Appendix A

We can put Equation (1) in terms of u_t, giving us $u_t = Y_t - \alpha Y_{t-1} - \beta X_t$. At time $t-1$, this expression will be: $u_{t-1} = Y_{t-1} - \alpha Y_{t-2} - \beta X_{t-1}$. Substituting this into Equation (3) gives us:

$$u_t = \phi Y_{t-1} - \alpha\phi Y_{t-2} - \beta\phi X_{t-1} + e_{2t}$$

This expression for u_t can be substituted back into Equation (1), giving us:

$$Y_t = (\alpha + \phi)Y_{t-1} + (-\alpha\phi)Y_{t-2} + \beta X_t + (-\beta\phi)X_{t-1} + e_{2t}$$

Equation (4) is a restricted version of the ADL (2,1) model. Both models have the same number of lags of Y_t and X_t, but the Equation (4) model introduces restrictions on the values of the coefficients, while the ADL (2,1) model has no restrictions on coefficient values. Hendry (1995) and De Boef and Keele (2008) recommend starting with a general model and "testing down," so if we were actually fitting this model to a real dataset (with an unknown data generating process), we would want to start with the ADL (2,1) model, rather than the restricted ADL (2,1) model that is Equation (4). Validating this modeling approach with a more general model is preferable to validating it with a more restricted model. Therefore, the Monte Carlo results estimated for EQ4 in the paper will be estimated using the general ADL (2,1) model. Table 8 shows the results of the Monte Carlo simulations of the percent bias and root mean square error using the restricted parameter estimates of the EQ4 model. The percent bias of the restricted EQ4 model is around -2 percent or $+2$ percent, which is slightly higher than for the general ADL(2,1) model, which tended to have biases around 1 percent or less (see Figures 1(a) and 1(c)). The RMSE in Table 8 is about the same as the RMSE shown in Figures 1(b) and 1(d), where the EQ4 estimate is from the general ADL(2,1) model.

Table 8: Percent Bias and RMSE in Restricted EQ4 Model (under varying levels of α and ϕ)

	α					
	0.0	0.1	0.2	0.3	0.4	0.5
Percent Bias in Estimates of β	-0.87	-2.25	-1.09	-2.09	-3.66	-2.02
RMSE in Estimates of β	0.10	0.10	0.10	0.10	0.10	0.10
	ϕ					
	0.0	0.1	0.2	0.5		
Percent Bias in Estimates of β	2.27	1.76	3.21	2.47	1.67	2.52
RMSE in Estimates of β	0.07	0.07	0.08	0.09	0.09	0.09

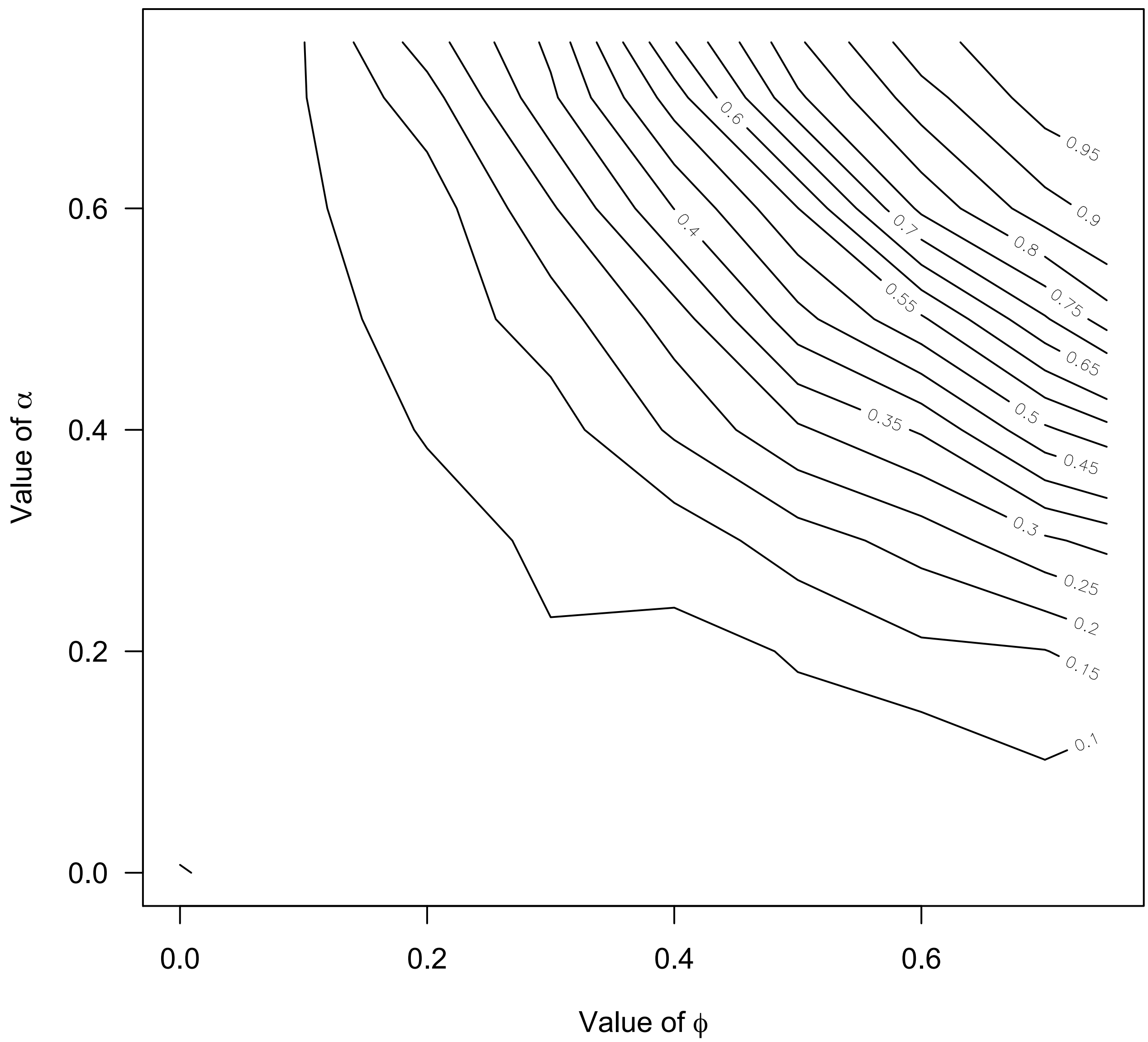
$\beta = 0.5$, $\rho = 0.95$
ϕ = 0.75 under varying levels of α
α = 0.75 under varying levels of ϕ

