Supplementary Material

to accompany

"A Universal Model for Growth of User Population of of Products and Services"

Derivation of General Model for Uncorrelated User Growth Network

Consider a network $G = (V, E)$ of users and prospective users of a product or services. We assume that

- the node degree of the network is $k_1 < k_2 < ... < k_I$;
- the number of nodes with degree k_i is N_i;
- at time t, the number of nodes (users) with degree k_i is m_i.

Define $\vec{m} = \{m_1, m_2, \cdots, m_I\}^T$, $\vec{k} = \{k_1, k_2, \cdots, k_I\}^T$ and $\vec{N} = \{N_1, N_2, \cdots N_I\}^T$. Let $P(X(t) = \vec{m})$ denote the probability that there are $\vec{m}$ users in the network at time t. $P(X(t + \Delta t) = \vec{n}\,|x(t) = \vec{m}\,)$ be the transition probability of $X(t) = \vec{n}$ at time $t + \Delta t$, conditioned upon $X(t) = \vec{m}$ at the immediate preceding time t. Then, the occurrence of the event $X(t + \Delta t) = \vec{n}$ can be thought of as the occurrence of the joint event involving $(X(t + \Delta t) = \vec{n}, X(t) = \vec{m})$ and $(X(t + \Delta t) = \vec{n}, X(t) = \vec{n})$. With no loss of generality, we assume that at infinitesimal time interval Δt, at most one prospective transition link undergoes a transition, i.e., at most one prospective user transits to a user. Then, we have

$$P(X(t+\Delta t) = \vec{n}) = \sum_{i=1}^{I} P(X(t) = \vec{n} - \vec{r}_i) \times P(X(t+\Delta t) = \vec{n}\,|X(t) = \vec{n} - \vec{r}_i\,)$$
$$+ P(X(t) = \vec{n}) \times (1 - \sum_{i=1}^{I} P(X(t) = \vec{n} + \vec{r}_i\,|X(t) = \vec{n}\,)$$

where $\vec{r}_i = \{0, 0, \cdots, \underset{ith}{1}, \cdots, 0, 0\}^T$.

Now, we need to find the exact form of $P(X(t + \Delta t) = \vec{n}\,|X(t) = \vec{n} - \vec{r}_i\,)$, which corresponds to a node with degree k_i transitting from state P to state U.

$$P(X(t+\Delta t) = \vec{n}\,|X(t) = \vec{n} - \vec{r}_i\,) = h_1(\vec{n} - \vec{r}_i)c_1\Delta t + h_2(\vec{n} - \vec{r}_i)c_2\Delta t,$$

where, for transition channel $T_\mu(\mu = 1, 2)$, the number of prospective transition links $h_\mu(\vec{n} - \vec{r}_i)$ represents the number of distinct prospective transition links with $(\vec{n} - \vec{r}_i)$ users. It is readily shown that $h_2(\vec{n} - \vec{r}_i) = n_i - 1$. For $h_1(\vec{n} - \vec{r}_i)$, we assume that the network is an uncorrelated network. In this case, the number of prospective users with degree k_i is $(N_i - n_i + 1)$. Hence, the total number of links from a prospective user with node degree k_i is $k_i \cdot (N_i - n_i + 1)$.

As the network is uncorrelated, the probability that a link points to a node with s connections is equal to $sP(s)/\langle k\rangle$. Furthermore, the number of nodes with degree s is n_s, and we assume that the users are homogeneously distributed. Then, the probability of a link from a prospective user with degree k_i points to a user with degree s is

$$\frac{sP(s)}{\langle k\rangle} \cdot \frac{n_s}{N_s} = \frac{sP(s)N}{N\langle k\rangle} \cdot \frac{n_s}{N_s} = \frac{sn_s}{N\langle k\rangle}.$$

