbb{C}} U_{\mu(\mathtt{E})} = h^{2,0}_\phi \in \{1,2\}\,,$$

or

$$\mu(\mathtt{E}) \;=\; -c \quad \text{and} \quad \dim_{\mathbb{C}} U^*_{1-\mu(\mathtt{E})} = h^{2,0}_\phi \in \{1,2\}\,.$$

The proofs of Theorems 4.1 and 4.3 now proceed by applying the observations of this section to each pair $(\mathfrak{g}_{\mathbb{C}}, \mathtt{E})$ corresponding to a row of Table 3.1.

We now turn to the simultaneous proofs of Theorems 4.1 and 4.3 in §§A.2–A.7, followed by the proofs of Theorems 4.4 and 4.6 in §A.8 and §A.9, respectively.

A.2. **Grassmannian Hodge domains.** We begin with the first row of Table 3.1 and the pair $(\mathfrak{g}_\mathbb{C}, \mathtt{E}) = (\mathfrak{sl}(a+b,\mathbb{C})\,,\, \mathtt{A}^a)$.[1]

The standard representation $U_{\omega_1} = \mathbb{C}^{a+b}$ of $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}_{a+b}\mathbb{C}$ admits a decomposition $\mathbb{C}^{a+b} = A \oplus B$ with $\dim A = a$ and $\dim B = b$ and such that A is an eigenspace of $\mathtt{E}$ with eigenvalue $b/(a+b)$, and B is an eigenspace with eigenvalue $-a/(a+b)$. It will be helpful to note that the $\mathtt{E}$–eigenspace decomposition of $\bigwedge^i\mathbb{C}^{a+b}$ is

$$\textstyle\bigwedge^i(A \oplus B) \ = \ \bigoplus_{\alpha+\beta=i} (\bigwedge^\alpha A) \otimes (\bigwedge^\beta B)\,. \tag{A.2}$$

Fix bases $\{e_1, \ldots, e_a\}$ and $\{e_{a+1}, \ldots, e_{a+b}\}$ of A and B, respectively.

We assume throughout §A.2 that $a + b = i + j = k + \ell = r + 1$. Consulting §A.1 and §B.2.1, we see that the pair $(\mu, \mathtt{E} = \mathtt{A}^a)$ must be one of the following:

(i) $a = 1$ and $\mu = \omega_i$, any $1 \le i \le r$;
(ii) $a = 1$ and $\mu = \omega_i + \omega_k$, any $1 \le i, k \le r$;
(iii) $a = 2$ and $\mu = \omega_i$, any $2 \le i \le r - 1$;
(iv) $\mu \in \{\omega_1, 2\omega_1\}$, any $2 \le a \le r - 1$;
(v) $\mu = \omega_2$ and any $2 \le a \le r - 1$.

(This list suppresses some cases that are essentially symmetric with those already listed. For example $\mathtt{E} = \mathtt{A}^r$ and $\mu = \omega_i$ is symmetric with (i).) We proceed to consider each of these five cases.

(i) Consulting (A.2) we see that

$$U_{\omega_i} \ = \ \textstyle\bigwedge^i(A \oplus B) \ = \ \big(A \otimes \bigwedge^{i-1} B\big) \ \oplus \ \big(\bigwedge^i B\big)\ .$$

These eigenspaces have dimensions $\left(\binom{r}{i-1}, \binom{r}{i}\right)$. In order to realize a Hodge representation with $h^{2,0}_\phi \in \{1, 2\}$, one of these dimensions must be 1 or 2.

The first dimension will be one if and only if $r = 1$ (which forces $i = 1$). But in this case the representation U_μ is real, and so the resulting Hodge representation will be weight $n = 1$, not the desired weight $n = 2$.

The second dimension will be one if and only if $i = r$. Then the dimensions of the $\mathtt{E}$–eigenspaces of U_{ω_r} and $U^*_{\omega_r} = U_{\omega_1}$ are $(r, 1)$ and $(1, r)$, respectively. The eigenvalue for $\bigwedge^r B \subset U_{\omega_r}$ is $-r/(r + 1)$. So setting $c = -1/(r + 1)$ gives us a Hodge representation with eigenvalues $\mathbf{h}_\phi = (1, 2r, 1)$, yielding Theorem 4.1(ii).

The first dimension will be two if and only if $i = r = 2$. In this case the dimensions are $(2, 1)$, and $U_\mu = U_{\omega_2}$ is complex with $U^*_\mu = U_{\omega_1} = \mathbb{C}^3$. The $\mathtt{E}$–eigenspaces of U^*_μ have dimensions $(1, 2)$. We have $\mu(\mathtt{E}) = 1/3$. Setting $c = -1/3$ yields a special case of Theorem 4.1(ii), and setting $c = 2/3$ yields a special case of Theorem 4.3(ii) (Remark 3.4).

The second dimension will be two if and only if $r = 2$ and $i = 1$. In this case the dimensions are $(1, 2)$, and $U_\mu = U_{\omega_1} = \mathbb{C}^3$ is complex with $U^*_\mu = U_{\omega_2} = \bigwedge^2\mathbb{C}^3$. The $\mathtt{E}$–eigenspaces of U^*_μ have dimensions $(2, 1)$. We have $\mu(\mathtt{E}) = 2/3$. Setting $c = 1/3$ yields a

[1] Despite what the reader might anticipate, this case/row is the most tedious and painstaking to work through. This is essentially due to the numerically more complicated relationship between the roots (dual to the basis $\mathtt{A}^a$ for the grading elements) and the weights (i.e. the complexity in the Cartan matrix) for $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(a + b, \mathbb{C})$. The other cases §§A.3–A.7 are easier to analyze.

special case of Theorem 4.1(ii) (Remark 3.4). Setting $c = -2/3$ yields a special case of Theorem 4.3(ii).

