# S Supplementary Appendix for "Deductions for Early Retirement"

In this supplementary appendix I offer a number of extensions of the basic set-up of the paper without long-run changes, i.e. extensions that are set in an environment where the crucial demographic variables (the retirement age, life expectancy etc.) are assumed to be stationary.[23] In particular, I discuss how the main results are affected in a set-up with mortality and with growing wages and (appendix S.1.1) and with heterogeneity of entry ages, employment histories and wages (appendix S.1.2).[24] In the first section of this supplementary appendix I summarize the main results of these extensions while in the next sections I offer the proofs of the results. They involve somewhat longer arguments and derivations.

## S.1 Main conclusions

The main conclusion from these extensions is that a stationary distribution of retirement ages is associated with a situation where the level of actuarial deductions is independent of the market interest rate.

### S.1.1 Mortality and growing economy

In sections 3 and 4 of the paper I have used a model with rectangular mortality (all cohort members die at the same age $\omega$) and with constant wages. In order to broaden this view I have also looked at a set-up with non-rectangular survivorship $S(a)$ (where $S(0) = 1$ and $S(\omega) = 0$) and a growing wage $W(t)$. Since this general case involves many derivations I have collected them in a separate supplementary appendix S.2 where I show how the main formulas have to be adapted. Furthermore, I demonstrate that all main findings continue to hold in this more general framework. In particular I show that for a stationary retirement distribution (i) a NDC system is stable without the use of additional deductions or supplements; (ii) the DB and AR systems are also compatible with balanced budgets if they are augmented by demographic deduction factors that are independent of the market interest rate; (iii) the discount rate that corresponds to these

[23]Extensions that also involve time-variability (e.g. increasing mean retirement age or increasing life expectancy) are sketched in sections 5.3 and 5.4 of the paper.

[24]In the working paper version of the paper I have also dealt with the case where a pension system is unbalanced by design (i.e. designed in a way such that it is not balanced in the stationary situation).

budget-neutral deduction factors is given by the internal rate of return (i.e. now the growth rate of wages). Quantitatively, I show that the actuarial deductions are lower for non-rectangular survivorship than for the rectangular case (as reported in table 2). For realistic assumptions they come out as 7% (DB system) and 4.9% (AR system) which is smaller than the corresponding rates for rectangular survival where they have been calculated as 8.33% and 6.25%, respectively.

### S.1.2 Additional heterogeneity

The real-world is more complex than reflected in the frameworks used so far. Individuals differ along many more dimensions including fertility, mortality, work history and wages. It can be shown that the main results of sections 3 and 4 of the paper will continue to hold in a set-up that allows for heterogeneity in labor market entry age and in the average lifetime wage. If these variables follow a stationary distribution then one can use the same arguments as above to conclude that a pure NDC system without additional deductions is compatible with a stable budget. In fact, one can regard the formulation with fixed $A$ and $W$ as referring to one specific constellation. Since the pure NDC system leads to a balanced budget for this (as for any other) specific subgroup one can conclude that also the aggregate budget will be in balance. Section S.3 shows this in more detail. Furthermore, following the arguments of section 5.2 one would guess that fluctuations around this stable joint distribution should also be compatible with an approximately balanced budget.

## S.2 Proofs of the extensions of appendix S.1.1

This generalizes the model of section 4 of the paper. The main results of the generalized model are summarized in section S.1.1 above.

### S.2.1 Set-up

**Demographic structure** I work with a model in continuous time. In every instant of time $t$ a new cohort is born. The maximum age that a member of cohort $t$ can reach is time-invariant and denoted by $\omega$. $S(a)$ gives the probability that an individual survives to age $a$. It holds that $S(0) = 1$, $S(\omega) = 0$ and that survivorship declines with age, i.e. $\frac{dS(a)}{da} \leq 0$ for $a \in [0, \omega]$. The mortality hazard rate is given by $\mu(a) \equiv -\frac{dS(a)}{da}\frac{1}{S(a)}$.

Therefore:

$$S(a) = e^{\int_0^a -\mu(x)\,\mathrm{d}x}. \tag{22}$$

An interesting benchmark case is given by rectangular survivorship where $S(a) = 1$ for $a \in [0, \omega]$. In this case there are no premature deaths and all members of a cohort reach the maximum age $\omega$. This corresponds to the assumption made in section 4.

Remaining life expectancy is given by:[25]

$$e(z) = \int_z^\omega e^{\int_z^a -\mu(x)\,\mathrm{d}x}\,\mathrm{d}a = \frac{\int_z^\omega S(a)\,\mathrm{d}a}{S(z)}. \tag{23}$$

The second equality follows from the fact that $e^{\int_z^a -\mu(x)\,\mathrm{d}x} = e^{\int_0^a -\mu(x)\,\mathrm{d}x} e^{\int_0^z \mu(x)\,\mathrm{d}x} = \frac{S(a)}{S(z)}$ where the last step uses equation (22).

The size of cohort $t$ at age $a$ is given by $N(a,t) = N(0,t)S(a)$, where $N(0,t)$ stands for the initial size of the cohort. For the sake of simplicity I assume constant sizes of birth cohorts, i.e. $N(0,t) = N, \forall t$. The entry age in the labor market is again assumed to be constant and given by $A$ while the age-specific probability to retire for generation $t$ is again given by $f(a,t)$ for $a \in [A, \omega]$. I assume that the mortality rates are independent from this probability. The cumulative function $F(a,t)$ then gives the percentage of the surviving members of cohort $t$ that are already retired at age $a$. It holds that $F(A,t) = 0$ and $F(\omega,t) = 1$. In this appendix I focus on a stationary retirement distribution, i.e. $f(a,t) = f(a)$ and $F(a,t) = F(a)$.

The total size of the active population $L$ and the retired population $M$ are constant and given by:

$$L = N \int_A^\omega S(a)\,(1 - F(a))\,\mathrm{d}a, \tag{24}$$

$$M = N \int_A^\omega S(a)F(a)\,\mathrm{d}a. \tag{25}$$

**Budget of the pension system** The contribution rate to the PAYG pension system is assumed to be fixed at $\tau$. I abstract from intragenerational wage differences and seniority profiles and simply assume that in a specific period $t$ each workers earns an identical wage $W(t)$. Wages grow at rate $g(t)$, i.e. $W(t) = W(0)e^{\int_0^t g(s)\,\mathrm{d}s}$.

Each retired member of generation $t$ receives a pension payment $P(R,a,t)$. The size of the pension can depend on the payment period $t+a$, on the individual's age $a$ and also

[25] See, e.g., Keyfitz & Caswell (2005) or Goldstein (2006).

on the time of his or her retirement $R \leq a$. Below I will say more about the determination of the pension payments in different systems.

