Appendix A: Derivation of option deltas

The first line of the call formula in Equation 13 is the Black-Scholes call, for which we know the delta is

$$\delta_C^{BS} = e^{-qT}\,\Phi(z_1). \tag{32}$$

Differentiating the $1/\theta$ *Adjustment* term in Equation 13 gives

$$\frac{\partial(Adj^{CALL})}{\partial S} = \frac{1}{\theta}\left\{\begin{array}{l} e^{-qT}b^{1+\theta}\left[-S^{-\theta}\,\Phi'(z_2)\times\dfrac{1}{\sigma S\sqrt{T}} - \theta S^{-\theta-1}\,\Phi(z_2)\right] \\ +\,e^{-rT}b^{1-\theta}K^{\theta}\,\Phi'(z_2-\theta\sigma\sqrt{T})\times\dfrac{1}{\sigma S\sqrt{T}} \end{array}\right\}. \tag{33}$$

Using $\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ and expanding $(z_2-\theta\sigma\sqrt{T})^2$, the term on the bottom line inside the braces is

$$\begin{aligned}
& e^{-rT}b^{1-\theta}K^{\theta}\,\frac{1}{\sigma S\sqrt{T}}\,\frac{1}{\sqrt{2\pi}}\,\exp\left\{\frac{-z_2^2}{2}\right\}.\,\exp\left\{\frac{2z_2\,\theta\sigma\sqrt{T}-\theta^2\sigma^2T}{2}\right\} \\
&= \left[e^{-qT}b^{1+\theta}S^{-\theta}\,\frac{1}{\sigma S\sqrt{T}}\,\Phi'(z_2).\,\exp\left\{-rT+qT+z_2\,\theta\sigma\sqrt{T}-\frac{\theta^2\sigma^2T}{2}\right\}\right]\times\left[\frac{KS}{b^2}\right]^{\theta} \\
&= e^{-qT}b^{1+\theta}S^{-\theta}\,\frac{1}{\sigma S\sqrt{T}}\,\Phi'(z_2).\,\exp\left\{(\theta-1)\left[r-q-\theta\,\frac{\sigma^2}{2}\right]T\right\} \quad \text{(put } -\theta\ln\left[\frac{b^2}{KS}\right] \text{ in exponent, expand } z_2\text{)} \\
&= e^{-qT}b^{1+\theta}S^{-\theta}\,\frac{1}{\sigma S\sqrt{T}}\,\Phi'(z_2) \qquad \text{(use } \theta = 2(r-q)/\sigma^2\text{)}
\end{aligned}$$

which cancels with the $\Phi'(z_2)$ term in the top line inside the braces.

Then combining Equations 32 and 33, the call delta is

$$\frac{\partial C_B}{\partial S} = e^{-qT}\left\{\Phi(z_1) - \left(\frac{b}{S}\right)^{1+\theta}\Phi(z_2)\right\}. \tag{34}$$

Turning now to the put, the first 2 lines of the formula in Equation 15 are the difference of the Black-Scholes puts for strikes K and b. Using standard Black-Scholes deltas, the delta of this is

$$\delta_P^{BS}(K) - \delta_P^{BS}(b) = e^{-qT}\left\{\left[\Phi(z_1)-1\right]-\left[\Phi(z_3)-1\right]\right\} = e^{-qT}\left\{\Phi(z_1)-\Phi(z_3)\right\}. \tag{35}$$

Differentiating the $1/\theta$ *Adjustment* term in Equation 14 gives

$$\frac{\partial(Adj^{PUT})}{\partial S} = \frac{1}{\theta}\left\{\begin{array}{l} -be^{-rT}\,\Phi'(-z_3+\sigma\sqrt{T})\times\dfrac{1}{\sigma S\sqrt{T}} \\ +e^{-qT}b^{1+\theta}\,S^{-\theta}\left[\Phi'(z_4)-\Phi'(z_2)\right]\times\dfrac{1}{\sigma S\sqrt{T}} \\ +e^{-qT}b^{1+\theta}\,\theta S^{-\theta-1}\left[\Phi(z_4)-\Phi(z_2)\right] \\ +e^{-rT}b^{1-\theta}K^{\theta}\,\Phi'(z_2-\theta\sigma\sqrt{T})\times\dfrac{1}{\sigma S\sqrt{T}} \end{array}\right\} \tag{36}$$

where the last line inside the braces was previously shown (in the call derivation above) to cancel with the $\Phi'(z_2)$ term in the second line.