(ii) We have $U_{\omega_i+\omega_k} \subset (\bigwedge^i \mathbb{C}^{r+1}) \otimes (\bigwedge^k \mathbb{C}^{r+1})$, with the latter having three distinct $\mathtt{E}$–eigenspaces

$$\begin{aligned}(\textstyle\bigwedge^i \mathbb{C}^{r+1}) \otimes (\bigwedge^k \mathbb{C}^{r+1}) \quad &= \quad \Big(A \otimes A \otimes (\textstyle\bigwedge^{i-1} B) \otimes (\bigwedge^{k-1} B)\Big) \\ &\oplus \; \left\{ \begin{array}{l} (A \otimes (\bigwedge^{i-1} B) \otimes (\bigwedge^k B)) \\ (A \otimes (\bigwedge^{i} B) \otimes (\bigwedge^{k-1} B)) \end{array} \right. \\ &\oplus \; \Big((\textstyle\bigwedge^i B) \otimes (\bigwedge^k B)\Big)\end{aligned}$$

The product $(e_1 \wedge \cdots \wedge e_i) \otimes (e_1 \wedge \cdots \wedge e_k) \in A \otimes A \otimes (\bigwedge^{i-1} B) \otimes (\bigwedge^{k-1} B)$ is a highest weight vector of $U_{\omega_i+\omega_k}$. Without loss of generality $i \le k$. The products

$$(e_1 \wedge \cdots \wedge e_i) \otimes (e_1 \wedge \cdots \wedge e_h)\,, \quad k \le h \le r+1\,,$$

are all elements of the first eigenspace $U_{\mu(\mathtt{E})} \subset A \otimes A \otimes (\bigwedge^{i-1} B) \otimes (\bigwedge^{k-1} B)$. Because this eigenspace may have dimension at most $h^{2,0}_\phi \le 2$, we see that $k = r$ (and the eigenspace has dimension at least 2). Likewise

$$(e_1 \wedge \cdots \wedge e_h) \otimes (e_1 \wedge \cdots \wedge e_r)\,, \quad i \le h \le r\,,$$

are also elements of this eigenspace; and dimension/Hodge number considerations again force $i = k = r$. The representation $U_{2\omega_r}$ is complex, unless $r = 1$; if complex, then the associated Hodge representation has $h^{2,0}_\phi > 2$, which is too large. So we must have $r = 1$, in which case $V_\mathbb{C} = U_{2\omega_1} = \mathrm{Sym}^2\mathbb{C}^2$ is real and we have $\mathbf{h}_\phi = (1,1,1)$. However, under the isomorphism $\mathfrak{sl}_2\mathbb{C} \simeq \mathfrak{so}(3,\mathbb{C})$, this is a special case of Theorem 4.1(i).

(iii) In this case we have $\mathtt{E}$–eigenspace decomposition

$$U_{\omega_i} \;=\; \textstyle\bigwedge^i(\mathbb{C}^{2+b}) \;=\; \Big((\bigwedge^2 A) \otimes (\bigwedge^{i-2} B)\Big) \;\oplus\; \Big(A \otimes (\bigwedge^{i-1} B)\Big) \;\oplus\; (\bigwedge^i B)\,.$$

The condition that the first and third eigenspaces $(\bigwedge^2 A) \otimes (\bigwedge^{i-2} B)$ and $\bigwedge^i B$ have dimensions 1 or 2 forces $i = b = 2$. Then $U_\mu = U_{\omega_1}$ is self-dual and real. This is a special case of Theorem 4.1(i) under the isomorphism $\mathfrak{sl}(4,\mathbb{C}) \simeq \mathfrak{so}(6,\mathbb{C})$.

(iv) If $\mu = \omega_1$, then $U_\mu = \mathbb{C}^{a+b} = A \oplus B$ is the standard representation. Recalling the discussion at the beginning of this section we see that we must have either $a = 2$ or $b = r + 1 - a = 2$. Taking $c = 2/(r+1)$ if $a = 2$, and $c = -2/(r+1)$ if $b = 2$, yields $\mathbf{h}_\phi = (2, 2r-2, 2)$ and Theorem 4.3(ii) (Remark 3.4).

If $\mu = 2\omega_1$, then $U_\mu = \mathrm{Sym}^2\mathbb{C}^{a+b} = (\mathrm{Sym}^2 A) \oplus (A \otimes B) \oplus (\mathrm{Sym}^2 B)$. In this case $\dim_\mathbb{C} \mathrm{Sym}^2 A \ge 3 > h^{2,0}_\phi$ is too large.

(v) If $\mu = \omega_2$, then $U_\mu = \bigwedge^2 \mathbb{C}^{a+b} = (\bigwedge^2 A) \oplus (A \otimes B) \oplus (\bigwedge^2 B)$. The first and third eigenspaces $\bigwedge^2 A$ and $\bigwedge^2 B$ are constrained to have dimension at most $h^{2,0}_\phi \le 2$. This forces $a = b = 2$. In this case U_μ is real and we have Hodge numbers $(1,4,1)$. This is the special case of Theorem 4.1(i) that we encountered above in part (iii) of the proof.

A.3. **Quadric hypersurface Hodge domains.** We next consider the second row of Table 3.1 and the pair $(\mathfrak{g}_\mathbb{C}, \mathtt{E}) = (\mathfrak{so}(d+2,\mathbb{C})\,,\, \mathtt{A}^1)$. Here we may assume that either $d=3$ or $d\geq 5$ (else we are in the case considered in §A.2).

A.3.1. *Period domains.* If $\mu=\omega_1$, so that $U_\mu = \mathbb{C}^{d+2}$ is the standard representation, and real with respect to $(\mathfrak{g}_\mathbb{C}, \mathtt{E})$, then $V_\mathbb{C} = U_{\omega_1}$ has eigenspace decomposition $\mathbb{C}\oplus\mathbb{C}^d\oplus\mathbb{C}$ with eigenvalues $(1,d,1)$. Of course, in this case the Hodge domain is the period domain $\mathcal{D}$ parameterizing Q–polarized Hodge structures with Hodge numbers $\mathbf{h}=(1,d,1)$.