Table 9: Percent of Simulations Detecting Autocorrelation with EQ4 Model

	α								
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.75
$\phi = 0.0$	5.70	5.50	6.10	6.40	7.20	6.60	6.70	6.50	7.60
$\phi = 0.1$	5.60	6.10	7.20	7.10	5.10	6.50	8.30	7.30	6.60
$\phi = 0.2$	4.60	5.80	4.70	6.00	4.90	6.80	5.40	7.10	6.20
$\phi = 0.3$	5.40	5.80	5.60	6.20	6.30	5.80	6.80	7.00	8.00
$\phi = 0.4$	5.10	7.30	5.00	4.90	7.00	5.70	5.70	7.60	8.30
$\phi = 0.5$	5.10	6.10	6.20	5.60	7.50	6.00	6.70	6.00	7.60
$\phi = 0.6$	5.00	5.70	5.80	7.00	4.90	5.50	4.70	5.90	6.80
$\phi = 0.7$	8.10	5.60	4.50	6.60	6.20	8.20	6.40	6.40	7.50
$\phi = 0.75$	5.70	4.90	6.20	5.80	6.20	6.50	7.00	7.10	7.90

$\beta = 0.5$, $\rho = 0.95$

Figure 5: Percent of Simulations Detecting Autocorrelation with ADL(1,1) Model, $\beta = 0.5$, $\rho = 0.95$



Appendix B

Table 10: Percent Bias **(Mean)** in Long-Run Estimates of X_t on Y Conditional Upon Varying Levels of α (dynamic parameter)

	α									
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
EQ4	1.82	1.71	3.94	3.48	3.40	4.78	5.36	6.56	−27.94	−56.26
ADL(1,1)	1.02	25.55	58.82	94.53	137.07	188.75	249.61	270.20	418.39	3808.31
LGDV2	179.69	182.30	187.84	191.13	194.66	200.39	206.48	214.68	231.92	295.43
LGDV	158.26	180.42	207.32	234.96	266.61	306.05	368.07	517.16	664.64	−6245.95
REG	1.05	−6.48	−14.36	−23.54	−32.69	−41.85	−52.32	−63.07	−74.45	−86.79

$\beta = 0.5$, $\rho = 0.25$, $\phi = 0.75$

Table 11: Percent Bias **(Mean)** in Long-Run Estimates of X_t on Y Conditional Upon Varying Levels of α (dynamic parameter)

	α									
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
EQ4	1.95	1.26	1.51	2.27	2.58	3.62	4.49	6.93	10.49	17.68
ADL(1,1)	1.48	15.76	35.26	60.21	89.99	128.28	188.94	250.76	445.43	1372.88
LGDV2	100.31	101.03	102.58	104.24	106.65	109.29	113.86	119.42	121.72	155.26
LGDV	94.00	108.05	124.15	142.96	165.49	193.84	253.97	309.42	−11553.13	784.56
REG	0.78	−3.53	−9.42	−15.40	−22.30	−30.13	−39.54	−50.30	−63.77	−80.32

$\beta = 0.5$, $\rho = 0.55$, $\phi = 0.75$

Table 12: Percent Bias **(Mean)** in Long-Run Estimates of X_t on Y Conditional Upon Varying Levels of α (dynamic parameter)

	α									
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
EQ4	6.79	16.63	2.51	9.12	5.61	6.95	21.95	6.00	8.03	−166.10
ADL(1,1)	6.77	19.09	6.69	17.69	19.26	25.55	45.72	39.57	41.93	56.49
LGDV2	17.42	29.39	12.38	20.03	16.79	15.92	34.72	14.50	14.27	9.65
LGDV	18.26	33.11	17.23	28.45	29.39	31.04	54.18	39.88	47.05	−412.31
REG	6.80	16.79	1.26	6.83	2.66	1.14	18.36	−5.45	−12.46	−32.15

$\beta = 0.5$, $\rho = 0.95$, $\phi = 0.75$

Table 13: Percent Bias **(Mean)** in Long-Run Estimates of X_t on Y Conditional Upon Varying Levels of ϕ (autoregressive error parameter)

	ϕ									
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
EQ4	1.97	2.29	1.98	3.06	3.57	4.19	5.10	6.63	12.37	13.79
ADL(1,1)	1.71	13.17	26.95	47.17	73.23	107.86	176.29	252.52	623.96	−12038.13
LGDV2	1.94	8.14	16.18	28.42	44.27	67.71	103.36	167.99	301.06	765.36
LGDV	1.74	15.16	31.77	55.49	86.65	131.80	217.28	305.84	−319.84	−425.61
REG	−68.64	−68.70	−68.75	−68.73	−68.75	−68.56	−68.70	−68.74	−68.94	−68.75

$\beta = 0.5$, $\rho = 0.25$, $\alpha = 0.75$

Table 14: Percent Bias **(Mean)** in Long-Run Estimates of X_t on Y Conditional Upon Varying Levels of ϕ (autoregressive error parameter)

	ϕ									
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
EQ4	1.41	1.54	1.94	2.11	2.49	3.39	4.34	9.05	10.52	19.62
ADL(1,1)	1.17	9.13	19.61	33.12	51.74	80.26	123.84	205.68	697.50	2568.33
LGDV2	1.42	3.87	8.03	13.68	21.98	35.05	55.58	91.41	166.29	425.91
LGDV	1.19	9.88	21.40	36.36	57.13	89.04	140.31	−988.27	310.10	1628.52
REG	−56.79	−56.86	−56.83	−56.92	−56.78	−56.93	−56.87	−56.94	−56.87	−56.67

$\beta = 0.5$, $\rho = 0.55$, $\alpha = 0.75$

Table 15: Percent Bias **(Mean)** in Long-Run Estimates of X_t on Y Conditional Upon Varying Levels of ϕ (autoregressive error parameter)

	ϕ									
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
EQ4	7.50	6.45	45.20	13.73	−117.48	0.53	7.64	8.70	6.47	10.92
ADL(1,1)	7.39	7.96	48.06	18.67	−114.43	10.77	25.66	39.73	66.58	439.82
LGDV2	7.53	6.45	45.07	13.81	−117.41	1.55	10.41	13.74	16.87	39.58
LGDV	7.43	7.96	47.72	18.23	−115.12	10.19	24.36	35.83	56.94	765.38
REG	−8.49	−9.34	23.50	−2.48	−122.57	−14.51	−7.94	−7.98	−10.93	−9.08

$\beta = 0.5$, $\rho = 0.95$, $\alpha = 0.75$

Appendix C

The Ljung-Box test is a group test of whether there is autocorrelation present in a time series between a lag of 1 and some lag L. The Ljung-Box test statistic is:[14]

$$H = T(T+2)\sum_{l=1}^{L}\frac{\hat{\rho}_l^2}{T-l}$$

L represents the total number of lags for the test, specified by the user. For the Ljung-Box tests in the paper, I set $L = 12$. T represents the length of the time series, and $\hat{\rho}_l$ represents the sample autocorrelation for the time series at lag l. Under the null hypothesis of no autocorrelation in the time series, $H \sim \chi^2_{L-p}$ as $T \to \infty$ (p is the number of lags of the dependent variable in the model used to estimate the time series).[15] The null hypothesis is rejected if $H > Q_{1-r}(\chi^2_{L-p})$, where $Q_{1-r}(\cdot)$ represents the $1-r$ quantile of the distribution and r is the significance level, which I set at .05.