In order to calculate the total expenditures of the pension system one can make the following considerations. First, focus on one particular retirement age $R$ and calculate the total of pension payments that is distributed to the group of pensioners that has retired at this age. This comprises individuals at different ages $a \in [R, \omega]$. For a person who is of age $a$ in period $t$ the pension payment is $P(R, a, t-a)$ and the size of this subgroup is $N \times S(a)$. The total payments to people with retirement age $R$ in period $t$ is thus given by: $P^{total}(R,t) = N \int_R^\omega P(R,a,t-a)S(a)\,\mathrm{d}a$. The same logic applies for any possible retirement age $R \in [A, \omega]$ where the relative frequency of the retirement age is given by $f(R)$. Total pension expenditures in period $t$ can thus be written as $E(t) = \int_A^\omega P^{total}(R,t)f(R)\,\mathrm{d}R$ or:[26]

$$E(t) = N \int_A^\omega \left( \int_R^\omega P(R,a,t-a)S(a)\,\mathrm{d}a \right) f(R)\,\mathrm{d}R. \tag{26}$$

Total revenues $I(t)$, on the other hand, are given by:

$$I(t) = \tau W(t) L = \tau W(t) N \int_A^\omega S(a)\left(1 - F(a)\right)\,\mathrm{d}a. \tag{27}$$

The total deficit (or surplus) of the pension system is given by $D(t) = E(t) - I(t)$ while the deficit ratio by $d(t) = \frac{D(t)}{I(t)} = \frac{E(t)}{I(t)} - 1$. A continuously balanced budget is thus characterized by $D(t) = d(t) = 0, \forall t$.

### S.2.2 Different PAYG systems

In the last section S.2.5 of this appendix I discuss in detail how the pension level $P_j(R,a,t)$ is determined in the three different pension systems $j \in \{\mathrm{DB}, \mathrm{AR}, \mathrm{NDC}\}$. Here I only summarize the main results. There are two main differences to the simple model of sections 3 and 4 that have to do with the assumptions of growth and mortality. All pension systems have to specify how pension claims that have been acquired in the past are revalued at the moment of retirement. In the NDC system, e.g., this is done by the choice of a "notional interest rate" $\rho(a,t)$. There exist two popular variants of this interest

[26] Alternatively, one could also reverse the order and look first at a fixed age $a$ of the retired population and calculate the total pension payments to those individuals that have different retirement ages $R$. In the second step one would then calculate the total pension payments to all possible ages $a$. This results in an equivalent expression. I focus on the formulation in (26) since it is more convenient for later calculations.

rate that are discussed in the literature and used in real-world systems:

$$\rho(a,t) = g(t) + \mu(a) \tag{28a}$$

or

$$\rho(a,t) = g(t). \tag{28b}$$

Both notional interest rates reflect the growth rate of average wages $g(t)$ while the first specification (28a) also corrects for the fact that each period some cohort members die. The account values of the deceased cohort members are regarded as "inheritance gains" that are distributed among surviving cohort members by granting an extra return $\mu(a)$.

For later reference it is also useful to define the following term:

$$h(R) \equiv \frac{\int_A^R S(a)\,\mathrm{d}a}{(R-A)S(R)}, \tag{29}$$

that stands for the "per capita inheritance gains premium", i.e. the factor by which the first pension at retirement age $R$ is higher if the revaluation takes inheritance gains into account. For rectangular survivorship there are no inheritance gains and thus the average premium is $h(R) = 1$.

Furthermore, due to the assumption of a growing economy one also has to specify how pension are adjusted over time. Here it is assumed (for simplicity and in line with the practice in many countries) that ongoing pensions are adjusted with the average growth rate of wages, i.e.%

$$\vartheta(t) = g(t). \tag{30}$$

In section S.2.5 I show that in this situation the three pension system are associated with the following pension levels:

$$P_{\mathrm{NDC}}(R,a,t) = \tau W(t+a)\frac{(R-A)h(R)}{e(R)}X_{\mathrm{NDC}}(R,t), \tag{31}$$

$$P_{\mathrm{DB}}(R,a,t) = q^* W(t+a)X_{\mathrm{DB}}(R,t), \tag{32}$$

$$P_{\mathrm{AR}}(R,a,t) = \kappa^* W(t+a)(R-A)X_{\mathrm{AR}}(R,t), \tag{33}$$

where the NDC system uses the notional interest including inheritance gains (i.e. equation (28a)) to revalue past contributions. This is in line with the approach used in Sweden. The other two system are based on indexations excluding these mortality adjustments (i.e.

on equation (28b)) which is also in line with the real-world systems. All three pension system also allow for the use of demographic adjustment factors $X_j(R,t)$ as discussed in section 3. In the case of non-stationary constellations the deduction factor might also be time-dependent as is indicated by the use of a time index $t$.

### S.2.3 Budget-neutral deductions

In this part I investigate how the deduction factors $X_j(R,t)$ have to be determined in order to guarantee a balanced PAYG system in the case of a stationary demographic situation. The following proposition is a generalization of proposition 1 and has already been stated in section 5.1 of the paper. It states that a standard NDC system with $\rho(a,t) = g(t) + \mu(a)$ leads to a balanced budget without the need for further deductions.

**Propositon 2** *Assume a stationary demographic situation where the size of birth cohorts is constant ($N(0,t) = N$), people start to work at age $A$, the maximum age is $\omega$, mortality is described by the survivorship function $S(a)$ for $a \in [0,\omega]$, retirement age is distributed according to the probability density function $f(R)$ for $R \in [A,\omega]$ and wages grow with rate $g(t)$. In this case a NDC system will be in continuous balance ($E(t) = I(t), \forall t$) if the notional interest rate and the adjustment factor are set according to $\rho(a,t) = g(t) + \mu(a)$ (equation (28a)) and $\vartheta(t) = g(t)$ (equation (30)), respectively, and if there are no additional deductions ($X_{NDC}(R,t) = 1$).*

**Proof.** For the NDC system one can insert the pension level from equation (31), i.e. $P_{\mathrm{NDC}}(R,a,t-a) = \tau W(t)\frac{(R-A)h(R)}{e(R)}X_{\mathrm{NDC}}(R,t)$ into (26) to conclude that:

$$\begin{aligned} E(t) &= \tau W(t) N \int_A^\omega \frac{(R-A)h(R)}{e(R)} X_{\mathrm{NDC}}(R,t) \left( \int_R^\omega S(a)\,\mathrm{d}a \right) f(R)\,\mathrm{d}R \\ &= \tau W(t) N \int_A^\omega \left( \int_A^R S(a)\,\mathrm{d}a \right) X_{\mathrm{NDC}}(R,t) f(R)\,\mathrm{d}R, \end{aligned} \tag{34}$$

where I use the definitions $h(R) = \frac{\int_A^R S(a)\,\mathrm{d}a}{(R-A)S(R)}$ and $e(R) = \frac{\int_R^\omega S(a)\,\mathrm{d}a}{S(R)}$.