A.3.2. *Exterior powers.* For the analysis that follows, it will be helpful to make the following observations about exterior powers of the standard representation. Given $2\leq i\leq r-1\leq \frac{1}{2}d$, the representation $\bigwedge^i\mathbb{C}^{d+2}$ is real, defines a Hodge representation, and has $\mathtt{E}$–eigenspace decomposition

$$\begin{aligned}
\textstyle\bigwedge^i(\mathbb{C}\oplus\mathbb{C}^d\oplus\mathbb{C}) \quad &= \quad \Big(\mathbb{C}\otimes(\textstyle\bigwedge^{i-1}\mathbb{C}^d)\Big)\\
&\oplus\ \Big((\mathbb{C}\otimes(\textstyle\bigwedge^{i-2}\mathbb{C}^d)\otimes\mathbb{C})\ \oplus\ (\textstyle\bigwedge^i\mathbb{C}^d)\Big)\\
&\oplus\ \Big((\textstyle\bigwedge^{i-1}\mathbb{C}^d)\otimes\mathbb{C}\Big)\,.
\end{aligned}$$

The dimension $h^{2,0}_\phi$ of the first eigenspace $\mathbb{C}\otimes(\bigwedge^{i-1}\mathbb{C}^d)$ is $\binom{d}{i-1}$. We have $h^{2,0}_\phi\in\{1,2\}$ if and only if $i=2$ and $d=2$. But we are assuming $d\geq 3$.

We assume $\mu\neq\omega_1$ for the remainder of §A.3. (The case $\mu=\omega_1$ is treated in §A.3.1.) The representation theory of $\mathfrak{g}_\mathbb{C}=\mathfrak{so}(d+2,\mathbb{C})$ depends on the parity of d; we begin with d odd.

A.3.3. *The case of d odd.* Assume $d\equiv 1 \bmod 2$. Consulting (A.1) and §B.2.2, we see that either $\mu=\omega_i$ with $2\leq i\leq r-1$, or $\mu\in\{\omega_r, 2\omega_r\}$. In the first case we have $U_{\omega_i}=\bigwedge^i\mathbb{C}^{d+2}$, which is treated in §A.3.2.

- The representation $U_{2\omega_r}=\bigwedge^r U_{\omega_1}=\bigwedge^r\mathbb{C}^{d+2}$ has $\mathtt{E}$–eigenspace decomposition

$$U_{2\omega_r}\ =\ \Big(\mathbb{C}\otimes(\textstyle\bigwedge^{r-1}\mathbb{C}^d)\Big)\ \oplus\ \textstyle\bigwedge^r\mathbb{C}^d\ \oplus\ \Big((\textstyle\bigwedge^{r-1}\mathbb{C}^d)\otimes\mathbb{C}\Big)\,.$$

The resulting Hodge representation has $h^{2,0}_\phi\geq \dim_\mathbb{C}\bigwedge^{r-1}\mathbb{C}^d\geq 3$, which is too large.

- Likewise, the dimensions $(2^{r-1},2^{r-1})$ of the $\mathtt{E}$–eigenspaces in the spinor representation U_{ω_r} are to large, unless $r=2$. But in this case that representation is real, and the Hodge representation is of weight 1 (§A.1(a)).

A.3.4. *The case of d even.* Assume $d\equiv 0\bmod 2$. Consulting (A.1) and §B.2.2, we see that either $\mu=\omega_i$ with $2\leq i\leq r-2$, or $\mu\in\{\omega_{r-1},\omega_r\}\cup\{2\omega_{r-1},\omega_{r-1}+\omega_r,2\omega_r\}$. In the first case we have $U_{\omega_i}=\bigwedge^i\mathbb{C}^{d+2}$, which is treated in §A.3.2. Likewise, $U_{\omega_{r-1}+\omega_r}=\bigwedge^{r-1}\mathbb{C}^{d+2}$ is treated in §A.3.2.

- The cases $\mu=\omega_{r-1}$ and $\mu=\omega_r$ are symmetric, so we treat $\mu=\omega_r$ here. The half-spin representation U_{ω_r} decomposes into two $\mathtt{E}$–eigenspaces of dimensions $(2^{r-2},2^{r-2})$. Since $r\geq 4$, these dimensions are too large to realize a Hodge representation (as in §A.1) with $h^{2,0}_\phi\in\{1,2\}$.

• Similarly the cases $\mu = 2\omega_{r-1}$ and $\mu = 2\omega_r$ are symmetric, and we treat $\mu = 2\omega_r$ here. We have $\bigwedge^r \mathbb{C}^{d+2} = \bigwedge^r \mathbb{C}^{2r} = U_{2\omega_{r-1}} \oplus U_{2\omega_r}$. The representation $U_{2\omega_r}$ decomposes into three $\mathtt{E}$–eigenspaces, the first and last of which have dimension $\frac{1}{2}\binom{2r-2}{r-1}$. Again, since $r \geq 4$, these dimensions are too large to realize a Hodge representation (as in §A.1) with $h^{2,0}_\phi \in \{1,2\}$.

A.4. **Lagrangian grassmannian Hodge domains.** Consider the third row of Table 3.1 and the pair $(\mathfrak{g}_\mathbb{C}, \mathtt{E}) = (\mathfrak{sp}(2r,\mathbb{C})\,,\, \mathtt{A}^r)$. Here we may assume $r \geq 3$ (else we are in the case considered in §A.3.) Consulting (A.1), §A.1(a) and §B.2.3, we see that μ must be one of $2\omega_1, \omega_2$; in each case $U_\mathbb{C}$ is real. The $\mathtt{E}$–eigenspace decomposition of the standard representation $U_{\omega_1} = \mathbb{C}^{2r}$ is $\mathbb{C}^r \oplus \mathbb{C}^r$; in particular, the dimensions of the eigenspaces are (r,r).

• In the case that $\mu = 2\omega_1$, the representation $U_{2\omega_1} = \mathrm{Sym}^2\mathbb{C}^r$ has $\mathtt{E}$–eigenspace decomposition $(\mathrm{Sym}^2\mathbb{C}^r) \oplus (\mathbb{C}^r \to \mathbb{C}^r) \oplus (\mathrm{Sym}^2\mathbb{C}^r)$. The dimensions of the eigenspaces are $(\frac{1}{2}r(r+1)\,,\, r^2\,,\, \frac{1}{2}r(r+1))$. The requirement $\frac{1}{2}r(r+1) = h^{2,0}_\phi \in \{1,2\}$ forces $r = 1$, a contradiction.