The convergence of the test statistic to a chi-square distribution under the null is an asymptotic property, so for finite T, the expected rate of Type I errors will not necessarily be 5% even with a significance level of .05. The rate of Type I errors was slightly higher than 5% in the Monte Carlo simulations considered in the paper.

[14]Shumway and Stoffer (2006)

[15]For ARIMA models, the number of autoregressive parameters (p) and moving average parameters (q) should be subtracted from the number of degrees of freedom.

Appendix D

When working with time series data, it is very important to consider whether the data are stationary, that is whether the expected value and variance of the data are independent of time t. If the data are non-stationary, it must be transformed to a stationary time series before proceeding with any time series modeling. The most common method of transforming a non-stationary time series into a stationary time series is to take a first difference ($y_t - y_{t-1}$ in the case of a *unit root*). Let us consider the stationarity conditions for the ADL(2, 1) model: the ADL(2, 1) is second-order autoregressive, or AR(2), so we will want to consider what conditions are required for an AR(2) model to be stationary.

The stationarity conditions for an AR(1) model are very straightforward, but the stationarity conditions for an AR(2) are slightly more complicated. In the case of an AR(1) model ($y_t = \alpha y_{t-1} + \epsilon_t$, where ϵ_t is a white noise error term), the stationarity condition is $|\alpha| < 1$. Consider a disturbance ϵ that occurs at time t. The total effect of the disturbance is $\epsilon + \alpha\epsilon + \alpha^2\epsilon + \alpha^3\epsilon + \ldots = \sum_{j=0}^{\infty} \alpha^j \epsilon$. If $|\alpha| > 1$, the time series is explosive, that is it goes to ∞ or $-\infty$, because any disturbance ϵ_t is magnified as t increases. That is because if $|\alpha| > 1$, then $\lim_{t\to\infty} \alpha^t \epsilon \to \infty$. But if $|\alpha| < 1$, then $\lim_{t\to\infty} \alpha^t \epsilon \to 0$ and $\sum_{j=0}^{\infty} \alpha^j \epsilon$ is a convergent geometric series equal to $\epsilon/(1-\alpha)$. In the case where $|\alpha| < 1$, the long run effect of any disturbance ϵ goes to zero as t increases.

An autoregressive time series will be stationary if all of its roots lie outside of the unit circle (Wei, 2005). Finding the roots of an AR(1) process is straightforward:

$$\begin{aligned} y_t &= \alpha y_{t-1} + \epsilon_t \\ (1 - \alpha L) y_t &= \epsilon_t \end{aligned}$$

L is the lag operator (defined as $Ly_t = y_{t-1}$). To lie outside the unit circle, the root must be greater than 1 or less than -1. The root for the AR(1) is $1/\alpha$. For this root to lie outside the unit circle, the following conditions must hold: $1/\alpha > 1$ or $1/\alpha < -1$. Putting these expressions in terms of α, the stationarity conditions are $\alpha < 1$ and $\alpha > -1$ or $|\alpha| < 1$. Now let us consider the roots of an AR(2) process:

$$\begin{aligned}
y_t &= \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \epsilon_t \\
(1 - \alpha_1 L - \alpha_2 L^2) y_t &= \epsilon_t \\
(1 - s_1 L)(1 - s_2 L) y_t &= \epsilon_t \\
(1 - s_1 L - s_2 L + s_1 s_2 L^2) y_t &= \epsilon_t
\end{aligned}$$

Variables s_1 and s_2 represent the inverse of the roots of the AR(2) process. If the roots of this process lie outside the unit circle, their inverses must lie in the unit circle for the AR(2) to be stationary, that is $|s_1| < 1$ and $|s_2| < 1$. Together, this implies that $|s_1 s_2| < 1$ and because $\alpha_2 = -s_1 s_2$, then $|\alpha_2| < 1$. This gives us our first stationarity condition for the AR(2) process. Note that $\alpha_1 = s_1 + s_2$ and because $|s_1 + s_2| < 2$ under stationarity conditions, then $|\alpha_1| < 2$. This gives us our second stationarity condition.

Using the quadratic formula, the roots of the AR(2) process are:

$$\frac{-\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2\alpha_2}$$

The roots must be greater than 1 or less than -1 to satisfy stationarity conditions. Let us consider the roots where the square root term is positive (the solutions are identical when the square root term is negative) and solve for the conditions needed for the roots to lie outside the unit circle:

$$
\begin{aligned}
\frac{-\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_2}}{2\alpha_2} &> 1 \\
\sqrt{\alpha_1^2 + 4\alpha_2} &> \alpha_1 + 2\alpha_2 \\
\alpha_1^2 + 4\alpha_2 &> \alpha_1^2 + 4\alpha_1\alpha_2 + 4\alpha_2^2 \\
\alpha_2^2 + \alpha_1\alpha_2 - \alpha_2 &< 0 \\
\alpha_1 + \alpha_2 &< 1
\end{aligned}
$$

The third condition for stationarity is that $\alpha_1 + \alpha_2 < 1$. Now let us solve for the conditions needed for the root to be less than -1:

$$
\begin{aligned}
\frac{-\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_2}}{2\alpha_2} &< -1 \\
\sqrt{\alpha_1^2 + 4\alpha_2} &> 2\alpha_2 - \alpha_1 \\
\alpha_1^2 + 4\alpha_2 &> \alpha_1^2 - 4\alpha_1\alpha_2 + 4\alpha_2^2 \\
\alpha_2^2 - \alpha_1\alpha_2 - \alpha_2 &< 0 \\
\alpha_2 - \alpha_1 &< 1
\end{aligned}
$$

This gives us the fourth condition required for stationarity. Putting these conditions together, the following must be satisfied for an AR(2) process to be stationary (the last three conditions are sufficient to derive the first condition):

$$\begin{aligned} |\alpha_1| &< 2 \\ |\alpha_2| &< 1 \\ \alpha_1 + \alpha_2 &< 1 \\ \alpha_2 - \alpha_1 &< 1 \end{aligned}$$

Wei (2005) also provides a derivation of the stationarity conditions for an AR(2) process.