Note that

$$\vec{n}-\vec{r}_i=\{n_1,n_2,\cdots,n_i-1,\cdots,n_I\}^T.$$

Hence, the total number of distinct prospective transition links with $\vec{n}-\vec{r}_i$ users of channel T_1 is

$$k_i(N_i-n_i+1)(\sum_{j=1,j\neq i}^{I}\frac{k_jn_j}{N\langle k\rangle}+\frac{k_i(n_i-1)}{N\langle k\rangle})=\frac{k_i(N_i-n_i+1)}{N\langle k\rangle}(-k_i+\sum\nolimits_{j=1}^{I}k_j\cdot n_j).$$

We have

$$\begin{aligned}&P(X(t+\Delta t)=\vec{n}\,|X(t)=\vec{n}-\vec{r}_i\,)\\&=(N_i-n_i+1)c_2\Delta t+\frac{k_i(N_i-n_i+1)}{N\langle k\rangle}(-k_i+\sum\nolimits_{j=1}^{I}k_j\cdot n_j)c_1\Delta t.\end{aligned}$$

Thus, the probability $P(X(t+\Delta t)=\vec{n})$ is

$$\begin{aligned}&P(X(t+\Delta t)=\vec{n})\\&=\sum\nolimits_{i=1}^{I}\left[P(X(t)=\vec{n}-\vec{r}_i)\times(N_i-n_i+1)c_2\Delta t+\frac{k_i(N_i-n_i+1)}{N\langle k\rangle}(-k_i+\sum\nolimits_{j=1}^{I}k_j\cdot n_j)c_1\Delta t\right]\\&+P(X(t)=\vec{n})\times(1-\left[\sum\nolimits_{i=1}^{I}(N_i-n_i)c_2\Delta t+(\frac{k_i(N_i-n_i)}{N\langle k\rangle}\cdot\sum\nolimits_{j=1}^{I}k_jn_j)c_1\Delta t\right]).\end{aligned}$$

Rearranging the previous formula, we get

$$\begin{aligned}P(X(t+\Delta t)=\vec{n})=P(X(t)=\vec{n})&\\&+c_2\Delta t\left[\sum\nolimits_{i=1}^{I}P(X(t)=\vec{n}-\vec{r}_i)\cdot(N_i-n_i+1)-P(X(t)=\vec{n})\sum\nolimits_{i=1}^{I}(N_i-n_i)\right]\\&+\frac{c_1\Delta t}{N\langle k\rangle}\left[\sum\nolimits_{i=1}^{I}P(X(t)=\vec{n}-\vec{r}_i)\cdot k_i(N_i-n_i+1)(-k_i+\sum\nolimits_{j=1}^{I}k_j\cdot n_j)\right.\\&\left.-P(X(t)=\vec{n})\cdot\sum\nolimits_{i=1}^{I}k_i(N_i-n_i)\cdot\sum\nolimits_{j=1}^{I}k_jn_j\right].\end{aligned}\tag{1}$$

The expectation of $\vec{n}$ is

$$\begin{aligned}\langle\vec{n}\rangle&=\sum\nolimits_{\vec{n}}\vec{n}P(X(t+\Delta t)=\vec{n})\\&=\sum\nolimits_{\vec{n}}\vec{n}P(X(t)=\vec{n})\cdots(\mathrm{I})\\&+\sum\nolimits_{\vec{n}}\vec{n}c_2\Delta t\left[\sum\nolimits_{i=1}^{I}P(X(t)=\vec{n}-\vec{r}_i).(N_i-n_i+1)-P(X(t)=\vec{n})\sum\nolimits_{i=1}^{I}(N_i-n_i)\right]\cdots(\mathrm{II})\\&+\sum\nolimits_{\vec{n}}\vec{n}\frac{c_1\Delta t}{N\langle k\rangle}\left[\sum\nolimits_{i=1}^{I}P(X(t)=\vec{n}-\vec{r}_i).k_i(N_i-n_i+1)(-k_i+\sum\nolimits_{j=1}^{I}k_jn_j)\cdots(\mathrm{III})\right.\\&\left.-P(X(t)=\vec{n}\cdot\sum\nolimits_{i=1}^{I}k_i(N_i-n_i)\sum\nolimits_{j=1}^{I}k_jn_j\right].\end{aligned}\tag{2}$$