For the assumptions of proposition 2 total expenditures in equation (34) can be written as:

$$E(t) = \tau W(t) N \int_A^\omega \left( \int_A^R S(a)\,\mathrm{d}a \right) f(R)\,\mathrm{d}R.$$

One can define $u(R) = \int_A^R S(a)\,\mathrm{d}a$ and $v(R) = 1 - F(R)$ with $u'(R) = S(R)$ and $v'(R) =$

$-f(R)$. Using integration by parts it holds that:

$$\int_A^\omega \left( \int_A^R S(a)\,\mathrm{d}a \right) f(R)\,\mathrm{d}R = -\int_A^\omega u(R)v'(R)\,\mathrm{d}R = -u(R)v(R) + \int_A^\omega u'(R)v(R)\,\mathrm{d}R.$$

The term $(-u(R)v(R))$ is given by $\left[ \left( \int_A^R S(a)\,\mathrm{d}a \right) (1 - F(R)) \right]_A^\omega$ which can be evaluated as

$$\left( \int_A^\omega S(a)\,\mathrm{d}a \right) (1 - F(\omega)) - \left( \int_A^A S(a)\,\mathrm{d}a \right) (1 - F(A)) = 0. \tag{35}$$

Since it holds that $\int_A^\omega u(R)v'(R)\,\mathrm{d}R = \int_A^\omega S(R)(1 - F(R))\,\mathrm{d}R$ one can conclude that:

$$E(t) = \tau W(t) N \int_A^\omega S(R)(1 - F(R))\,\mathrm{d}R. \tag{36}$$

This is equal to total revenues $I(t) = \tau W(t) N \int_A^\omega S(a)\,(1 - F(a))\,\mathrm{d}a$ (see (27)) and thus $E(t) = I(t)$. ■

Proposition 2 generalizes proposition 1 and it confirms the previous findings. For a stationary economic and demographic situation a NDC system that includes a correction for inheritance gains is stable if one uses the benchmark NDC formula:

$$P_{\mathrm{NDC}}(R, a, t) = \tau W(t + a) \frac{(R - A)h(R)}{e(R)}. \tag{37}$$

There is no need for an additional adjustment factor and it holds (as in section 4.2) that $X_{\mathrm{NDC}}(R, t) = \Psi_{\mathrm{NDC}}(R) = 1$.

In this case it is also possible to make the DB and the AR systems stable by just using demographic deduction factors $\Psi_{\mathrm{DB}}(R)$ and $\Psi_{\mathrm{AR}}(R)$ that are independent of the discount rate $\delta$ and of time $t$. These are calculated in appendix S.2.5.[27] As in appendix A I again invoke the "balanced target condition", i.e. the condition that the target replacement rate $q^*$ is chosen in such a way that if everybody retires at the target retirement age $R^*$ there will be no deductions ($\Psi_{\mathrm{DB}}(R^*) = 1$) and the system will be in balance. The demographic adjustment factors come out as:

$$\Psi_{\mathrm{DB}}(R) = \frac{e(R^*)}{e(R)} \frac{h(R)}{h(R^*)} \frac{R - A}{R^* - A}, \tag{38}$$

[27] I also discuss the case of a NDC system that uses (28b) instead of (28a), i.e. that excludes the correction for inheritance gains in the notional interest rate. In this case one also needs a demographic adjustment factor in order to implement a balanced system.

Table S.1: Three PAYG systems for a stationary demography

| | (1) | (2) | (3) | (4) |
|---|---|---|---|---|
| Type ($j$) | $\widehat{P}_j(R,a,t)$ | $\widehat{P}_j(R,a,t)$ (for BTC) | $\Psi_j$ | $P_j(R,a,t)$ (for BTC) |
| **DB** | $q^* W(t+a)$ | $\tau W(t+a)\frac{(R^*-A)h(R^*)}{e(R^*)}$ | $\frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}\frac{R-A}{R^*-A}$ | $\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$ |
| **AR** | $\kappa^*(R-A)W(t+a)$ | $\tau W(t+a)\frac{(R-A)h(R^*)}{e(R^*)}$ | $\frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}$ | $\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$ |
| **NDC** | $\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$ | $\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$ | 1 | $\tau W(t+a)\frac{(R-A)h(R)}{e(R)}$ |

*Note:* The table shows the formula pension $\widehat{P}_j(R,a,t)$, the demographic deduction factor $\Psi_j$ and the total pension $P_j(R,a,t) = \widehat{P}_j(R,a,t)\Psi_j$ for three variants of PAYG schemes: DB (Defined Benefit), AR (Accrual Rates), NDC (Notional Defined Contribution). The balanced target condition (BTC) has to hold if the system has a balanced budget in the case that all individuals retire at the target retirement age $R = R^*$. These are specified in the text. Column (4) is the multiple of columns (2) and (3).

$$\Psi_{\mathrm{AR}}(R) = \frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}. \tag{39}$$

In table S.1 I collect important formulas for the three PAYG systems. In particular, it contains the formula pension $\widehat{P}_j(R,a,t)$ (both in its basic form and after invoking the balanced target condition), the demographic deduction factor $\Psi_j$ and the final pension $P_j(R,a,t) = \widehat{P}_j(R,a,t)\Psi_j$. Note that for rectangular survivorship it holds that $h(R) = 1$ and $e(R) = \omega - R$ . In this case the results of table S.1 coincide with the ones of table 1. In particular, $\Psi_{\mathrm{DB}}(R) = \frac{\omega-R^*}{\omega-R}\frac{R-A}{R^*-A}$ and $\Psi_{\mathrm{AR}}(R) = \frac{\omega-R^*}{\omega-R}$.

### S.2.4 The choice of discount rates

So far I have shown that for a stationary economic and demographic situation a standard NDC system implements a stable PAYG pension system. By using the correct demographic adjustment factors $\Psi_{\mathrm{DB}}$ and $\Psi_{\mathrm{AR}}$ also the DB and a AR systems can be amended to guarantee a continuous budgetary balance. This implies that it is not necessary to refer to the market interest rate in order to design the budget-neutral deduction rates in this stationary constellation.

It is interesting to look at this issue from the viewpoint of the standard deduction framework presented in section 3.2 and ask a number of questions. First, which choice of the discount rate will give rise to the budget-neutral demographic deduction factors $\Psi_j$? Second, what deductions are implied if the discount rate is set to higher levels? Third, under which conditions will higher discount rates also be compatible with a balanced budget? Answers to these questions will be provided in the next three subsections.

**The appropriate budget-neutral discount rate for a stationary demography** In order to find the discount rate that is compatible with the budget-neutral demographic deduction factors $\Psi_j$ one has to adapt the neutrality condition (1) of section 3.2 for the general framework. It comes out as:

$$\begin{aligned} &\textstyle\int_R^{R^*} \left(\tau W(t+a) + \widehat{P}_j(R,a,t)X_j\right) e^{-\delta(a-R)} S(a)\,\mathrm{d}a = \\ &\textstyle\int_{R^*}^{\omega} \left(\widehat{P}_j(R^*,a,t) - \widehat{P}_j(R,a,t)X_j\right) e^{-\delta(a-R)} S(a)\,\mathrm{d}a. \end{aligned} \tag{40}$$

I want to know for which choice of $\delta$ the total deduction factor will collapse to the demographic factor, i.e. for which $\delta$ it holds that $X_j = \Psi_j$. In general, one cannot solve (40) for $X_j$ in closed form. In the following I show, however, analytically that for the case of constant growth (i.e. $g(t) = g$) the choice of $\delta = g$ leads to the result that $X_j = \Psi_j$.