• In the case that $\mu = \omega_2$, we have $U_{\omega_2} \oplus \mathrm{span}_\mathbb{C}\{Q\} = \bigwedge^2\mathbb{C}^{2r}$, and the dimensions of the $\mathtt{E}$–eigenspaces are $(\frac{1}{2}r(r-1)\,,\, r^2-1\,,\, \frac{1}{2}r(r-1))$. The requirement $\frac{1}{2}r(r-1) = h^{2,0}_\phi \in \{1,2\}$ forces $r = 2$, yielding $\mathbf{h}_\phi = (1,3,1)$. This case is covered by Theorem 4.1(i).

A.5. **Spinor Hodge domains.** Let $(\mathfrak{g}_\mathbb{C}, \mathtt{E}) = (\mathfrak{so}(2r,\mathbb{C})\,,\, \mathtt{A}^r)$. We may assume without loss of generality that $r \geq 4$. Consulting §A.1 and §B.2.4, we see that μ is restricted to be one:

(i) $\mu \in \{\omega_1, 2\omega_1, \omega_2\}$, any $r \geq 4$;
(ii) $r = 4$ and $\omega \in \{\omega_3, \omega_1+\omega_3, 2\,\omega_3, \omega_4\}$;
(iii) $r = 5$, $\mu \in \{\omega_4, \omega_5\}$;
(iv) $r = 6$, $\mu = \omega_5$.

We consider each of these cases below.

(i) If $\mu = \omega_1$, then U_μ is the standard representation $\mathbb{C}^{2r}$, with $\mathtt{E}$–eigenspace decomposition $\mathbb{C}^r \oplus \mathbb{C}^r$. The dimensions (r,r) of the $\mathtt{E}$–eigenspaces are too large ($r \geq 4 > 2 \geq h^{1,1}_\phi$). If $\mu = 2\omega_1$, then $U_{2\omega_1} \oplus \mathrm{span}_\mathbb{C}\{Q\} = \mathrm{Sym}^2\mathbb{C}^{2r}$, and the dimensions $(\frac{1}{2}r(r+1), r^2-1, \frac{1}{2}r(r+1))$ of the $\mathtt{E}$–eigenspaces are again too large. If $\mu = \omega_2$, then $U_{\omega_2} = \bigwedge^2\mathbb{C}^{2r} = (\bigwedge^2\mathbb{C}^r) \oplus (\mathbb{C}^r \otimes \mathbb{C}^r) \oplus (\bigwedge^2\mathbb{C}^r)$ and the dimensions $(\frac{1}{2}r(r-1), r^2, \frac{1}{2}r(r-1))$ are again too large.

(ii) Now suppose that $r = 4$. Then the dimensions of the $\mathtt{E}$–eigenspaces for the representations in (ii) are $(8,8)$, $(15,26,15)$, $(10,15,10)$ and $(1,6,1)$, respectively. The requirement that $h^{1,1}_\phi \in \{1,2\}$, restricts us to $\mu = \omega_4$. In this case U_μ is real, and we have Hodge numbers $\mathbf{h}_\phi = (1,6,1)$. This is a special case of Theorem 4.1(i) under an outer automorphism (triality) of $\mathfrak{so}(8,\mathbb{C})$ that permutes the weight $\{\omega_1,\omega_3,\omega_4\}$. of

(iii) Next take $r = 5$. The $\mathtt{E}$–eigenspaces of U_{μ_4} and U_{ω_5} have dimensions $(5,10,1)$ and $(1,10,5)$, respectively. These are too large for our desired Hodge numbers $\mathbf{h}_\phi$.

(iv) Finally, we consider $r = 6$ and $\mu = \omega_5$. In this case U_{ω_5} is quaternionic, and the the $\mathtt{E}$–eigenspaces have dimensions $(6, 20, 6)$ so that the Hodge numbers of the associated Hodge representation $V_{\mathbb{R}} = U_{\omega_5} \oplus U_{\omega_5}$ are $\mathbf{h} = (12, 40, 12)$; again these are too large.

A.6. **Cayley Hodge domains.** Let $(\mathfrak{g}_{\mathbb{C}}, \mathtt{E}) = (\mathfrak{e}_6\,,\, \mathtt{A}^6)$. Consulting §A.1 and §B.2.5, we see that μ is restricted to be one of $\{\omega_1\,,\ \omega_2\,,\ \omega_6\}$. In each case $(\mu + \mu^*)(\mathtt{E}) = 2$, so that U_μ has three nontrivial eigenvalues. The dimensions of the $\mathtt{E}$–eigenspaces are $(10, 16, 1)$, $(16, 46, 16)$ and $(1, 16, 10)$, respectively. In each case the first/last is too large ($> 2 \geq h^{2,0}_\phi$) to yield a Hodge representation satisfying the desired constraints.

A.7. **Freudenthal Hodge domains.** Let $(\mathfrak{g}_{\mathbb{C}}, \mathtt{E}) = (\mathfrak{e}_7\,,\, \mathtt{A}^7)$. Consulting §A.1 and §B.2.6, we see that μ is restricted to be the first fundamental weight $\mu = \omega_1$. In this case the representation U_{ω_1} is real (with respect to $\mathfrak{g}_{\mathbb{R}}$) and we have Hodge numbers $(27, 79, 27)$; the first is too large ($> 2 \geq h^{2,0}_\phi$).

This completes the proofs of Theorems 4.1 and 4.3.

