

Spencer et al. Model E(B)

$\mu1 = .; \mu2 = .; T = .; t = .; \tau = .; v = .; En = .; c1 = .; cd1 = .; d = .; c2 = .;$

$$w = e^{-\mu1 (t + \tau + v)} e^{-\mu2 (T - t - \tau - v)} (En - (c1 t + cd1 t + c1 \tau + d + c2 (T - t - \tau))) \\ e^{-\mu2 (-t + T - v - \tau) - \mu1 (t + v + \tau)} (-d + En - c1 t - cd1 t - c2 (-t + T - \tau) - c1 \tau)$$

$delwbydelt = D[w, t]$

$$(-c1 + c2 - cd1) e^{-\mu2 (-t + T - v - \tau) - \mu1 (t + v + \tau)} + \\ e^{-\mu2 (-t + T - v - \tau) - \mu1 (t + v + \tau)} (-\mu1 + \mu2) (-d + En - c1 t - cd1 t - c2 (-t + T - \tau) - c1 \tau)$$

$delwbydel\tau = D[w, \tau]$

$$(-c1 + c2) e^{-\mu2 (-t + T - v - \tau) - \mu1 (t + v + \tau)} + \\ e^{-\mu2 (-t + T - v - \tau) - \mu1 (t + v + \tau)} (-\mu1 + \mu2) (-d + En - c1 t - cd1 t - c2 (-t + T - \tau) - c1 \tau)$$

$sols = \text{Solve}[\{delwbydelt == 0, delwbydel\tau == 0\}, \{t, \tau\}]$

$\{\}$

No general solution : must look on boundaries

Boundary 1 : $t = 0$

$w1 = \text{Simplify}[w /. t \rightarrow 0]$

$$-e^{-v \mu1 - T \mu2 + v \mu2 - \mu1 \tau + \mu2 \tau} (d - En + c2 T + c1 \tau - c2 \tau)$$

$delw1bydel\tau = D[w1, \tau]$

$$-(c1 - c2) e^{-v \mu1 - T \mu2 + v \mu2 - \mu1 \tau + \mu2 \tau} - e^{-v \mu1 - T \mu2 + v \mu2 - \mu1 \tau + \mu2 \tau} (-\mu1 + \mu2) (d - En + c2 T + c1 \tau - c2 \tau)$$

$\tau_{crit1} = \text{Solve}[delw1bydel\tau == 0, \tau]$

$$\{\{\tau \rightarrow (c1 - c2 - d \mu1 + En \mu1 - c2 T \mu1 + d \mu2 - En \mu2 + c2 T \mu2) / ((c1 - c2) (\mu1 - \mu2))\}\}$$

$w1v = w1 /. \{v \rightarrow 16 - \tau\}$

$$-e^{-T \mu2 - \mu1 (16 - \tau) + \mu2 (16 - \tau) - \mu1 \tau + \mu2 \tau} (d - En + c2 T + c1 \tau - c2 \tau)$$

$wmax1 = \text{Simplify}[w1v /. \tau_{crit1}[[1]]]$

$$\frac{(c1 - c2) e^{-16 \mu1 - (-16 + T) \mu2}}{-\mu1 + \mu2}$$

$wmax1 /. \{$

$En \rightarrow 50, c1 \rightarrow 0.1, cd1 \rightarrow 0.05, c2 \rightarrow 0.2, d \rightarrow 0.3, \mu1 \rightarrow 0.3, \mu2 \rightarrow 0.4, T \rightarrow 20\}$

-0.00166156

Boundary 2 : $\tau = 0$

$w2 = \text{Simplify}[w /. \tau \rightarrow 0]$

$$-e^{-(t + v) \mu1 + (t - T + v) \mu2} (d - En + c1 t - c2 t + cd1 t + c2 T)$$

delw2bydelt = D[w2, t]

$$- (c1 - c2 + cd1) e^{-(t+v) \mu1 + (t-T+v) \mu2} - e^{-(t+v) \mu1 + (t-T+v) \mu2} (d - En + c1 t - c2 t + cd1 t + c2 T) (-\mu1 + \mu2)$$

tcrit2 = Solve[delw2bydelt == 0, t]

$$\{ \{t \rightarrow (c1 - c2 + cd1 - d \mu1 + En \mu1 - c2 T \mu1 + d \mu2 - En \mu2 + c2 T \mu2) / ((c1 - c2 + cd1) (\mu1 - \mu2))\} \}$$