Appendix E

Let us consider why we can accurately estimate the coefficient of X_t with the ADL(1,1) model even though it improperly excludes Y_{t-2}. In order for an omitted variable to result in a biased estimate of a particular coefficient, it must be the case that the omitted variable are correlated with the variable of interest. Consider the following regression:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\epsilon}$$

Variable $\mathbf{X}$ and variable $\mathbf{Z}$ are matricies of explanatory variables for $\mathbf{y}$, while variable $\boldsymbol{\epsilon}$ is an error term that is uncorrelated with $\mathbf{X}$ and $\mathbf{Z}$. Suppose we regress $\mathbf{y}$ on $\mathbf{X}$, leaving $\mathbf{Z}$ out of the regression. Our estimate of $\boldsymbol{\beta}$ will be:

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\epsilon}) \\
&= \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}\boldsymbol{\gamma} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\epsilon}
\end{aligned}$$

In expectation, $\mathbf{X}'\boldsymbol{\epsilon}$ will be zero. If there is no correlation between the explanatory variables and the omitted explanatory variables, then the expected value of the second term will be zero and $\hat{\boldsymbol{\beta}}$ will be an unbiased estimator of $\boldsymbol{\beta}$. But, if there is any correlation between the explanatory variables and the omitted explanatory variables, i.e. $E(\mathbf{X}'\mathbf{Z}) \neq 0$, then the second term will not be equal to zero and the estimate of $\boldsymbol{\beta}$ will suffer from omitted variable bias equal to:

$$E(\hat{\boldsymbol{\beta}}) - \boldsymbol{\beta} = (\mathbf{X}'\mathbf{X})^{-1}E(\mathbf{X}'\mathbf{Z})\boldsymbol{\gamma}$$

If we use the ADL(1,1) model to estimate parameters with data generated from Equation (4), then $\mathbf{X} = [Y_{t-1}|X_{t-1}|X_t]$ and $\mathbf{Z} = [Y_{t-2}]$ and $\mathbf{X}'\mathbf{Z} = [Y'_{t-1}Y_{t-2}|X'_{t-1}Y_{t-2}|X'_tY_{t-2}]$. To determine the omitted variable bias for the coefficient of X_t, we can multiply the third row of matrix

$(\mathbf{X}'\mathbf{X})^{-1}$ by $\mathbf{X}'\mathbf{Z}$. For simplicity, we can use the third row from the matrix of cofactors for $(\mathbf{X}'\mathbf{X})$, which we will be noted as $(\mathbf{X}'\mathbf{X})^{CF}$

$$
\begin{aligned}
(\mathbf{X}'\mathbf{X})^{CF}_{3,}\mathbf{X}'\mathbf{Z} \;=\; & \left[(X'_{t-1}Y_{t-1})(X'_tX_{t-1}) - (X'_{t-1}X_{t-1})(X'_tY_{t-1})\right] Y'_{t-1}Y_{t-2} \\
& + \left[(Y'_{t-1}X_{t-1})(X'_tY_{t-1}) - (Y'_{t-1}Y_{t-1})(X'_tX_{t-1})\right] X'_{t-1}Y_{t-2} \\
& + \left[(Y'_{t-1}Y_{t-1})(X'_{t-1}X_{t-1}) - (Y'_{t-1}X_{t-1})(X'_{t-1}Y_{t-1})\right] X'_tY_{t-2}
\end{aligned}
$$

From Equation 2, we know that $X_t = \rho X_{t-1} + e_{1t}$. Let us make this substitution into the first line of the above formula for $(\mathbf{X}'\mathbf{X})^{-1}_{3,}\mathbf{X}'\mathbf{Z}$.

$$
\begin{aligned}
& \left[(X'_{t-1}Y_{t-1})(X'_tX_{t-1}) - (X'_{t-1}X_{t-1})(X'_tY_{t-1})\right] Y'_{t-1}Y_{t-2} \\
= \; & \left[(X'_{t-1}Y_{t-1})(X'_{t-1}X_{t-1}\rho + e'_{1t}X_{t-1}) - (X'_{t-1}X_{t-1})(X'_{t-1}Y_{t-1}\rho + e'_{1t}Y_{t-1})\right] Y'_{t-1}Y_{t-2} \\
= \; & [(X'_{t-1}Y_{t-1})(X'_{t-1}X_{t-1})\rho - (X'_{t-1}X_{t-1})(X'_{t-1}Y_{t-1})\rho \\
& +(X'_{t-1}Y_{t-1})(e'_{1t}X_{t-1}) - (X'_{t-1}X_{t-1})(e'_{1t}Y_{t-1})]Y'_{t-1}Y_{t-2} \\
= \; & \left[(X'_{t-1}Y_{t-1})(e'_{1t}X_{t-1}) - (X'_{t-1}X_{t-1})(e'_{1t}Y_{t-1})\right] Y'_{t-1}Y_{t-2}
\end{aligned}
$$

The second line of $(\mathbf{X}'\mathbf{X})^{CF}_{3,}\mathbf{X}'\mathbf{Z}$ will be:

$$
\begin{aligned}
& \left[(Y'_{t-1}X_{t-1})(X'_tY_{t-1}) - (Y'_{t-1}Y_{t-1})(X'_tX_{t-1})\right] X'_{t-1}Y_{t-2} \\
= \; & \left[(Y'_{t-1}X_{t-1})(X'_{t-1}Y_{t-1}\rho + e'_{1t}Y_{t-1}) - (Y'_{t-1}Y_{t-1})(X'_{t-1}X_{t-1}\rho + e'_{1t}X_{t-1})\right] X'_{t-1}Y_{t-2} \\
= \; & \left[(Y'_{t-1}X_{t-1})(X'_{t-1}Y_{t-1})\rho - (Y'_{t-1}Y_{t-1})(X'_{t-1}X_{t-1})\rho\right] X'_{t-1}Y_{t-2} \\
& + \big(Y'_{t-1}X_{t-1})(e'_{1t}Y_{t-1}) - (Y'_{t-1}Y_{t-1})(e'_{1t}X_{t-1})\big] X'_{t-1}Y_{t-2}
\end{aligned}
$$

The third line of $(\mathbf{X}'\mathbf{X})_{3,}^{CF}\mathbf{X}'\mathbf{Z}$ will be:

$$
\begin{aligned}
& \left[(Y_{t-1}'Y_{t-1})(X_{t-1}'X_{t-1}) - (Y_{t-1}'X_{t-1})(X_{t-1}'Y_{t-1})\right] X_t'Y_{t-2} \\
= \quad & \left[(Y_{t-1}'Y_{t-1})(X_{t-1}'X_{t-1}) - (Y_{t-1}'X_{t-1})(X_{t-1}'Y_{t-1})\right] (X_{t-1}'Y_{t-2}\rho + e_{1t}'Y_{t-2}) \\
= \quad & \left[(Y_{t-1}'Y_{t-1})(X_{t-1}'X_{t-1})\rho - (Y_{t-1}'X_{t-1})(X_{t-1}'Y_{t-1})\rho\right] X_{t-1}'Y_{t-2} \\
= \quad & + \left[(Y_{t-1}'Y_{t-1})(X_{t-1}'X_{t-1}) - (Y_{t-1}'X_{t-1})(X_{t-1}'Y_{t-1})\right] e_{1t}'Y_{t-2}
\end{aligned}
$$