A.8. **When horizontality fails.** In this section we prove Theorem 4.4. Computationally the identification of Hodge subdomains for which horizontality fails entails dropping the assumption that the grading element $\mathtt{E}$ is of the form listed in Table 3.1. Fortunately, the period domain $\mathcal{D}_{\mathbf{h}}$ parameterizing weight two, polarized Hodge structures with $p_g = 2$ is "close enough" to the classical Hermitian period domains (for principally polarized abelian varieties and K3s) that we still have strong restrictions on the possible grading elements. For this period domain the horizontal subbundle $F^{-1}(T\mathcal{D}_{\mathbf{h}}) \subset T\mathcal{D}_{\mathbf{h}}$ (also known as the *infinitesimal period relation* (IPR)) is a contact subbundle.[2]

Any Hodge structure $\varphi \in \mathcal{D}_{\mathbf{h}}$ induces a Hodge structure on the Lie algebra

$$\tilde{\mathfrak{g}}_{\mathbb{C}} \;:=\; \mathrm{End}(V_{\mathbb{C}}, Q) \;=\; \bigoplus_p \tilde{\mathfrak{g}}^{p,-p}_\varphi$$

of $\mathcal{G}_{\mathbb{R}}$ as in §2.2. Assuming the normalization of §2.3.4, the period domain $\mathcal{D}_{\mathbf{h}}$ is Hermitian if and only if $\tilde{\mathfrak{g}}^{p,-p}_\varphi = 0$ for all $|p| \geq 2$ (§3.2). And the IPR is contact (as in the present example) if and only if $\tilde{\mathfrak{g}}^{p,-p}_\varphi = 0$ for all $|p| \geq 3$, and $\dim \tilde{\mathfrak{g}}^{2,-2}_\varphi = 1$. Given a Hodge representation (2.1), since the induced Hodge structure (2.11) on $\mathfrak{g}_{\mathbb{R}}$ is given by $\mathfrak{g}^{p,-p}_\phi = \mathfrak{g}_{\mathbb{C}} \cap \tilde{\mathfrak{g}}^{p,-p}_\phi$, it follows (from the discussion of §3.2) that horizontality will fail for the Hodge subdomain $D \subset \mathcal{D}_{\mathbf{h}}$ if and only if the induced Hodge decomposition (2.9) is of the form

$$\mathfrak{g}_{\mathbb{C}} \;=\; \mathfrak{g}^{2,-2}_\phi \,\oplus\, \mathfrak{g}^{1,-1}_\phi \,\oplus\, \mathfrak{g}^{0,0}_\phi \,\oplus\, \mathfrak{g}^{-1,1}_\phi \,\oplus\, \mathfrak{g}^{-2,2}_\phi\,, \tag{A.3}$$

with $\dim_{\mathbb{C}} \mathfrak{g}^{2,-2}_\phi = 1$. In this case the grading element is necessarily of the form listed in Table A.1, cf. [ČS09, Proposition 3.2.4].[3] (See [Kna02] for remaining notation.) For each of the five exceptional cases, the compact dual $\check{D} = \mathcal{G}/\mathcal{P}_{\mathtt{i}} \hookrightarrow \mathbb{P}V_{\omega_{\mathtt{i}}}$ is a rational homogeneous

[2] In particular, it has corank one. In the classical case that the period domain is Hermitian the subbundle $F^{-1}(T\mathcal{D}_{\mathbf{h}}) = T\mathcal{D}_{\mathbf{h}}$ has corank zero. This is the sense in which the period domain $\mathcal{D}_{\mathbf{h}}$ with $\mathbf{h} = (2, h^{1,1}, 2)$ is as "close as one can get to the classical/Hermitian case." (We use the notation $F^{-1}(T\mathcal{D}_{\mathbf{h}})$ for the horizontal subbundle because it is the first subspace in a natural filtration of the holomorphic tangent bundle $T\check{\mathcal{D}}_{\mathbf{h}} \supset T\mathcal{D}_{\mathbf{h}}$.)

[3] Be aware there are typos in the table of that proposition.

variety with isotropy group $\mathcal{P}_{\mathtt{i}}$ the maximal parabolic subgroup associated with the grading element $\mathtt{E} = \mathtt{A}^{\mathtt{i}}$.

The proof of Theorem 4.4 now proceeds as outlined in §A.1, the single exception being that we work with Table A.1 (not Table 3.1). As it is a straightforward variation on the proof of Theorems 4.1 and 4.3, the proof is left to the reader; for easy reference, the relevant eigenvalues are listed in §B.3.

A.9. **When simplicity fails.** Here we prove Theorem 4.6. To begin, suppose that $\mathfrak{g}_\mathbb{C} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ factors into the direct sum of two nontrivial ideals. Then $U = U_\mu$ is necessarily of the form $T_1 \otimes T_2$ with T_i an irreducible representation of $\mathfrak{g}_i$ of highest weight μ_i and $\mu = \mu_1 + \mu_2$. Likewise, $\mathtt{E} = \mathtt{E}_1 + \mathtt{E}_2$, with $\mathtt{E}_i$ a grading element of $\mathfrak{g}_i$. We write

$$(\mathfrak{g}_\mathbb{C}, \mathtt{E}, \mu) \ = \ (\mathfrak{g}_1, \mathtt{E}_1, \mu_1) \oplus (\mathfrak{g}_2, \mathtt{E}_2, \mu_2) \,.$$

The Hodge representation will have weight/level $n = 2$ if and only if

$$1 \ = \ c + \mu(\mathtt{E}) \ = \ c + \mu_1(\mathtt{E}_1) + \mu_2(\mathtt{E}_2) \,.$$

Recall Remark 2.21, and note that $e(\mu, \mathtt{E}) = e(\mu_1, \mathtt{E}_1) + e(\mu_2, \mathtt{E}_2) \geq 2$. The hypothesis $e(\mu, \mathtt{E}) = 2$ forces $e(\mu_i, \mathtt{E}_i) = 1$ and $\mathfrak{g}_i$ to be simple.