First, we consider part (II) of Eq. (2)

$$\begin{aligned}&\sum\nolimits_{\vec{n}}\vec{n}\sum\nolimits_{i=1}^{I}P(X(t)=\vec{n}-\vec{r}_i)\cdot(N_i-n_i+1)-P(X(t)=\vec{n})\sum\nolimits_{i=1}^{I}(N_i-n_i)\\&=\sum\nolimits_{i=1}^{I}\sum\nolimits_{\vec{n}}(\vec{n}-\vec{r}_i)P(X(t)=\vec{n}-\vec{r}_i)\cdot(N_i-n_i+1)+\sum\nolimits_{i=1}^{I}\vec{r}_i\sum\nolimits_{\vec{n}}P(X(t)=\vec{n}-\vec{r}_i)\cdot(N_i-n_i+1)\\&-\sum\nolimits_{i=1}^{I}\sum\nolimits_{\vec{n}}\vec{n}\cdot P(X(t)=\vec{n})(N_i-n_i).\end{aligned}$$

Note that

$$\sum_{\vec{n}} \vec{n} \cdot P(X(t)=\vec{n}) = \langle \vec{n} \rangle$$
$$\sum_{i=1}^{I} \sum_{\vec{n}} (\vec{n}-\vec{r}_i) P(X(t)=\vec{n}-\vec{r}_i) \cdot (N_i - n_i + 1) = \sum_{i=1}^{I} \sum_{\vec{n}} \vec{n} P(X(t)=\vec{n}) \cdot (N_i - n_i)$$
$$\sum_{i=1}^{I} \vec{r}_i \cdot n_i = \vec{n}.$$

Then, we have

$$\begin{aligned}
&\sum_{\vec{n}} \vec{n} \sum_{i=1}^{I} P(X(t)=\vec{n}-\vec{r}_i) \cdot (N_i - n_i + 1) - P(X(t)=\vec{n}) \sum_{i=1}^{I} (N_i - n_i) \\
&= \sum_{\vec{n}} P(X(t)=\vec{n})(\vec{N}-\vec{n}).
\end{aligned}$$

Similarly, for part (III) of Eq. (2)

$$\begin{aligned}
&\sum_{i=1}^{I} P(X(t)=\vec{n}-\vec{r}_i) \cdot k_i (N_i - n_i + 1)(-k_i + \sum_{j=1}^{I} k_j \cdot n_j) \\
&- P(X(t)=\vec{n}) \cdot \sum_{i=1}^{I} k_i (N_i - n_i) \cdot \sum_{j=1}^{I} k_j n_j.
\end{aligned}$$

The expectation is

$$\begin{aligned}
&\sum_{\vec{n}} \vec{n} \Big[\sum_{i=1}^{I} P(X(t)=\vec{n}-\vec{r}_i) \cdot k_i (N_i - n_i + 1)(-k_i + \sum_{j=1}^{I} k_j \cdot n_j) \\
&-P(X(t)=\vec{n}) \cdot \sum_{i=1}^{I} k_i (N_i - n_i) \cdot \sum_{j=1}^{I} k_j n_j \Big] \\
=&\sum_{\vec{n}} \sum_{i=1}^{I} (\vec{n}-\vec{r}_i) P(X(t)=\vec{n}-\vec{r}_i) \cdot k_i (N_i - n_i + 1) \vec{k}^T (\vec{n}-\vec{r}_i) \\
&- \sum_{\vec{n}} \sum_{i=1}^{I} \vec{n} P(X(t)=\vec{n}) k_i (N_i - n_i) \vec{k}^T \vec{n} \\
&+ \sum_{\vec{n}} \sum_{i=1}^{I} \vec{r}_i P(X(t)=\vec{n}-\vec{r}_i) \cdot k_i (N_i - n_i + 1)(-k_i + \sum_{j=1}^{I} k_j \cdot n_j).
\end{aligned} \tag{3}$$