In order to do so I focus on formula pensions that are proportional to $\tau W(t+a)$. Therefore I write $\widehat{P}(R,a,t) = \tau W(t+a)\check{P}(R)$. Furthermore, noting that $W(t+a) = W(t+R)e^{\int_R^a g(t+s)\,\mathrm{d}s}$ equation (40) can also be written as:

$$\begin{aligned} \tau W(t+R) &\int_R^{R^*} \left(1 + \check{P}(R)X\right) e^{\int_R^a g(t+s)\,\mathrm{d}s} e^{-\delta(a-R)} S(a)\,\mathrm{d}a = \\ \tau W(t+R) &\int_{R^*}^{\omega} \left(\check{P}(R^*) - \check{P}(R)X\right) e^{\int_R^a g(t+s)\,\mathrm{d}s} e^{-\delta(a-R)} S(a)\,\mathrm{d}a. \end{aligned}$$

For constant wage growth $g(s) = g$ this can be simplified to:[28]

$$\begin{aligned} &\textstyle\int_R^{R^*} \left(1 + \check{P}(R)X\right) e^{-(\delta-g)(a-R)} S(a)\,\mathrm{d}a = \\ &\textstyle\int_{R^*}^{\omega} \left(\check{P}(R^*) - \check{P}(R)X\right) e^{-(\delta-g)(a-R)} S(a)\,\mathrm{d}a. \end{aligned} \tag{41}$$

I want to show that for the choice of $\delta = g$ the deductions $X$ coincide with the demographic deduction factor $\Psi$. I focus first on the NDC system. In this case it holds that $\Psi = 1$ and the conjecture is that $X = \Psi = 1$. Furthermore, $\widehat{P}(R,a,t) = \tau W(t+a)\frac{(R-A)h(R)}{e(R)}$, i.e. $\check{P}(R) = \frac{(R-A)h(R)}{e(R)}$. Noting that $h(R) = \frac{\int_A^R S(a)\,\mathrm{d}a}{(R-A)S(R)}$ and $e(R) = \frac{\int_R^{\omega} S(a)\,\mathrm{d}a}{S(R)}$ one can thus

[28] Similarly, one could also assume a time-varying discount rate $\delta(s)$. Under the assumption that $\delta(s) = g(s)$ one could then derive the same result as in the following.

write: $\check{P}(R) = \frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^\omega S(a)\,\mathrm{d}a}$. Inserting this into (41) leads to:

$$\left(1 + \frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^\omega S(a)\,\mathrm{d}a}\right) \int_R^{R^*} S(a)\,\mathrm{d}a = \left(\frac{\int_A^{R^*} S(a)\,\mathrm{d}a}{\int_{R^*}^\omega S(a)\,\mathrm{d}a} - \frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^\omega S(a)\,\mathrm{d}a}\right) \int_{R^*}^\omega S(a)\,\mathrm{d}a.$$

This can be simplified to:

$$\left(\int_R^\omega S(a)\,\mathrm{d}a + \int_A^R S(a)\,\mathrm{d}a\right) \int_R^{R^*} S(a)\,\mathrm{d}a =$$
$$\left(\int_A^{R^*} S(a)\,\mathrm{d}a \int_R^\omega S(a)\,\mathrm{d}a - \int_A^R S(a)\,\mathrm{d}a \int_{R^*}^\omega S(a)\,\mathrm{d}a\right).$$

Collecting terms this leads to:

$$\int_R^\omega S(a)\,\mathrm{d}a \left(\int_R^{R^*} S(a)\,\mathrm{d}a - \int_A^{R^*} S(a)\,\mathrm{d}a\right) = \int_A^R S(a)\,\mathrm{d}a \left(-\int_{R^*}^\omega S(a)\,\mathrm{d}a - \int_R^{R^*} S(a)\,\mathrm{d}a\right).$$

Combining integrals one can conclude:

$$-\left(\int_R^\omega S(a)\,\mathrm{d}a \int_A^R S(a)\,\mathrm{d}a\right) = -\left(\int_R^\omega S(a)\,\mathrm{d}a \int_A^R S(a)\,\mathrm{d}a\right).$$

This proves the conjecture that for $\delta = g$ the implied deduction $X$ equals the demographic deduction $\Psi$ which is just $\Psi = 1$ for the NDC system. Since the demographic deduction factors $\Psi_{\mathrm{DB}}$ and $\Psi_{\mathrm{AR}}$ are just determined in such a way as to transform the DB and AR systems into a NDC system the same conclusion also holds for these systems.

The stated result implies that the appropriate budget-neutral discount rate for a stationary situation is given by the internal rate of return. This has often been claimed in the related literature but the present framework allows to formulate it in a precise manner and to state the exact conditions (in particular demographic stationarity) under which it holds.

**Deductions for different discount rates** In a next step one can investigate which deductions are implied by choices of the discount rate $\delta > g$. Although these choices are not necessary from the viewpoint of budgetary stability it is nevertheless instructive to see the magnitudes involved. To do so I use illustrative numerical examples. In particular, I assume a Gompertz survival curve of the form $S(a) = e^{\frac{\alpha}{\beta}\left(1-e^{\beta a}\right)}$.[29]

[29]The associated mortality rate is given by $\mu(a) = \alpha e^{\beta a}$ (i.e. the logarithm of mortality rates increases linearly in age). The Gompertz-function delivers a good description of empirical mortality data. For the

Table S.2: Deductions for $R = 64$ and $R^* = 65$

| | | | $\hat{\delta} = 0$ | | | $\hat{\delta} = 0.02$ | | | $\hat{\delta} = 0.05$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Type $j$ | $\widehat{P}_j$ | $\Psi_j$ | $X_j$ | $x_j$(in%) | $P_j$ | $X_j$ | $x_j$(in%) | $P_j$ | $X_j$ | $x_j$(in%) | $P_j$ |
| **DB** | 67.0 | 0.93 | 0.93 | -7.04 | 62.3 | 0.91 | -8.71 | 61.1 | 0.89 | -11.49 | 59.3 |
| **AR** | 65.5 | 0.95 | 0.95 | -4.92 | 62.3 | 0.93 | -6.64 | 61.1 | 0.91 | -9.48 | 59.3 |
| **NDC** | 62.3 | 1 | 1 | 0 | 62.3 | 0.98 | -1.81 | 61.1 | 0.95 | -4.79 | 59.3 |

*Note:* The table shows the actuarial deduction factors $X_j$, the annual deductions rates $x_j$ (based on the linear relation $x_j = \frac{X_j - 1}{R^* - R}$) and the final pension $P_j(R, R^*) = \widehat{P}_j(R, R^*)X_j$ for three pension schemes and three (net) discount rates $\hat{\delta} \equiv \delta - g$. For the sake of comparison also the values of the pure demographic deduction factors $\Psi_j$ are reported. The numerical values are: $A = 20$, $\tau = 0.25$, $g = 0.02$, $R^* = 65$ and $R = 64$. For all three schemes the target pension is $P^* = 67$. Mortality follows a Gompertz distribution with $\alpha = 0.000025$ and $\beta = 0.096$.