Proposition A.4. *Any semisimple algebra $\mathfrak{g}_\mathbb{R}$ admitting a Hodge representation of level $n = 2$ is either simple, or decomposes as the sum $\mathfrak{g}_\mathbb{R} = \mathfrak{g}_{1,\mathbb{R}} \oplus \mathfrak{g}_{2,\mathbb{R}}$. In the latter case, the triples $(\mathfrak{g}_i, \mathtt{E}_i, \mu_i)$ are necessarily one of:*

(i) $(\mathfrak{sl}_{r+1}\mathbb{C}, \mathtt{A}^a, \omega_1)$. *The (standard) representation $U_{\omega_1} = \mathbb{C}^{r+1}$ is real if $r = 1$, and complex otherwise. The $\mathtt{A}^a$–eigenspace decomposition is $\mathbb{C}^{r+1} = \mathbb{C}^a \oplus \mathbb{C}^{r+1-a}$.*

(ii) $(\mathfrak{sl}_{r+1}\mathbb{C}, \mathtt{A}^1, \omega_a)$. *The representation $U_{\omega_a} = \bigwedge^a \mathbb{C}^{r+1}$ is complex unless $r + 1 = 2a$, in which case the representation is real if and only if a is odd. The $\mathtt{A}^1$–eigenspace decomposition $\bigwedge^a \mathbb{C}^{r+1} = (\mathbb{C}^1 \otimes \bigwedge^{a-1} \mathbb{C}^r) \oplus (\bigwedge^a \mathbb{C}^r)$ is induced by that of $\mathbb{C}^{r+1} = \mathbb{C} \oplus \mathbb{C}^r$.*

(iii) $(\mathfrak{sp}(2r, \mathbb{C}), \mathtt{A}^r, \omega_1)$. *The (standard) representation $U_{\omega_1} = \mathbb{C}^{2r}$ is real, and has $\mathtt{A}^r$–eigenspace decomposition $\mathbb{C}^{2r} = \mathbb{C}^r \oplus \mathbb{C}^r$.*

(iv) $(\mathfrak{so}(2r, \mathbb{C}), \mathtt{A}_r, \omega_1)$. *The (standard) representation $U_{\omega_1} = \mathbb{C}^{2r}$ is quaternionic, and has $\mathtt{A}^r$–eigenspace decomposition $\mathbb{C}^{2r} = \mathbb{C}^r \oplus \mathbb{C}^r$.*

Table A.1. Data underlying irreducible contact Hodge domains

$\mathfrak{g}_\mathbb{C}$	$\mathtt{E}$	$\check{D} = G_\mathbb{C}/P_\mathtt{E}$	$\mathfrak{g}_\mathbb{R}$	$\mathfrak{k}$
$\mathfrak{sl}(r+1, \mathbb{C})$	$\mathtt{A}^1 + \mathtt{A}^r$	$\mathrm{Flag}(1, r; \mathbb{C}^{r+1})$	$\mathfrak{su}(2, r-1)$	$\mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(r-1))$
$\mathfrak{so}(d+4, \mathbb{C})$	$\mathtt{A}^2$	$\mathrm{Gr}^Q(2, \mathbb{C}^{d+4})$	$\mathfrak{so}(4, d)$	$\mathfrak{s}(\mathfrak{o}(4) \oplus \mathfrak{o}(d))$
$\mathfrak{sp}(2r, \mathbb{C})$	$\mathtt{A}^1$	$\mathbb{P}^{2r-1}$	$\mathfrak{sp}(1, r-1)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(r-1)$
$\mathfrak{e}_6$	$\mathtt{A}^2$		EII	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_7$	$\mathtt{A}^1$		EVI	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_8$	$\mathtt{A}^8$		EIX	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$
$\mathfrak{f}_4$	$\mathtt{A}^1$		FI	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$
$\mathfrak{g}_2$	$\mathtt{A}^2$		G	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)\,.$

(v) $(\mathfrak{so}(2r+1,\mathbb{C}), \mathtt{A}^1, \omega_r)$. *The (spin) representation* U_{ω_r} *is real if* $\frac{1}{2}r(r-1)$ *is even, and quaternionic otherwise. The* $\mathtt{A}^1$*–eigenspace decomposition is* $U_{\omega_r} = \mathbb{C}^{2^{r-1}} \oplus \mathbb{C}^{2^{r-1}}$.

(vi) $(\mathfrak{so}(2r,\mathbb{C}), \mathtt{A}^1, \omega_r)$. *The (half-spin) representation* U_{ω_r} *is complex if* r *is odd. If* r *is even, then the representation is real if* $\frac{1}{2}(r+1)(r-2)$ *is even, and quaternionic otherwise. The* $\mathtt{A}^1$*–eigenspace decomposition is* $U_{\omega_r} = \mathbb{C}^{2^{r-2}} \oplus \mathbb{C}^{2^{r-2}}$.

Proof. The proof proceeds as outlined in §A.1 and demonstrated in §§A.2–A.7; details are left to the reader. □

Let

$$T_i \;=\; T_{i,\mu_i(\mathtt{E}_i)} \;\oplus\; T_{i,\mu_i(\mathtt{E}_i)-1} \;=\; T_i' \;\oplus\; T_i''$$

be the $\mathtt{E}_i$–eigenspace decompositions. Then the $\mathtt{E}$–eigenspace decomposition

$$U \;=\; U_{\mu(\mathtt{E})} \;\oplus\; U_{\mu(\mathtt{E})-1} \;\oplus\; U_{\mu(\mathtt{E})-2}$$

is given by

$$\begin{aligned} U_{\mu(\mathtt{E})} &= T_1' \otimes T_2' \\ U_{\mu(\mathtt{E})-1} &= (T_1' \otimes T_2'') \;\oplus\; (T_1'' \oplus T_2') \\ U_{\mu(\mathtt{E})-2} &= T_1'' \otimes T_2'' \,. \end{aligned}$$

So in order to obtain a Hodge representation (2.1) with $p_g = h^{2,0} = 2$ we must have

$$1 \;\le\; \dim T_1' \otimes T_2' \,, \quad \dim T_1'' \otimes T_2'' \;\le\; 2\,;$$

in particular,

$$\dim T_i \;\le\; 4\,.$$

Modulo isomorphisms of (low-rank) Lie algebras, this leaves us with (i) and (iii) of Proposition A.4. Theorem 4.6 now follows from the discussion of §A.1; details are left to the reader.