It is readily shown that

$$\begin{aligned}
&\sum_{\vec{n}} \sum_{i=1}^{I} (\vec{n}-\vec{r}_i) P(X(t)=\vec{n}-\vec{r}_i) \cdot k_i (N_i - n_i + 1) \vec{k}^T (\vec{n}-\vec{r}_i) \\
=&\sum_{\vec{n}} \sum_{i=1}^{I} \vec{n} P(X(t)=\vec{n}) k_i (N_i - n_i) \vec{k}^T \vec{n}.
\end{aligned}$$

Thus, Eq. (3) can be simplified as

$$\begin{aligned}
&\sum_{\vec{n}} \vec{n} \Big[\sum_{i=1}^{I} P(X(t)=\vec{n}-\vec{r}_i) \cdot k_i (N_i - n_i + 1)(-k_i + \sum_{j=1}^{I} k_j \cdot n_j) \\
&-P(X(t)=\vec{n}) \cdot \sum_{i=1}^{I} k_i (N_i - n_i) \cdot \sum_{j=1}^{I} k_j n_j \Big] \\
=&\sum_{\vec{n}} \sum_{i=1}^{I} \vec{r}_i P(X(t)=\vec{n}-\vec{r}_i) \cdot k_i (N_i - n_i + 1)(-k_i + \sum_{j=1}^{I} k_j \cdot n_j).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\sum_{\vec{n}} \vec{n} P(X(t+\Delta t)=\vec{n}) =& \sum_{\vec{n}} \vec{n} P(X(t)=\vec{n}) \\
&+ c_2 \Delta t \sum_{\vec{n}} P(X(t)=\vec{n})(N-\vec{n}) \\
&+ \frac{c_1 \Delta t}{N \langle k \rangle} \sum_{\vec{n}} \sum_{i=1}^{I} \vec{r}_i P(X(t)=\vec{n}) \cdot k_i (N_i - n_i) \vec{k}^T \vec{n}.
\end{aligned} \tag{4}$$

The ith vector of $\sum_{\vec{n}}\sum_{i=1}^{I}\vec{r}_i P(X(t)=\vec{n})\cdot k_i(N_i-n_i)\vec{k}^T\vec{n}$ is

$$\begin{aligned}&\sum\nolimits_{\vec{n}} P(X(t)=\vec{n})k_i(N_i-n_i)\sum\nolimits_{j=1}^{I} k_j n_j\\&=k_i N_i\sum\nolimits_{j=1}^{I} k_j\langle n_j\rangle - k_i\langle n_i\rangle\sum\nolimits_{j=1}^{I} k_j\langle n_j\rangle(1+\delta_{ij}),\end{aligned}$$

where $\delta_{ij}=\frac{\langle n_i n_j\rangle-\langle n_i\rangle\langle n_j\rangle}{\langle n_i\rangle\langle n_j\rangle}$. Hence, we have

$$\begin{aligned}&\frac{c_1\Delta t}{N\langle k\rangle}\sum\nolimits_{\vec{n}} P(X(t)=\vec{n})k_i(N_i-n_i)\sum\nolimits_{j=1}^{I} k_j n_j\\&=\frac{c_1\Delta t k_i}{N\langle k\rangle}\times(N_i\sum\nolimits_{j=1}^{I} k_j\langle n_j\rangle-\langle n_i\rangle\sum\nolimits_{j=1}^{I} k_j\langle n_j\rangle(1+\delta_{ij})).\end{aligned}\tag{5}$$

From Eqs. (4) and (5) and taking the limit $\Delta t\to 0$, we get the I**th-order universal growth equation** as

$$\frac{\partial x_i}{\partial t}=\frac{c_1 k_i}{N\langle k\rangle}\left(N_i\sum_{j=1}^{I}k_j x_j - x_i\sum_{j=1}^{I}k_j x_j(1+\delta_{ij})\right)+c_2(N_i-x_i),\quad (i=1,2,\cdots,I),\tag{6}$$

where $x_i\triangleq\langle\vec{n}_i\rangle$ and $\delta_{ij}=\frac{\mathrm{cov}[n_i,n_j]}{\mathrm{E}[n_i,n_j]}$.