In Table S.2 I report the deduction factors $X_j$ and annual deduction rates $x_j$ for various assumption concerning the (net) discount rate $\hat{\delta} \equiv \delta - g$. It corresponds to table 2 from section 3.2 which was based on rectangular survivorship. In addition, I also report the demographic deduction factors $\Psi_j$ that are sufficient for budgetary stability. For all three systems the target pension level at the target retirement age $R^* = 65$ is given by $P^* = 67$ (which implies a target replacement rate for the DB system of $q^* = 0.67$). This is lower than for the case of rectangular survivorship since —due to premature deaths—remaining life expectancy $e(R^*)$ at $R^* = 65$ is now larger (18.6 vs. 15). The step-up of pensions due to inheritance gains is non-trivial and given by $h(R^*) = 1.11$ (or 11%). This is due to the fact that only 88% of all members of a cohort survive up to this age ($S(R^*) = 0.88$).

The results are qualitatively similar to the ones for rectangular survivorship in table 2 The first thing to note is that for $\hat{\delta} = 0$ (i.e. $\delta = g$) the total deductions correspond exactly to the demographic adjustment factors, i.e. $X_j = \Psi_j$ as has already been shown in section S.2.4. For the DB system the correct annual deduction rate for a retirement at the age of 64 is 7% while it is 4.9% for the AR system if mortality follows a Gompertz pattern. This is smaller than the corresponding rates for rectangular survival where they have been calculated as 8.33% and 6.25%, respectively. For larger discount rates, however, the difference shrinks and for $\hat{\delta} = 0.05$, e.g., the annual deduction rates for the Gompertz

following examples I use a parameterization of $\alpha = 0.000025$ and $\beta = 0.096$. This is roughly based on the Austrian life tables from 2005 which have the convenient property that the (unisex) life expectancy at birth was almost exactly 80 which facilitates comparisons to the numerical examples of section 3 with rectangular survivorship.

Table S.3: Deficit ratios $d = E/I$

| Type $j$ | Distribution 1 $\hat{\delta}=0$ | Distribution 1 $\hat{\delta}=0.02$ | Distribution 1 $\hat{\delta}=0.05$ | Distribution 2 $\hat{\delta}=0$ | Distribution 2 $\hat{\delta}=0.02$ | Distribution 2 $\hat{\delta}=0.05$ | Distribution 3 $\hat{\delta}=0$ | Distribution 3 $\hat{\delta}=0.02$ | Distribution 3 $\hat{\delta}=0.05$ |
|---|---|---|---|---|---|---|---|---|---|
| **DB** | 1 | 1 | 1.004 | 1 | 0.982 | 0.954 | 1 | 1.001 | 1.003 |
| **AR** | 1 | 1 | 1.004 | 1 | 0.982 | 0.954 | 1 | 1 | 1.003 |
| **NDC** | 1 | 1 | 1.004 | 1 | 0.982 | 0.954 | 1 | 1 | 1.003 |

*Note:* The table shows the deficit ratio for three pension schemes, three assumptions of the discount rate $\hat{\delta} \equiv \delta - g$ and three assumed distributions of the retirement age. These are $R^L = 60$, $R^{mod} = 65$, $R^H = 70$ (distribution 1), $R^L = 60$, $R^{mod} = 65$, $R^H = 67$ (distribution 2) and $R^L = 60$, $R^{mod} = 67$, $R^H = 68$ (distribution 3). The mean retirement age is $\overline{R} = 65$ for distributions 1 and 3 and $\overline{R} = 64$ for distribution 2. Mortality follows a Gompertz distribution with $\alpha = 0.000025$ and $\beta = 0.096$ and the target retirement age is always $R^* = 65$.

and the rectangular case are very similar.

**Balanced and unbalanced budgets for different discount rates** In the simple framework of sections 3 and 4 I have shown that even for discount rates that are larger than necessary it might still be the case that the system runs a balanced budget. In particular, I have shown there that the balanced budget condition depends on the choice of both the discount rate $\delta$ and the target retirement age $R^*$. This can be repeated for the general framework. It is not possible to derive closed-form solutions for the balanced budget and one has to resort to numerical calculations. In particular, I assume that retirement ages follow a triangular distribution that is defined by the minimum and maximum retirement ages $R^L$ and $R^H$, respectively and also by the mode $R^{mod}$. The density function is then given by $f(R) = \frac{2(R-R^L)}{(R^H-R^L)(R^{mod}-R^L)}$ for $R^L \leq R \leq R^{mod}$ and $f(R) = \frac{2(R^H-R)}{(R^H-R^L)(R^H-R^{mod})}$ for $R^{mod} \leq R \leq R^H$.[30]

In table S.3 I show three distributions that differ in the shape and the average retirement age $\overline{R} = \frac{R^L+R^{mod}+R^H}{3}$. In all three distributions I assume that the earliest retirement age is given by $R^L = 60$. In the first distribution $R^H = 70$ and the modus, median and mean coincide at $R^{mod} = \overline{R} = 65$. In the second distribution the modus is again $R^{mod} = 65$ while the maximum retirement age is $R^H = 67$ implying a mean of $\overline{R} = 64$. Finally, for the third distribution I assume a non-symmetric case with $\overline{R} = 68$ and $R^{mod} = 67$ which implies an average retirement age of $\overline{R} = 65$.

The first result is that the budget is in balance for all three distributions of retirement

[30] I have also analyzed different retirement distributions. The results are robust.

ages as long as the discount rate is equal to the growth rate of wages (i.e. $\hat{\delta} = 0$). This is of course an expected result that follows from the analytical findings of section S.2.4. As a second result one can see that the budget is also (approximately) balanced for situations where $\hat{\delta} > 0$ as long as $R^* = \overline{R}$ (which is the case for distributions 1 and 3). For the second distribution, however, for which $R^* = 65 > 64 = \overline{R}$ this is different. In this case the pension system runs a permanent surplus if the discount rate is above the growth rate of wages ($d = 0.98$ for $\hat{\delta} = 0.02$ and $d = 0.95$ for $\hat{\delta} = 0.05$).

These results are completely parallel to the ones of section 4. In particular, the budget is balanced as long as the discount rate is set equal to the growth rate of wages (as has already been shown analytically in section S.2.4). Furthermore, the budget also turns out to be (approximately) balanced for situations where the discount rate differs from this benchmark value as long as $R^* = \overline{R}$.

### S.2.5 Derivation of the pension formulas for the different systems

In this section I derive the formulas for the different pension systems that have been stated and used above in section S.2.2.

**Notional defined contribution pension system** I start with the discussion of how the pension level $P(R, a, t)$ is determined in NDC systems. This provides again a useful benchmark case to derive the necessary deductions and supplements for the two other PAYG pension systems (AR and DB).