Exactly the same as Solution for τ when $t = 0$

w2v = w2 /. {v → 16 - t}

$$- e^{-16 \mu1 + (16-T) \mu2} (d - En + c1 t - c2 t + cd1 t + c2 T)$$

wmax2 = Simplify[w2v /. tcrit2[[1]]]

$$- \frac{(c1 - c2 + cd1) e^{-16 \mu1 - (-16+T) \mu2}}{\mu1 - \mu2}$$

wmax2 /. {

$$En \rightarrow 50, c1 \rightarrow 0.1, cd1 \rightarrow 0.05, c2 \rightarrow 0.2, d \rightarrow 0.3, \mu1 \rightarrow 0.3, \mu2 \rightarrow 0.4, T \rightarrow 20\}$$

$$- 0.000830779$$

Boundary 3 : $v = 0$

w3 = Simplify[w /. v → 0]

$$- e^{-\mu1 (t+\tau) + \mu2 (t-T+\tau)} (d - En + c1 t - c2 t + cd1 t + c2 T + c1 \tau - c2 \tau)$$

delw3bydelt = D[w3, t]

$$- (c1 - c2 + cd1) e^{-\mu1 (t+\tau) + \mu2 (t-T+\tau)} - e^{-\mu1 (t+\tau) + \mu2 (t-T+\tau)} (-\mu1 + \mu2) (d - En + c1 t - c2 t + cd1 t + c2 T + c1 \tau - c2 \tau)$$

tcrit3 = Simplify[Solve[delw3bydelt == 0, t]]

$$\{ \{t \rightarrow (cd1 - d \mu1 + En \mu1 + d \mu2 - En \mu2 + c2 (-1 - T \mu1 + T \mu2 + \mu1 \tau - \mu2 \tau) + c1 (1 - \mu1 \tau + \mu2 \tau)) / ((c1 - c2 + cd1) (\mu1 - \mu2))\} \}$$

w3v = w3 /. {τ → 16 - t}

$$- e^{-16 \mu1 + (16-T) \mu2} (d - En + c1 (16 - t) - c2 (16 - t) + c1 t - c2 t + cd1 t + c2 T)$$

wmax3 = Simplify[w3v /. tcrit3[[1]]]

$$\left(e^{-16 \mu1 - (-16+T) \mu2} \left(-cd1^2 - 16 c1^2 (\mu1 - \mu2) + c2^2 (-16 + T) (\mu1 - \mu2) + c2 ((d - En) (\mu1 - \mu2) + cd1 (1 - \mu1 (-16 + \tau) + \mu2 (-16 + \tau))) + c1 (- (d - En + c2 (-32 + T)) (\mu1 - \mu2) + cd1 (-1 + 16 \mu2 + \mu1 (-16 + \tau) - \mu2 \tau)) \right) \right) / ((c1 - c2 + cd1) (\mu1 - \mu2))$$

wmax3 /. {

$$En \rightarrow 50, c1 \rightarrow 0.1, cd1 \rightarrow 0.05, c2 \rightarrow 0.2, d \rightarrow 0.3, \mu1 \rightarrow 0.3, \mu2 \rightarrow 0.4, T \rightarrow 20\}$$

$$0.332311 (-0.0025 + 0.2 (4.97 + 0.05 (1 + 0.1 (-16 + \tau))) +$$

$$0.1 (-5.21 + 0.05 (5.4 + 0.3 (-16 + \tau) - 0.4 \tau)))$$

Need to look in the corners, too

w00 = Simplify[w /. {t → 0, τ → 0}]