Adding the second and third lines together, we get:

$$
\begin{aligned}
& \big(Y_{t-1}'X_{t-1})(e_{1t}'Y_{t-1}) - (Y_{t-1}'Y_{t-1})(e_{1t}'X_{t-1})\big] X_{t-1}'Y_{t-2} \\
& + \left[(Y_{t-1}'Y_{t-1})(X_{t-1}'X_{t-1}) - (Y_{t-1}'X_{t-1})(X_{t-1}'Y_{t-1})\right] e_{1t}'Y_{t-2}
\end{aligned}
$$

Notice that all of the terms in the first line and the sum of the second and third lines contain $e_{1t}'Y_{t-1}$, $e_{1t}'Y_{t-2}$, or $e_{1t}'X_{t-1}$. In expectation, $e_{1t}'Y_{t-1}$, $e_{1t}'Y_{t-2}$, and $e_{1t}'X_{t-1}$ are all equal to zero. But because the determinant of $\mathbf{X}'\mathbf{X}$ contains X_t (which itself contains e_{1t}), the expected value will not be equal to zero. In practice, this bias is very tiny for fairly large values of T, as can be seen in the Monte Carlo simulations. As T gets larger, $e_{1t}'Y_{t-1}$, $e_{1t}'Y_{t-2}$, and $e_{1t}'X_{t-1}$ will go to zero. Thus the numerator of $(\mathbf{X}'\mathbf{X})_{3,}^{-1}\mathbf{X}'\mathbf{Z}$ will go to zero asymptotically, meaning that the omitted variable bias in X_t from leaving out Y_{t-2} will go to zero asymptotically. We can show that $\frac{1}{T}e_{1t}'Y_{t-1}$, $\frac{1}{T}e_{1t}'Y_{t-2}$, and $\frac{1}{T}e'_{1t}X_{t-1}$ will go to zero asymptotically as follows (using $\frac{1}{T}e'_{1t}Y_{t-1}$ as an example).

$$\begin{aligned} E\left(\frac{1}{T}e'_{1t}Y_{t-1}\right) &= \frac{1}{T}E\left(\sum_{t=1}^{T}e_{1t}Y_{t-1}\right) = \frac{1}{T}\sum_{t=1}^{T}E(e_{1t}Y_{t-1}) = \frac{1}{T}\sum_{t=1}^{T}E(e_{1t})E(Y_{t-1}) \\ &= E(e_{1t})E(Y_{t-1}) = 0 \end{aligned}$$

$$\begin{aligned} V\left(\frac{1}{T}e'_{1t}Y_{t-1}\right) &= \frac{1}{T^2}V\left(\sum_{t=1}^{T}e_{1t}Y_{t-1}\right) = \frac{1}{T^2}\left[\sum_{t=1}^{T}V(e_{1t}Y_{t-1}) + \sum_{t\neq u}\text{Cov}(e_{1t}Y_{t-1}, e_{1u}Y_{u-1})\right] \\ &= \frac{1}{T}V(e_{1t}Y_{t-1}) = \frac{1}{T}\sigma_1^2 E(Y_{t-1}^2) \\ \lim_{T\to\infty}\frac{1}{T}\sigma_1^2 E(Y_{t-1}^2) &= 0 \end{aligned}$$

We can divide both the numerator and denominator of $(\mathbf{X'X})^{-1}_{3,}\mathbf{X'Z}$ by $1/T$. Every term in the numerator contains $\frac{1}{T}e'_{1t}Y_{t-1}$, $\frac{1}{T}e'_{1t}Y_{t-2}$, or $\frac{1}{T}e'_{1t}X_{t-1}$ (by distributing $1/T$ through the numerator), all of which go to zero as $T \to \infty$. Therefore, the numerator itself goes to zero as $T \to \infty$.

$$\begin{aligned} \lim_{T\to\infty}(\mathbf{X'X})^{-1}_{3,}\mathbf{X'Z} &= \lim_{T\to\infty}\frac{1}{\det(\mathbf{X'X})}(\mathbf{X'X})^{CF}_{3,}\mathbf{X'Z} \\ &= \lim_{T\to\infty}\frac{(1/T)(\mathbf{X'X})^{CF}_{3,}\mathbf{X'Z}}{(1/T)\det(\mathbf{X'X})} = 0 \end{aligned}$$

That means the ADL(1,1) model estimate of the coefficient of X_t will be almost same as the expected value of the coefficient for the ADL(2,1) model and will converge to the ADL(2,1) estimate as $T \to \infty$. Let us calculate the bias in the coefficient of X_t if we used the LGDV2 model. In this case, the matrix of regressors is $\mathbf{X} = [Y_{t-1}|Y_{t-2}|X_t]$ and the matrix of omitted variables is $\mathbf{Z} = [X_{t-1}]$. For notational simplicity, let us use the following: $a = X'_tX_t$, $b = X'_{t-1}X_t$, $c = X'_tY_t$,

$d = Y_{t-1}'Y_t$, and $f = Y_t'Y_t$. Matrix $\mathbf{X'Z}$ will be: $\mathbf{X'Z} = [c|\rho c + e_{1t}'Y_{t-2}|b]^T$. Matrix $\mathbf{X'X}$ will be:

$$\begin{bmatrix} f & d & \rho c + e_{1t}'Y_{t-1} \\ d & f & \rho^2 c + \rho e_{1,t-1}'Y_{t-2} + e_{1t}'Y_{t-2} \\ \rho c + e_{1t}'Y_{t-1} & \rho^2 c + \rho e_{1,t-1}'Y_{t-2} + e_{1t}'Y_{t-2} & a \end{bmatrix}$$

As we saw earlier, the terms containing e_{1t} will converge to zero as T grows to infinity. Therefore this calculation of the bias will set aside those terms containing e_{1t}, so we will approximate $\mathbf{X'Z}$ as $\mathbf{X'Z} \approx [c|\rho c|b]^T$ and approximate $\mathbf{X'X}$ as:

$$\begin{bmatrix} f & d & \rho c \\ d & f & \rho^2 c \\ \rho c & \rho^2 c & a \end{bmatrix}$$

The first, second and third terms of $(\mathbf{X'X})_{3,}^{CF}\mathbf{X'Z}$ will be:

$$\begin{aligned}
\left[(Y_{t-2}'Y_{t-1})(X_t'Y_{t-2}) - (Y_{t-2}'Y_{t-2})(X_t'Y_{t-1})\right](Y_{t-1}'X_{t-1}) &\approx (\rho^2 cd - \rho cf)c = \rho^2 c^2 d - \rho c^2 f \\
\left[(Y_{t-1}'Y_{t-2})(X_t'Y_{t-1}) - (Y_{t-1}'Y_{t-1})(X_t'Y_{t-2})\right](Y_{t-2}'X_{t-1}) &\approx (\rho cd - \rho^2 cf)\rho c = \rho^2 c^2 d - \rho^3 c^2 f \\
\left[(Y_{t-1}'Y_{t-1})(Y_{t-2}'Y_{t-2}) - (Y_{t-1}'Y_{t-2})(Y_{t-2}'Y_{t-1})\right](X_t'X_{t-1}) &\approx (f^2 - d^2)\rho a = \rho a f^2 - \rho a d^2
\end{aligned}$$