The contributions in a NDC system are credited to a notional account and they are revalued with a "notional interest rate" $\rho(a, t)$ (that is allowed to change over time and across ages). The total value of this account is called the "notional capital" that accumulates over the working periods of an insured person. When the individual retires at age $R$ the final notional capital is given by:

$$K(R, t) = \int_A^R \tau W(t + x) e^{\int_x^R \rho(s, t+s)\, \mathrm{d}s}\, \mathrm{d}x, \tag{42}$$

where the cumulative factor $e^{\int_x^R \rho(s,t+s)\, \mathrm{d}s}$ indicates how the contribution $\tau W(t + x)$ that is paid into the pension system in period $t + x$ is revalued when calculating the final amount of the notional capital in period $t + R$ (the period of retirement). The notional interest rate is a crucial magnitude in a NDC system as I discuss in a different paper (see Knell 2018). In section S.2.2 I use two standard definitions that can be found in real-world NDC systems. Both notional interest rates are related to the growth rate of productivity (or of average wages) while one also contains a correction for the fast that the cohort size decreases with the mortality rate $\mu(a)$. The account values of the deceased cohort members are regarded as “inheritance gains” that are distributed among surviving cohort members by granting an extra return $\mu(a)$. In particular:

$$\rho(a,t) = g(t) + \mu(a)$$

or

$$\rho(a,t) = g(t),$$

which corresponds to equations (28a) and (28b) in section S.2.2. Using these definitions for the notional interest rate in (42) one can conclude that $K(R,t) = \tau W(t+R)(R-A)$ if one uses the value of $\rho(a,t)$ without inheritance gains (equation (28b)) or $K(R,t) = \tau W(t+R)\int_A^R e^{\int_a^R \mu(s)\,\mathrm{d}s}\,\mathrm{d}a$ if one uses the specification that includes the inheritance gains (equation (28a)). This can also be written as $K(R,t) = \tau W(t+R)(R-A)h(R)$, where $h(R) \equiv \frac{\int_A^R S(a)\,\mathrm{d}a}{(R-A)S(R)}$ as expressed in equation (29). The term $h(R)$ stands for the per capita “inheritance gains premium” to the “normal” notional capital, averaged over the $(R-A)$ contribution periods and distributed among the mass $S(R)$ of surviving members.[31]

The first pension that is received by a member of cohort $t$ who retires at the age $R$ is given by:

$$P_{\mathrm{NDC}}(R,R,t) = \frac{K(R,t)}{e(R)} X_{\mathrm{NDC}}(R,t). \tag{44}$$

The first pension is calculated by taking the final notional capital $K(R,t)$ and turning it into an annual pension payment by using remaining life expectancy $e(R)$ as the annuity conversion factor. In addition, there might also be a deduction factor $X_{\mathrm{NDC}}(R,t)$ that is applied to secure a balanced budget if the formula pension $\frac{K(R,t)}{e(R)}$ is not sufficient. For simplicity I write this demographic deduction factor only as a function of the actual retirement age $R$ and time although—in general—it will also depend on other demographic and policy variables (like $R^*$ and $A$). The dependence on time is relevant for the case of non-stationary retirement distributions.

[31] The relation follows by noting that $e^{\int_a^R \mu(s)\,\mathrm{d}s}$ can be written as: $e^{\int_a^R \mu(s)\,\mathrm{d}s} = e^{\int_0^R \mu(s)\,\mathrm{d}s} e^{\int_0^a -\mu(s)\,\mathrm{d}s}$. Therefore $\int_A^R e^{\int_a^R \mu(s)\,\mathrm{d}s}\,\mathrm{d}a = e^{\int_0^R \mu(s)\,\mathrm{d}s} \int_A^R e^{\int_0^a -\mu(s)\,\mathrm{d}s}\,\mathrm{d}a = (S(R))^{-1}\int_A^R S(a)\,\mathrm{d}a$, where I use (22).

From equation (23) it is known that:

$$e(R) = \frac{\int_R^\omega S(a)\,\mathrm{d}a}{S(R)}. \tag{45}$$

This can be used together with $K(R,t) = \tau W(t+R)(R-A)h(R) = \tau W(t+R)\frac{\int_A^R S(a)\,\mathrm{d}a}{S(R)}$ to derive the first pension in the case that the notional interest rate includes a correction for the inheritance gains (i.e. $\rho(a,t) = g(t) + \mu(a)$):

$$\begin{aligned} P_{\mathrm{NDC}}(R,R,t) &= \tau W(t+R)\frac{(R-A)h(R)}{e(R)} X_{\mathrm{NDC}}(R,t) \\ &= \tau W(t+R)\frac{\int_A^R S(a)\,\mathrm{d}a}{\int_R^\omega S(a)\,\mathrm{d}a} X_{\mathrm{NDC}}(R,t). \end{aligned} \tag{46}$$

This expression is quite intuitive. At the moment of retirement the first pension payment is proportional to the wage level in the period of retirement $t+R$. This is due to the fact that past contributions are indexed to average wage growth. The inclusion of inheritance gains raises the notional capital (which is captured by the expression in the numerator) while the period pension payment depends on remaining life expectancy (which is captured by the expression in the denominator). In addition there might be a correction for early or late retirement $X_{\mathrm{NDC}}(R,t)$.

For the situation where inheritance gains are not included and where the notional interest rate is simply given by $\rho(t) = g(t)$ the first pension is:

$$\begin{aligned} P_{NDC'}(R,R,t) &= \tau W(t+R)\frac{R-A}{e(R)} X_{NDC'}(R,t) \\ &= \tau W(t+R)\frac{R-A}{\frac{\int_R^\omega S(a)\,\mathrm{d}a}{S(R)}} X_{NDC'}(R,t). \end{aligned} \tag{47}$$

For the case of rectangular survivorship ($S(a) = 1$ for $a \in [A,\omega]$) one gets that both (46) and (47) lead to the same result that $P_{\mathrm{NDC}}(R,R,t) = \tau W(t+R)\frac{R-A}{\omega-R} X_{\mathrm{NDC}}(R,t)$. This is the same expression that was used in section 3 (see table 1).

Existing pensions are adjusted according to:

$$P_j(R,a,t) = P_j(R,R,t)e^{\int_R^a \vartheta(t+s)\,\mathrm{d}s}, \tag{48}$$

for $a \in [R, \omega^c(t)]$ and where I use the index $j$ since the adjustment in (48) is valid for all three pension systems $j \in \{\text{DB}, \text{AR}, \text{NDC}\}$. The variable $\vartheta(t)$ stands for the adjustment rate in period $t$ and the cumulative adjustment factor $e^{\int_R^a \vartheta(t+s)\,\mathrm{d}s}$ indicates how the first pension $P(R,R,t)$ is adjusted to give the pension payment in period $t+a$. In section S.2.2 I assume that ongoing pensions are adjusted with the average growth rate of wages as expressed in equation (30) stating that $\vartheta(t) = g(t)$.

In this case one can use (46) and (48) to conclude that with $\rho(a,t) = g(t) + \mu(a)$ the ongoing pension is:

$$P_{\text{NDC}}(R,a,t) = \tau W(t+a)\frac{(R-A)h(R)}{e(R)} X_{\text{NDC}}(R,t),$$

which is equation (31) in section S.2.2. For $\rho(a,t) = g(t)$ it holds that $P_{NDC'}(R,a,t) = \tau W(t+a)\frac{R-A}{e(R)}\Psi_{NDC'}(R)$. In the following I focus on the first case with a compensation for inheritance gains.