B. Duality and eigenvalues for Hodge representations

For the computations of this paper it is useful to make some general observations about the irreducible $\mathfrak{g}_{\mathbb{C}}$–representations U_μ that yield Hodge representations $V_{\mathbb{R}}$ for each such pair. (Those that follow are all elementary consequences of the representation theory of complex, simple Lie algebras and may be found in any standard text.) Throughout we let $\{\omega_1, \ldots, \omega_r\} \subset \mathfrak{h}^*$ denote the fundamental weights of $\mathfrak{g}_{\mathbb{C}}$, and write the dominant integral weight $\mu = \mu^i \omega_i$ with $0 \le \mu^i \in \mathbb{Z}$.

Motivated by the considerations of §2.5, we define

$$\mathtt{T}_i \;:=\; 2 \sum_{j \ne i} \mathtt{A}^j\,.$$

B.1. **Duality.** Every representation U_μ of $\mathfrak{g}_\mathbb{C} = \mathfrak{so}(2r+1,\mathbb{C})$, $\mathfrak{sp}(2r,\mathbb{C})$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$ and $\mathfrak{g}_2$ is self-dual.

(1) A $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(r{+}1,\mathbb{C})$ representation U_μ is self-dual if and only if $\mu^i = \mu^j$ for all $i{+}j = r{+}1$.
(2) A $\mathfrak{g}_\mathbb{C} = \mathfrak{so}(2r,\mathbb{C})$ representation U_μ fails to be self-dual if and only if r is odd and $\mu^{r-1} \neq \mu^r$.
(3) For $\mathfrak{g}_\mathbb{C} = \mathfrak{e}_6$, we have $\omega_1^* = \omega_6$, $\omega_2^* = \omega_2$, $\omega_3^* = \omega_5$, $\omega_4^* = \omega_4$.

B.2. **Hermitian symmetric domains.** Let $(\mathfrak{g}_\mathbb{C}, \mathtt{E})$ be the data underlying the irreducible Hermitian symmetric domains (Table 3.1).

B.2.1. *Grassmannian Hodge domains.* With the notation in the first row of Table 3.1, we have $r+1 = a+b$. Given Remark 2.21 it will be useful to note that, given $i+j = r+1$, we have

$$\begin{aligned}
(\omega_i + \omega_j)(\mathtt{A}^a) &= \begin{cases} a\,, & a \le i \le j\,,\\ i\,, & i \le a \le j\,,\\ b\,, & i \le j \le a\,; \end{cases}\\
(\omega_i + \omega_j)(\mathtt{T}_a) &= 2ij \,-\, (\omega_i+\omega_j)(2\,\mathtt{A}^a)\,.
\end{aligned}$$

B.2.2. *Quadric Hodge domains.* Here (the second row of Table 3.1) the rank r of $\mathfrak{g}_\mathbb{C}$ is given by $d+2 \in \{2r, 2r+1\}$.

In the case that $d \equiv 1 \bmod 2$, we have

$$\begin{aligned}
\omega_i(\mathtt{A}^1) &= \begin{cases} 1\,, & i \le r-1\,,\\ \tfrac12\,, & i = r\,. \end{cases}\\
\omega_i(\mathtt{T}_1) &\equiv 0 \mod 2\,, \quad i \le r-1\,,\\
\omega_r(\mathtt{T}_1) &= \tfrac12 (r-1)(r+2)\,.
\end{aligned}$$

In the case that $d \equiv 0 \bmod 2$, we have

$$\begin{aligned}
\omega_i(\mathtt{A}^1) &= \begin{cases} 1\,, & i \le r-2\,,\\ \tfrac12\,, & i = r-1, r\,. \end{cases}\\
\omega_i(\mathtt{T}_1) &\equiv 0 \mod 2\,, \quad i \le r-2\,,\\
\omega_{r-1}(\mathtt{T}_1) = \omega_r(\mathtt{T}_1) &= \tfrac12 (r-2)(r+1)\,.
\end{aligned}$$

B.2.3. *Lagrangian Grassmannian Hodge domains.* We have

$$\begin{aligned}
\omega_i(\mathtt{A}^r) &= \tfrac12 i\,,\\
\omega_i(\mathtt{T}_r) &\equiv 0 \mod 2\,.
\end{aligned}$$

In particular, every representation U_μ is real (§2.5), with respect to the data $(\mathfrak{g}_\mathbb{C}, \mathtt{E}) = (\mathfrak{sp}(2r,\mathbb{C})\,,\, \mathtt{A}^r)$.

B.2.4. *Spinor Hodge domains.* We have

$$\begin{aligned}
\omega_i(\mathtt{A}^r) &= \tfrac12 i\,, & \omega_i(\mathtt{T}_r) &\equiv i \mod 2\,, \quad i \le r-2\,;\\
\omega_{r-1}(\mathtt{A}^r) &= \tfrac14 (r-2)\,, & \omega_{r-1}(\mathtt{T}_r) &= \tfrac12 (r^2 - 2r + 2)\,;\\
\omega_r(\mathtt{A}^r) &= \tfrac14 r\,, & \omega_r(\mathtt{T}_r) &= \tfrac12 r(r-2)\,.
\end{aligned}$$

B.2.5. *Cayley Hodge domains.* We have

$$\begin{array}{rcl@{\qquad}rcl}
(\omega_1+\omega_6)(\mathtt{A}^6) & = & 2\,, & (\omega_1+\omega_6)(\mathtt{T}_6) & \equiv & 0 \mod 2\,,\\
\omega_2(\mathtt{A}^6) & = & 1\,, & \omega_2(\mathtt{T}_6) & \equiv & 0 \mod 2\,,\\
(\omega_3+\omega_5)(\mathtt{A}^6) & = & 3\,, & (\omega_3+\omega_5)(\mathtt{T}_6) & \equiv & 0 \mod 2\,,\\
\omega_4(\mathtt{A}^6) & = & 2\,, & \omega_4(\mathtt{T}_6) & \equiv & 0 \mod 2\,.
\end{array}$$

In particular, every representation is either real or complex with respect to the data $(\mathfrak{g}_\mathbb{C}, \mathtt{E}) = (\mathfrak{e}_6, \mathtt{A}^6)$.