The terms in the determinant of $\mathbf{X'X}$ will be:

$$\begin{aligned}
(Y'_{t-1}Y_{t-1})(Y'_{t-2}Y_{t-2})(X'_tX_t) &= af^2 \\
(Y'_{t-2}Y_{t-1})(X'_tY_{t-2})(Y'_{t-1}X_t) &\approx \rho^3c^2d \\
(X'_tY_{t-1})(Y'_{t-1}Y_{t-2})(Y'_{t-2}X_t) &\approx \rho^3c^2d \\
-(Y'_{t-1}Y_{t-1})(X'_tY_{t-2})(Y'_{t-2}X_t) &\approx -\rho^4c^2f \\
-(X'_tY_{t-1})(Y'_{t-2}Y_{t-2})(Y'_{t-1}X_t) &\approx -\rho^2c^2f \\
-(Y'_{t-2}Y_{t-1})(Y'_{t-1}Y_{t-2})(X'_tX_t) &= -ad^2
\end{aligned}$$

Putting this all together, the ommitted variable bias $(\mathbf{X'X})^{-1}_{3,}\mathbf{X'Z}\boldsymbol{\gamma}$ (as T approaches infinity and noting that $\boldsymbol{\gamma} = [-\beta\phi]$) will be:

$$-\beta\phi \cdot \frac{2\rho^2c^2d - \rho c^2f - \rho^3c^2f + \rho af^2 - \rho ad^2}{af^2 + 2\rho^3c^2d - \rho^4c^2f - \rho^2c^2f - ad^2}$$

This solution is only an approximate (though a very close one) for finite T. Now let us calculate the omitted variable bias for the coefficient of X_t if we use the REG regression. In this case, the matrix of regressors is $\mathbf{X} = [X_t]$ and the matrix of omitted variables is $\mathbf{Z} = [Y_{t-1}|Y_{t-2}|X_{t-1}]$. Matrix $(\mathbf{X'X})^{-1}$ is $1/(X'_tX_t) = 1/a$ and matrix $\mathbf{X'Z}$ is $[\rho c + e'_{1t}Y_{t-1}|\rho^2c + \rho e'_{1,t-1}Y_{t-2} + e'_{1t}Y_{t-2}|\rho a + e'_{1t}X_{t-1}]$. The omitted variable bias for the coefficient of X_t (as T approaches infinity) is:

$$(\mathbf{X'X})^{-1}\mathbf{X'Z}\boldsymbol{\gamma} = \frac{\rho c(\alpha+\phi)}{a} + \frac{\rho^2c(-\alpha\phi)}{a} + \rho(-\beta\phi)$$

Finally, let us calculate the omitted variable bias for the coefficient of X_t if we use the LGDV regression. In this case, the matrix of regressors is $\mathbf{X} = [X_t|Y_{t-1}]$ and the matrix of omitted variables is $\mathbf{Z} = [X_{t-1}|Y_{t-2}]$. Matrix $\mathbf{X}'\mathbf{X}$ will be:

$$\begin{bmatrix} a & \rho c + \rho e'_{1t} Y_{t-1} \\ \rho c + \rho e'_{1t} Y_{t-1} & f \end{bmatrix}$$

As $T \to \infty$, we can approximate $\mathbf{X}'\mathbf{X}$ as:

$$\begin{bmatrix} a & \rho c \\ \rho c & f \end{bmatrix}$$

The determinant of $\mathbf{X}'\mathbf{X}$ is $af - \rho^2 c^2$. Matrix $\mathbf{X}'\mathbf{Z}$ will be:

$$\begin{bmatrix} \rho a + e'_{1t} X_{t-1} & \rho^2 c + \rho e'_{1,t-1} Y_{t-2} + e'_{1t} Y_{t-2} \\ c & d \end{bmatrix}$$

As $T \to \infty$, we can approximate $\mathbf{X}'\mathbf{Z}$ as:

$$\begin{bmatrix} \rho a & \rho^2 c \\ c & d \end{bmatrix}$$

The omitted variable bias for the estimate of the coefficient of X_t in the LGDV model will be $(\mathbf{X}'\mathbf{X})^{-1}_{1,}\mathbf{X}'\mathbf{Z}\boldsymbol{\gamma}$, where $\boldsymbol{\gamma} = [-\beta\phi| - \alpha\phi]$, which is:

$$-\beta\phi \cdot \frac{f\rho a - \rho c^2}{af - \rho^2 c^2} - \alpha\phi \frac{f\rho^2 c - \rho c d}{af - \rho^2 c^2}$$

Appendix F

One of the characteristics of a dynamic model is that a shift in the independent variable at time t does not just have an effect on the dependent variable at time t, rather a shift at time t has an effect in subsequent periods (this effect must decay to zero as T goes to infinity for the time series to be stationary). Up to now, we have focused on how X_t affects Y_t at time t but we have not discussed the total effect of X_t on Y_t. For this discussion, let us use an ADL(2, 1) model, that is $y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \beta_1 X_t + \beta_2 X_{t-1}$ where $X_t = \rho X_{t-1} + \varepsilon_t$. Now let us suppose that there is some shock ε in variable X at time t. The effect of this shock, ε, on variable Y will be the following in each period:

$$
\begin{aligned}
[\textbf{time 1}] &= \beta_1 \varepsilon \\
[\textbf{time 2}] &= \beta_1 \rho \varepsilon + \beta_2 \varepsilon + \alpha_1 [\textbf{time 1}] \\
[\textbf{time 3}] &= \beta_1 \rho^2 \varepsilon + \beta_2 \rho \varepsilon + \alpha_1 [\textbf{time 2}] + \alpha_2 [\textbf{time 1}] \\
[\textbf{time 4}] &= \beta_1 \rho^3 \varepsilon + \beta_2 \rho^2 \varepsilon + \alpha_1 [\textbf{time 3}] + \alpha_2 [\textbf{time 2}] \\
[\textbf{time 5}] &= \beta_1 \rho^4 \varepsilon + \beta_2 \rho^3 \varepsilon + \alpha_1 [\textbf{time 4}] + \alpha_2 [\textbf{time 3}]
\end{aligned}
$$

We can write the following formula for the effect of the shock ε on Y at time t, where $w_t = [\textbf{time t}]$:

$$
w_t = \beta_1 \rho^{t-1} \varepsilon + \beta_2 \rho^{t-2} \varepsilon + \alpha_1 w_{t-1} + \alpha_2 w_{t-2}
$$