$$-e^{-v\mu_1 - T\mu_2 + v\mu_2} (d - En + c_2 T)$$

w00 /. {En → 50, c1 → 0.1, cd1 → 0.05,
c2 → 0.2, d → 0.3, μ1 → 0.3, μ2 → 0.4, v → 16, T → 20}
0.0759332

w0T = Simplify[w /. {t → 0, v → 0}]

$$-e^{-T\mu_2 - \mu_1\tau + \mu_2\tau} (d - En + c_2 T + c_1 \tau - c_2 \tau)$$

w0T /. {En → 50, c1 → 0.1, cd1 → 0.05,
c2 → 0.2, d → 0.3, μ1 → 0.3, μ2 → 0.4, τ → 16, T → 20}
0.0785917

wT0 = Simplify[w /. {v → 0, τ → 0}]

$$-e^{-t\mu_1 + t\mu_2 - T\mu_2} (d - En + c_1 t - c_2 t + cd_1 t + c_2 T)$$

wT0 /. {En → 50, c1 → 0.1, cd1 → 0.05,
c2 → 0.2, d → 0.3, μ1 → 0.3, μ2 → 0.4, t → 16, T → 20}
0.0772624

{τcrit1[[1]], τcrit2[[1]]} /.
{En → 50, c1 → 0.1, cd1 → 0.05, c2 → 0.2, d → 0.3, μ1 → 0.3, μ2 → 0.4, T → 20}
{τ → -467.}, {t → -924.}}

Not biological solutions

{wmax1, wmax2, w00, w0T, wT0} /.
{En → 50, c1 → 0.1, cd1 → 0.05, c2 → 0.2, d → 0.3, μ1 → 0.3, μ2 → 0.4, T → 20}
{-0.00166156, -0.000830779, 45.7 e^{-8.+0.1v},
-e^{-8.+0.1τ} (-45.7 - 0.1 τ), -e^{-8.+0.1t} (-45.7 - 0.05 t)}

Maximize[
{w /. {En → 50, c1 → 0.1, cd1 → 0.05, c2 → 0.2, d → 0.3, μ1 → 0.3, μ2 → 0.4,
T → 20, v → 16 - t - τ}, 0 ≤ t, 0 ≤ τ, τ ≤ 16 - t}, {t, τ}]
{0.0785917, {t → 0., τ → 16.}}

{wmax1, wmax2, w00, w0T, wT0} /. {L → 0.9, c1 → 0.01,
cd1 → 0.05, c2 → 0.2, cd → 0.3, μ1 → 0.3, μ2 → 0.2, μd → 0.25, T → 20}
{0.00702594, 0.00517701, -e^{-4.-0.1v} (4. + d - En),
-e^{-4.-0.1τ} (4. + d - En - 0.19 τ), -e^{-4.-0.1t} (4. + d - En - 0.14 t)}

{wmax1, wmax2, w00, w0T, wT0} /. {L → 0.9, c1 → 0.01, cd1 → 0.05,
c2 → 0.2, cd → 0.21, μ1 → 0.3, μ2 → 0.2, μd → 0.29, T → 20}
{0.00702594, 0.00517701, -e^{-4.-0.1v} (4. + d - En),
-e^{-4.-0.1τ} (4. + d - En - 0.19 τ), -e^{-4.-0.1t} (4. + d - En - 0.14 t)}

```
w /. {En → 50, c1 → 0.1, cd1 → 0.05, c2 → 0.2,
      d → 0.3, μ1 → 0.3, μ2 → 0.4, T → 20, v → 0, t → 8, τ → 8}
0.077927
```

```
w /. {En → 50, c1 → 0.1, cd1 → 0.05, c2 → 0.2,
      d → 0.3, μ1 → 0.3, μ2 → 0.4, T → 20, v → 0, t → 3, τ → 13}
0.0783424
```

```
Maximize[
  {w /. {En → 50, c1 → 0.1, cd1 → 0.05, c2 → 0.2, d → 0.3, μ1 → 0.3, μ2 → 0.4,
        T → 20, v → 16 - t - τ}, 0 ≤ t, 0 ≤ τ, τ ≤ 16 - t}, {t, τ}]
{0.0785917, {t → 0., τ → 16.}}
```