**Defined benefit pension system** In a similar vein one can look at the two alternative pension systems discussed in section 3. The defined benefit (DB) system promises a target replacement rate $q^*$ if an individual retires at the target retirement age $R^*$. The replacement rate is related to average lifetime income, where past incomes are revalued at a rate $\rho(a,t)$ and where there are correction for early/late retirement. In particular, instead of (44) it now holds that:

$$P_{\text{DB}}(R,R,t) = q^* \overline{W}^{LT}(R,t) X_{\text{DB}}(R,t), \tag{49}$$

where $\overline{W}^{LT}(R,t) = \frac{\int_A^R W(t+x) e^{\int_x^R \rho(s,t+s)\,\mathrm{d}s}\,\mathrm{d}x}{R-A}$. This expression is closely related to the notional capital (42) for NDC systems. I know of no existing DB system that includes a correction for inheritance gains when indexing past wage levels. Therefore the benchmark DB system is characterized by the indexation $\rho(a,t) = g(t)$. From this is follows that $P_{\text{DB}}(R,R,t) = q^* W(t+R) X_{\text{DB}}(R,t)$. As above I assume that existing pension are adjusted with the growth rate of average wages according to (48) and (30) and thus:

$$P_{\text{DB}}(R,a,t) = q^* W(t+a) X_{\text{DB}}(R,t),$$

which corresponds to equation (32) in section S.2.2.

**Accrual rate pension system** Finally, one can look at the accrual rate system in the general set-up. The AR system promises a pension that is proportional to the revalued average lifetime income. In particular, for each year of work the system promises a certain percentage $\kappa^*$ (the accrual rate) of this lifetime average that can be claimed at the target retirement age $R^*$. For early retirement there exists a deduction $X_{\text{AR}}(R,t)$. In particular, the first pension is now defined as:

$$P_{\text{AR}}(R,R,t) = \kappa^*(R-A)\overline{W}^{LT}(R,t)X_{\text{AR}}(R,t), \tag{50}$$

where $\overline{W}^{LT}(R,t)$ stands for lifetime income (as defined in section S.2.5) for which past incomes are revalued at a rate $\rho(a,t)$. As before and in line with existing AR system I assume that indexation follows the growth rate of average wages, i.e. $\rho(a,t) = g(t)$. From this is follows that $P_{\text{AR}}(R,R,t) = \kappa^* W(t+R)(R-A)X_{\text{AR}}(R,t)$. For pension adjustment according to (48) and $\vartheta(t) = g(t)$ one can conclude that:

$$P_{\text{AR}}(R,a,t) = \kappa^* W(t+a)(R-A)X_{\text{AR}}(R,t).$$

which corresponds to equation (33) in section S.2.2.

### S.2.6 Demographic adjustment factors

Proposition 2 shows that for a stationary retirement distribution $f(R)$ the NDC system is stable for $X(R,t) = \Psi(R) = 1$. It is now straightforward to discuss the demographic deductions $\Psi_j(R)$ that are necessary to establish balanced budget for pension systems that deviate from the NDC benchmark. This can be seen by looking at equation (34). If the pension payments of the alternative system can be written as $P_j(R,a,t) = \tau W(t+a)(\cdots)$ then the correction $\Psi_j(R)$ just has to be set in a way such that it mimics the benchmark NDC-pension given in (37). As a first example one can look at a $NDC'$ system that does not include the compensation for inheritance gains (as is the case for most existing NDC systems with the notable exception of Sweden) and that sets $\rho(a,t) = g(t)$. In this case it has been shown above that the pension is given by $P_{NDC'}(R,a,t) = \tau W(t+a)\frac{(R-A)}{e(R)}\Psi_{NDC'}(R)$. It is immediately apparent that an adjustment with $\Psi_{NDC'}(R) = h(R)$ leads to a balanced budget. Otherwise, the pension system would run a surplus since $h(R) > 1$, i.e. the system would keep the inheritance gains for itself instead of distributing them among the surviving population.

For the defined benefit system I invoke as in section 3 the "balanced target condition", i.e. I assume that the target replacement rate $q^*$ is associated with a situation that there will not be any deductions ($\Psi_{\mathrm{DB}}(R^*) = 1$) if everybody retires at the target retirement age $R^*$. For the NDC system one knows from equation (37) that a balanced budget with $R = R^*, \forall i$ requires that everybody gets a pension equal to $P_{\mathrm{NDC}}(R^*, a, t) = \tau W(t + a)\frac{(R^*-A)h(R^*)}{e(R^*)}$. This should be equal to the DB pension with $R = R^*, \forall i$, i.e. to $P_{\mathrm{DB}}(R^*, a, t) = q^* W(t + a)$. From these two expressions it follows that $q^* = \tau\frac{(R^*-A)h(R^*)}{e(R^*)}$. Inserting this into equation (32) for $P_{\mathrm{DB}}(R, a, t)$ leads to the DB pension after evoking the stability condition:

$$P_{\mathrm{DB}}(R, a, t) = \tau W(t + a)\frac{(R^* - A)h(R^*)}{e(R^*)}\Psi_{\mathrm{DB}}(R). \tag{51}$$

This expression can now be compared to the pension of the benchmark NDC system (37) (that leads to a balanced budget) to conclude that $\Psi_{\mathrm{DB}}(R) = \frac{R-A}{R^*-A}\frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}$ as stated in equation (38).

One can use similar steps for the AR pension system. In particular, I assume that the target accrual rate $\kappa^*$ is chosen in such a way that the system is balanced if everybody retires at the target retirement age $R = R^*, \forall i$. Inserting the implied target accrual rate $\kappa^* = \tau\frac{h(R^*)}{e(R^*)}$ into equation (33) for $P_{\mathrm{AR}}(R, a, t)$ leads to the AR pension after evoking the stability condition:

$$P_{\mathrm{AR}}(R, a, t) = \tau W(t + a)\frac{(R - A)h(R^*)}{e(R^*)}\Psi_{\mathrm{AR}}(R). \tag{52}$$

The deduction rate that is necessary for a balanced AR system can be calculated by setting (52) equal to the NDC pension (37) and solving for $\Psi_{\mathrm{AR}}(R)$. This leads to $\Psi_{\mathrm{AR}}(R) = \frac{e(R^*)}{e(R)}\frac{h(R)}{h(R^*)}$ as stated in equation (39). These and other important formulas are collected in table S.1 of appendix S.2.2.

## S.3 Proofs of the extensions of appendix S.1.2

In the main part of the paper I have assumed that all individuals have the same entry age $A$, the same wage level $W$ and the same life expectancy $\omega$. In section S.1.2 I have already summarized the results for some extensions of this basic framework in the case that there are no long-run demographic changes (to life expectancy or to the average retirement age). In this appendix I now provide the proofs for these already stated results.