B.2.6. *Freudenthal Hodge domains.* Every representation is self-dual. We have

$$\begin{array}{rcl@{\quad}rcl@{\quad}rcl@{\quad}rcl}
\omega_1(\mathtt{A}^7) & = & 1\,, & \omega_2(\mathtt{A}^7) & = & 3/2\,, & \omega_3(\mathtt{A}^7) & = & 2\,, & \omega_4(\mathtt{A}^7) & = & 3\,,\\
\omega_5(\mathtt{A}^7) & = & 5/2\,, & \omega_6(\mathtt{A}^7) & = & 2\,, & \omega_7(\mathtt{A}^7) & = & 3/2\,,
\end{array}$$

and $\omega_i(\mathtt{T}_7) \equiv 0 \mod 2$, for all $1 \le i \le 7$. So every representation is real (§2.5) with respect to the data $(\mathfrak{g}_\mathbb{C}, \mathtt{E}) = (\mathfrak{e}_7, \mathtt{A}^7)$.

B.3. **Contact domains.** Let $(\mathfrak{g}_\mathbb{C}, \mathtt{E})$ be the data underlying the irreducible contact Hodge domains (Table A.1). (Duality of representations is as in §B.2 and so will not be repeated here.)

B.3.1. *Special Linear.* We begin with the first row of Table A.1. We have $\omega_i(\mathtt{A}^1 + \mathtt{A}^r) = 1$ for all $1 \le i \le r$. We have $\mathtt{T}_\phi = 2(\mathtt{A}^2 + \cdots + \mathtt{A}^{r-1})$. If $r = 3$, then $\omega_1(\mathtt{T}_\phi) = 1 = \omega_3(\mathtt{T}_\phi)$ and $\omega_2(\mathtt{T}_\phi) = 2$. If $r \ge 4$, then $\omega_i(\mathtt{T}_\phi) \equiv 0 \mod 2$.

B.3.2. *Orthogonal.* Here (the second row of Table A.1) the rank r of $\mathfrak{g}_\mathbb{C}$ is given by $d+4 \in \{2r, 2r+1\}$.

In the case that $d \equiv 1 \mod 2$, we have

$$\begin{array}{rcl}
\omega_i(\mathtt{A}^2) & = & \begin{cases} 1\,, & i = 1, r\,,\\ 2\,, & 2 \le i \le r-1\,. \end{cases}\\
\omega_i(\mathtt{T}_2) & \equiv & 0 \mod 2\,, \quad i \le r-1\,,\\
\omega_r(\mathtt{T}_2) & \equiv & \tfrac12 r(r+1) \mod 2\,.
\end{array}$$

In the case that $d \equiv 0 \mod 2$, we have

$$\begin{array}{rcl}
\omega_i(\mathtt{A}^2) & = & \begin{cases} 1\,, & i = 1, r-1, r\,,\\ 2\,, & 2 \le i \le r-2\,. \end{cases}\\
\omega_i(\mathtt{T}_2) & \equiv & 0 \mod 2\,, \quad i \le r-2\,,\\
\omega_{r-1}(\mathtt{T}_2) = \omega_r(\mathtt{T}_2) & \equiv & \tfrac12 r(r+1) \mod 2\,.
\end{array}$$

B.3.3. *Symplectic.* We have

$$\begin{array}{rcl}
\omega_i(\mathtt{A}^1) & = & 1\,,\\
\omega_i(\mathtt{T}_1) & \equiv & i \mod 2\,.
\end{array}$$

B.3.4. *Exceptional, rank 6.* We have $(\mathfrak{g}_\mathbb{C}, \mathtt{E}) = (\mathfrak{e}_6, \mathtt{A}^2)$ and

$$\begin{array}{rcl}(\omega_1+\omega_6)(\mathtt{A}^2) & = & 2\,, \\ \omega_2(\mathtt{A}^2) & = & 2\,, \\ (\omega_3+\omega_5)(\mathtt{A}^2) & = & 4\,, \\ \omega_4(\mathtt{A}^2) & = & 3\,,\end{array} \qquad \begin{array}{rcl}(\omega_1+\omega_6)(\mathtt{T}_2) & \equiv & 0 \mod 2\,, \\ \omega_2(\mathtt{T}_2) & \equiv & 0 \mod 2\,, \\ (\omega_3+\omega_5)(\mathtt{T}_2) & \equiv & 0 \mod 2\,, \\ \omega_4(\mathtt{T}_2) & \equiv & 0 \mod 2\,.\end{array}$$

B.3.5. *Exceptional, rank 7.* We have $(\mathfrak{g}_\mathbb{C}, \mathtt{E}) = (\mathfrak{e}_7, \mathtt{A}^1)$, with $\omega_7(\mathtt{A}^1) = 1$ and $\omega_i(\mathtt{A}^1) \geq 2$ for all $1 \leq i \leq 6$. Also $\omega_1(\mathtt{T}_1) \equiv 0 \bmod 2$.

B.3.6. *Exceptional, rank 8.* For $(\mathfrak{g}_\mathbb{C}, \mathtt{E}) = (\mathfrak{e}_8, \mathtt{A}^8)$ we have $\omega_i(\mathtt{A}^8) \geq 2$ for all $1 \leq i \leq 8$.

B.3.7. *Exceptional, rank 4.* We have $(\mathfrak{g}_\mathbb{C}, \mathtt{E}) = (\mathfrak{f}_4, \mathtt{A}^1)$ and $\omega_i(\mathtt{A}^1) \geq 2$ for all $1 \leq i \leq 4$.

B.3.8. *Exceptional, rank 2.* We have $(\mathfrak{g}_\mathbb{C}, \mathtt{E}) = (\mathfrak{g}_2, \mathtt{A}^2)$, $\omega_1(\mathtt{A}^2) = 1$ and $\omega_2(\mathtt{A}^2) = 2$, and $\omega_i(\mathtt{T}_1) \equiv 0 \bmod 2$ for all $1 \leq i \leq 2$.

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