The total effect of the shock ε on Y is $\sum_{t=1}^{\infty} w_t$, which can be expressed as (note that $|\rho| < 1$

for X_t to be stationary, which also means that two of the terms will be convergent geometric series):

$$\begin{aligned}
\sum_{t=1}^{\infty} w_t &= \sum_{t=1}^{\infty} \beta_1 \rho^{t-1}\varepsilon + \sum_{t=2}^{\infty} \beta_2 \rho^{t-2}\varepsilon + \sum_{t=2}^{\infty} \alpha_1 w_{t-1} + \sum_{t=3}^{\infty} \alpha_2 w_{t-2} \\
\sum_{t=1}^{\infty} w_t(1-\alpha_1-\alpha_2) &= \beta_1\varepsilon \sum_{t=1}^{\infty} \rho^{t-1} + \beta_2\varepsilon \sum_{t=2}^{\infty} \rho^{t-2} \\
\sum_{t=1}^{\infty} w_t(1-\alpha_1-\alpha_2) &= \frac{\beta_1\varepsilon}{1-\rho} + \frac{\beta_2\varepsilon}{1-\rho} \\
\sum_{t=1}^{\infty} w_t &= \frac{\beta_1+\beta_2}{(1-\rho)(1-\alpha_1-\alpha_2)} \cdot \varepsilon
\end{aligned}$$

Because the model in Equation (4) is a special case of the ADL(2,1) model, we can substitute $\beta_1 = \beta$, $\beta_2 = -\beta\phi$, $\alpha_1 = \alpha + \phi$, and $\alpha_2 = -\alpha\phi$ into the solution for $\sum_{t=0}^{\infty} w_t$:

$$\begin{aligned}
\sum_{t=1}^{\infty} w_t &= \frac{\beta(1-\phi)}{(1-\rho)(1-\alpha-\phi+\alpha\phi)} \cdot \varepsilon = \frac{\beta(1-\phi)}{(1-\rho)(1-\alpha)(1-\phi)} \cdot \varepsilon \\
&= \frac{\beta}{(1-\rho)(1-\alpha)} \cdot \varepsilon
\end{aligned}$$

If we are curious about the long-run effect of X_t on Y for an ADL(1,1) model where $X_t = \rho X_{t-1} + \varepsilon_t$, we can repeat this derivation above, but setting $\alpha_2 = 0$. It is easy to see that in that case, the long-run effect formula would be:

$$\frac{\beta_1+\beta_2}{(1-\rho)(1-\alpha_1)} \cdot \varepsilon$$

If we want to consider the case where X_t is not a dynamic process, we can set $\rho = 0$ and repeat the steps of the derivation. For the ADL(2,1), this would yield $(\beta_1+\beta_2)/(1-\alpha_1-\alpha_2)$. For the ADL(1,1) we would get the formula $(\beta_1+\beta_2)/(1-\alpha_1)$, which De Boef and Keele (2008) use to calculate the long-run effects of X_t on Y for this model. Thus, unless it is actually the case that X_t is not a dynamic time series, this formula would provide an inaccurate estimate of the long-run effect.

Appendix G

In Appendix E, I derived the asymptotic biases in the estimate of β under the LGDV, LGDV2, and REG models and demonstrated that the ADL(1,1) model provides an asymptotically unbiased estimate of β, even though it excludes Y_{t-2}. However, given how we defined the biases in Appendix E, we would still have to use Monte Carlo simulations to calculate the bias because the formulas were defined in terms of X_t, X_{t-1}, Y_t, and Y_{t-1}. In this section, I calculate the values of a, b, c, d, and f as expected values in terms of the parameters of an ADL(2,1) model where X_t is a dynamic time series, α_1, α_2, β_1, β_2, ρ, σ_1^2 (the variance of e_{1t}), and σ_2^2 (the variance of e_{2t}). When comparing the calculated values of a, b, c, d, and f in this section, note that the values of these variables in Appendix E would correspond to the values here multiplied through by T. As an example, in this section, $a = E(X_t^2)$, which is equal to $\frac{\sigma_1^2}{1-\rho^2}$. But in Appendix E, I defined a as $X_t'X_t$, so if we were calculating this value using the formula for a in this section, it would actually be $T \times \frac{\sigma_1^2}{1-\rho^2}$. However, the T terms cancel out when we are calculating the biases, so we can plug in the values of a, b, c, d, and f in this section to the bias formulas in Appendix E. One other difference is that we use expected values in the formulas for a, b, c, d, and f in this section, but not in Appendix E. But since we are considering the asymptotic bias, the formulas for a, b, c, d, and f in Appendix E will converge to their expected values.

$$
\begin{aligned}
Y_t &= \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \beta_1 X_t + \beta_2 X_{t-1} + e_{2t} \\
X_t &= \rho X_{t-1} + e_{1t} \\
a &= E(X_t^2) = E[\rho^2 X_{t-1}^2 + 2\rho X_{t-1} e_{1t} + e_{1t}^2] \\
a &= \rho^2 a + \sigma_1^2 \\
a &= \frac{\sigma_1^2}{1-\rho^2} \\
b &= E(X_{t-1}X_t) = E[\rho X_{t-1}^2 + X_{t-1}e_{1t}] \\
b &= \rho a \\
c &= E(X_t Y_t) = E[\alpha_1 X_t Y_{t-1} + \alpha_2 X_t Y_{t-2} + \beta_1 X_t^2 + \beta_2 X_{t-1} X_t + X_t e_{2t}] \\
c &= \alpha_1 E[\rho X_{t-1} Y_{t-1} + \rho e_{1t} Y_{t-1}] + \alpha_2 E[\rho^2 X_{t-2} Y_{t-2} + \rho e_{1,t-1} Y_{t-2} + e_{1t} Y_{t-2}] + \beta_1 a + \beta_2 b \\
c &= \alpha_1 \rho c + \alpha_2 \rho^2 c + \beta_1 a + \beta_2 b \\
c &= \frac{\beta_1 a + \beta_2 b}{1 - \alpha_1 \rho - \alpha_2 \rho^2} \\
d &= E(Y_{t-1} Y_t) = E[\alpha_1 Y_{t-1}^2 + \alpha_2 Y_{t-1} Y_{t-2} + \beta_1 Y_{t-1} X_t + \beta_2 Y_{t-1} X_{t-1} + Y_{t-1} e_{2t}] \\
d &= \alpha_1 E[Y_{t-1}^2] - \alpha_2 E[Y_{t-1} Y_{t-2}] + \beta_1 [\rho X_{t-1} Y_{t-1} + e_{1t} Y_{t-1}] + \beta_2 E[Y_{t-1} X_{t-1}] \\
d &= \alpha_1 f - \alpha_2 d + \beta_1 \rho c + \beta_2 c \\
d &= \frac{\alpha_1 f + \beta_1 \rho c + \beta_2 c}{1 - \alpha_2} = \frac{\alpha_1 f + \beta_3 c}{1 - \alpha_2} \quad \text{where} \quad \beta_3 = \beta_1 \rho + \beta_2 \\
g &= E[Y_{t-1} Y_t] = E[\alpha_1 Y_{t-1}^2 + \alpha_2 Y_{t-2} Y_{t-1} + \beta_3 X_{t-1} Y_{t-1} + \beta_1 e_{1t} Y_{t-1} + Y_{t-1} e_{2t}] \\
g &= \alpha_1 E[Y_{t-1}^2] + \alpha_2 E[Y_{t-2} Y_{t-1}] + \beta_3 E[X_{t-1} Y_{t-1}] \\
g &= \alpha_1 f + \alpha_2 d + \beta_3 c \\
h &= E[Y_{t-2} Y_t] = E[\alpha_1 Y_{t-1} Y_{t-2} + \alpha_2 Y_{t-2}^2 + \beta_3 X_{t-1} Y_{t-2} + \beta_1 e_{1t} Y_{t-2} + Y_{t-2} e_{2t}] \\
h &= \alpha_1 E[Y_{t-1} Y_{t-2}] + \alpha_2 E[Y_{t-2}^2] + \beta_3 E[\rho X_{t-2} Y_{t-2} + e_{1,t-1} Y_{t-2}] \\
h &= \alpha_1 d + \alpha_2 f + \beta_3 \rho c
\end{aligned}
$$