In order to do so I have to broaden the basic framework and I now assume that the society is split in $j = 1, 2, \dots J$ groups (with respective weight $\pi_j$) where each group is characterized by specific values for $A_j$, $W_j$ and possibly also $R_j^*$ (i.e. the system might prescribe group-specific target retirement ages).[32] In a previous version of the paper I also discussed the case with group-specific maximum ages $\omega_j$, This discussion is left out in the following but the results are available upon request. A possible correlation between income, entry age etc. is captured by the pattern of relative frequencies $\pi_j$ of the different groups with $\sum_{j=1}^{J} \pi_j = 1$. Although members of each group are identical along all the above mentioned dimensions they might differ in their retirement age. In particular, I assume that within each group $j$ there are $k = 1, 2, \dots K$ subgroups of individuals with a retirement age $R_j^k$ and a relative frequency $\phi_j^k$ with $\sum_{k=1}^{K} \phi_j^k = 1$. The within-group differences in the retirement age are important in order to calculate the appropriate deductions for early retirement at the age of $R_j^k$ instead of $R_j^*$. I now allow for the case of a "biased NDC system", i.e. $\widehat{P}_j^k = \tau W_j \eta_j \frac{R_j^k - A_j}{\omega - R_j^k}$. For $eta_j > 1$ the system is, e.g., more generous than the benchmark NDC system.

For the benchmark case with $R_j^k = R_j^*$ for all members $k$, revenues and expenditures are given by $I_j^* = \tau W_j (R_j^* - A_j)$ and $E_j^* = P_j^* (\omega - R_j^*) = \tau W_j \eta_j (R_j^* - A_j)$ (where I normalize the cohort size to $N_j = 1$). The deficit therefore comes out as $D_j^* = E_j^* - I_j^* = \tau W_j (\eta_j - 1)(R_j^* - A_j)$ and the deficit ratio as $d_j^* = E_j^* / I_j^* = \eta_j - 1$.

Members of group $j$ differ, however, in their retirement ages $R_j^k$. One can use the same steps as in section 3.2 of the paper to calculate the appropriate level of deductions $X_j^k$ that leaves the present value of payment streams unchanged for early or late retirement. This leads to:

$$X_j^k = 1 + \frac{R_j^* - R_j^k}{R_j^k - A_j} \frac{\eta_j - 1}{\eta_j} - \frac{\delta}{2} (R^* - R_j^k) \frac{\omega - R_j^* + \eta_j (R_j^* - A_j)}{\eta_j (R_j^k - A_j)}. \tag{53}$$

For $\eta_j = 1$ this is the same expression as (10) for the NDC system. The influence of $\eta_j$ on the budget-neutral deduction is not huge but still non-negligible. E.g., for $\delta = 0$, $A_j = 20$, $\omega = 80$, $R_j^* = 65$ and $\eta_j = 1.25$ one gets that $X_j^k = 1.0046$ (for $R_j^k = 64$). The early retiree receives an extra *supplement* of about 0.5%. The reason for this is that the

[32] In fact, one could also introduce heterogeneous wage profiles. Assume that group $j$ of the cohort born in period $t$ earned a wage $W_j(a, t) = \xi_j(a, t) W(t + a)$ in each of its working period between age $A_j$ and age $R_j$ where $W(t)$ now is the aggregate wage in period $t$. For the NDC system the notional capital would then be given by $K_j(R_j, t) = \tau W(t + R)(R_j - A_j) \overline{EP}_j$, where $\overline{EP}_j \equiv \left( \int_{A_j}^{R_j} \xi_j(a, t + a) \, \mathrm{d}a \right) / (R_j - A_j)$ stands for the average lifetime "earnings points" (to use an expression from the German pension system). If group $j$ had always earned average wages then $\xi_j(a, t + a) = 1, \forall a$ and $\overline{EP}_j = 1$.

normal NDC adjustment for early retirement lowers the annual pension payment and the multiplicative reward $\eta_j$ is thus applied to a smaller "base" which would—in the absence of $X_j^k$—reduce the total pension payments for early retirees in group $j$.

Total revenues for the group $j$ are given by $I_j = \sum_{k=1}^{K} \phi_j^k \tau W_j(R_j^k - A_j) = \tau W_j(\overline{R}_j - A_j)$, where $\overline{R}_j \equiv \sum_{k=1}^{K} \phi_j^k R_j^k$ stands for the average retirement age among the members of group $j$. Total expenditures, on the other hand, are given by $E_j = \sum_{k=1}^{K}(\omega - R_j^k)P_j^k$, where $P_j^k = \widehat{P}_j^k X_j^k$ is the pension payment associated with retirement age $R_j^k$ (including the additional deduction $X_j^k$). Using (53) one can conclude that (for $\delta = 0$) $(\omega - R_j^k)P_j^k = \tau W_j(\eta_j(R_j^k - A_j) + (R_j^* - R_j^k)(\eta_j - 1))$ and thus $E_j = \tau W_j(\eta_j(\overline{R}_j - A_j) + (R_j^* - \overline{R}_j)(\eta_j - 1))$. It follows that $D_j = E_j - I_j = \tau W_j(\eta_j - 1)(R_j^* - A_j)$ which is the same as in the benchmark with $R_j^k = R_j^*, \forall k$. As far as the deficit ratio is concerned one has to note that total revenues are given by $I_j = \tau W_j(\overline{R}_j - A_j)$. For $R_j^* = \overline{R}_j$ it is the case that $d_j = (\eta_j - 1)$ (which is the same as in the benchmark). For a homogeneous target age $R_j^* = R^*$, however, this is only possible if the mean retirement age is the same across all groups $j$ which cannot be taken for granted.

The size of the overall deficit $\overline{D} = \sum_{j=1}^{J} \pi_j D_j$ depends on the correlation between variables $A_j$, $W_j$ and $R_j^*$ since $\overline{D} = \sum_{j=1}^{J} \pi_j \tau W_j(\eta_j - 1)(R_j^* - A_j)$. In the absence of a "bias" (i.e for $\eta_j$=1) one can observe, however, that these possible correlations do not matter since each subgroup will have a balanced budget with $D_j = 0$ and thus also the overall budget $\overline{D}$ will be balanced.

The "bonus" (or "social security wealth") of the "biased NDC system" to different individuals $k$ in group $j$ is given by:

$$B_j^k = (\omega - R_j^k)P_j^k - \tau W_j(R_j^k - A_j) = \tau W_j(\eta_j - 1)(R_j^* - A_j).$$

This is the same for different members of group $j$ (i.e. it does not vary by the individual retirement age $R_j^k$).

The use of the "normal" deduction factor $X_j^k = 1$ instead of (53) leads to $(\omega - R_j^k)P_j^k = \tau W_j \eta_j (R_j^k - A_j)$ and thus $E_j = \tau W_j \eta_j (\overline{R}_j - A_j)$ and $D_j = \tau W_j(\eta_j - 1)(\overline{R}_j - A_j)$. This is the same deficit as in the benchmark. Now, however, the "bonus" that the system pays to different individuals is not identical for all members $k$ but rather $B_j^k = \tau W_j(\eta_j - 1)(R_j^k - A_j)$. The difference is not huge (about 2.3% for $R_j^k = 64$ and 12.5% for $R_j^k = 60$), but certainly not completely irrelevant.