$$
\begin{aligned}
w &= E[X_{t-1}Y_t] = E[\alpha_1 Y_{t-1}X_{t-1} + \alpha_2 Y_{t-2}X_{t-1} + \beta_3 X_{t-1}X_{t-1} + \beta_1 e_{1t}X_{t-1} + e_{2t}X_{t-1}] \\
w &= \alpha_1 E[Y_{t-1}X_{t-1}] + \alpha_2 E[\rho Y_{t-2}X_{t-2} + Y_{t-2}e_{1,t-1}] + \beta_3 E[X_{t-1}X_{t-1}] \\
w &= \alpha_1 c + \alpha_2 \rho c + \beta_3 a \\
f &= E[Y_t^2] = E[\alpha_1 Y_{t-1}Y_t + \alpha_2 Y_{t-2}Y_t + \beta_3 X_{t-1}Y_t + \beta_1 e_{1t}Y_t + Y_t e_{2t}] \\
f &= \alpha_1 E[Y_{t-1}Y_t] + \alpha_2 E[Y_{t-2}Y_t] + \beta_3 E[X_{t-1}Y_t] + \beta_1 E[e_{1t}Y_t] + E[Y_t e_{2t}] \\
f &= \alpha_1 g + \alpha_2 h + \beta_3 w + \beta_1^2\sigma_1^2 + \sigma_2^2 \\
f &= \alpha_1^2 f + \alpha_1\alpha_2 d + \alpha_1\beta_3 c + \alpha_1\alpha_2 d + \alpha_2^2 f + \alpha_2\beta_3\rho c + \beta_3\alpha_1 c + \beta_3\alpha_2\rho c + \beta_3^2 a + \beta_1^2\sigma_1^2 + \sigma_2^2 \\
f &= \alpha_1^2 f + \alpha_1\alpha_2 d + \alpha_1\beta_3 c + \alpha_1\alpha_2 d + \alpha_2^2 f + z \\
& \text{where } \; z = \alpha_2\beta_3\rho c + \beta_3\alpha_1 c + \beta_3\alpha_2\rho c + \beta_3^2 a + \beta_1^2\sigma_1^2 + \sigma_2^2 \\
f(1-\alpha_2) &= \alpha_1^2 f(1-\alpha_2) + \alpha_1\alpha_2(\alpha_1 f + \beta_3 c) + \alpha_1\beta_3 c(1-\alpha_2) \\
& +\alpha_1\alpha_2(\alpha_1 f + \beta_3 c) + \alpha_2^2 f(1-\alpha_2) + z(1-\alpha_2) \\
f - \alpha_2 f &= \alpha_1^2 f - \alpha_2\alpha_1^2 f + \alpha_1^2\alpha_2 f + \alpha_1\alpha_2\beta_3 c + \alpha_1\beta_3 c - \alpha_1\alpha_2\beta_3 c \\
& +\alpha_1^2\alpha_2 f + \alpha_1\alpha_2\beta_3 c + \alpha_2^2 f - \alpha_2^3 f + z(1-\alpha_2)
\end{aligned}
$$

$$
\begin{aligned}
f - \alpha_2 f - \alpha_2^2 f + \alpha_2^3 f - \alpha_1^2 f - \alpha_1^2\alpha_2 f &= \alpha_1\beta_3 c + \alpha_1\alpha_2\beta_3 c + z(1-\alpha_2) \\
f &= \frac{\alpha_1\beta_3 c(1+\alpha_2) + z(1-\alpha_2)}{1 - \alpha_2 - \alpha_2^2 + \alpha_2^3 - \alpha_1^2 - \alpha_1^2\alpha_2}
\end{aligned}
$$

Appendix H: An Aside on Panel Data Modeling

The data-generating process and modeling approaches considered in this paper involve a single time series ($N = 1$) where we have a fairly large number of observations ($T = 100$). Researchers using panel data who are interested in how this paper's findings apply to their work should be aware that there are additional complications beyond those involved with single time series models. A particularly vexing problem arises when we are dealing with panel data with small T and a model that includes a lagged dependent variable with fixed effects.

It is common to use fixed effects when modeling panel data, so as to control for fundamental differences between cases (such as countries) that are not captured by the independent variables. If there are fixed effects that are correlated with the independent variables, failure to include fixed effects in the regression model will lead to omitted variable bias, so fixed effects are widely used in modeling for panel data. The use of fixed effects leads to something known as the "incidental parameters" problem (Neyman and Scott, 1948). By estimating "incidental parameters" such as fixed effects, even as $N \to \infty$, we cannot take advantage of asymptotic unbiasedness because more incidental parameters must be estimated as N increases (with fixed T).

That means that in a model with lagged dependent variables and fixed effects, the bias identified by Hurwicz (1950) will not go to zero with fixed T, even as $N \to \infty$ (Nickell, 1981). Thus, if we had a dataset covering 5 years and 40 countries, we would not reduce the Hurwicz bias compared to a dataset with 5 years and 20 countries. Wawro (2002) observes that the political science literature has often been inattentive to this problem, though there is extensive literature in econometrics and statistics about how to construct estimators that are consistent when T is fixed and small, and $N \to \infty$. Wawro (2002) provides an overview of the major methods used to obtain consistent coefficient estimates in this context, including the Anderson and Hsiao (1982) method and also a generalized method of moments (GMM) estimator developed by Arellano and Bond (1991) that improves on the Anderson and Hsiao (1982) estimator by using a larger set of instrumental variables. A set of recommendations on how to approach dealing with the incidental parameters problem is beyond the scope of this paper, but researchers using panel data should be aware of this issue and consult the relevant literature for how to choose appropriate models for their data. The potential for complications does not mean that political scientists should shy away from using lagged dependent variables; in fact, such an approach would make potential problems worse, not better, as the Monte Carlo analysis showed.

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