Now note that

$$\begin{aligned} -z_3+\sigma\sqrt{T} &= \frac{1}{\sigma\sqrt{T}}\left[-\ln\left(\frac{S}{b}\right)-\left(r-q+\frac{\sigma^2}{2}\right)T+\sigma^2T\right] \\ &= \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{b}{S}\right)+\left(r-q+\frac{\sigma^2}{2}\right)T-2(r-q)T\right] \\ &= z_4-\frac{2(r-q)T}{\sigma\sqrt{T}}. \end{aligned} \tag{37}$$

Substituting from Equation 37 into the first line inside the braces in Equation 36 gives

$$\begin{aligned} &-e^{-qT}b^{1+\theta}S^{-\theta}\frac{1}{\sigma S\sqrt{T}}\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-\left(z_4-\frac{2(r-q)T}{\sigma\sqrt{T}}\right)^2}{2}\right\}.\exp\left\{-rT+qT-\theta\ln\frac{b}{S}\right\} \\ &= -e^{-qT}b^{1+\theta}S^{-\theta}\frac{1}{\sigma S\sqrt{T}}\,\Phi'(z_4).\exp\left\{2\left(z_4.\frac{(r-q)T}{\sigma\sqrt{T}}-\frac{(r-q)^2T}{\sigma^2}\right)-rT+qT-\theta\ln\frac{b}{S}\right\} \\ &= -e^{-qT}b^{1+\theta}S^{-\theta}\frac{1}{\sigma S\sqrt{T}}\,\Phi'(z_4) \qquad \text{(exponent = 0 after expanding } z_4 \text{ and } \theta) \end{aligned} \tag{38}$$

which cancels with the $\Phi'(z_4)$ term in the second line inside the braces.

Then combining Equations 35 and 36 (with all but the 3rd line now cancelled), the put delta is

$$\frac{\partial P_B}{\partial S} = e^{-qT}\left\{\Phi(z_1)-\Phi(z_3)+\left(\frac{b}{S}\right)^{1+\theta}\left[\Phi(z_4)-\Phi(z_2)\right]\right\}. \tag{39}$$

Appendix B: Black-Scholes replication in the presence of the barrier

As well as the direct and synthetic replication strategies discussed in Sections 3 to 5, there is another way of exactly replicating the payoff of a put (or call) on the observed price S_t, albeit at unnecessarily high expense. This is to start with wealth equal to the Black-Scholes price and then delta-trade using the Black-Scholes delta (say δ_{BS}) based on the observed price S_t. That this approach works in the presence of the barrier may initially seem surprising, but it can be verified by simulation. One way of understanding it is to consider Equation 10, which is reproduced below:

$$\mathrm{d}S_t = (\mu - q) S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t + \mathrm{d}L_t \tag{40}$$

Note that whenever $\mathrm{d}L_t = 0$, this has the usual Black-Scholes dynamics. Therefore in the initial period before S_t first touches the barrier, the Black-Scholes put replicating portfolio has the same dynamics as in the ordinary Black-Scholes world, and so earns the risk-free rate under $\mathbb{Q}_N$. When S_t touches the barrier, the replicating portfolio has exposure of δ_{BS} to the asset, and so receives an appropriately scaled infinitesimal increment (or decrement for a put, because the delta is negative). Once S_t departs from the barrier, it resumes its Black-Scholes dynamics. It therefore again earns the risk-free rate under $\mathbb{Q}_N$, until the next time S_t touches the barrier, when another appropriately scaled increment (or decrement for a put) is applied.

Overall, the sequence just described means that the Black-Scholes replicating portfolio for a put starts off at time zero with a value above the discounted $\mathbb{Q}_N$-expectation of the payoff of the put option on S_T. The dynamics over the full term – asset price sub-martingale, hence Black-Scholes put replicating portfolio super-martingale – then ensure that the Black-Scholes replicating portfolio converges with the option payoff at maturity.

Black-Scholes is "the barrier model with $b = 0$". By extension of the argument just given, replication using a delta calculated from the barrier model with an assumed barrier b' anywhere between 0 and the true barrier level b will also exactly replicate option payoffs, albeit at a higher cost than necessary. This can be verified by simulation.

The feasibility of replication with a delta based on any b' for $0 \le b' \le b$ adds insight into why a delta based on the true level b is the cheapest possible replication strategy for a put. When the spot price touches the barrier, the put replicating portfolio calculated using $b' < b$ has a negative delta; it therefore incurs an unhelpful decrement from the positive intervention at the barrier. But the delta, and therefore the decrement, are of smaller magnitude than Black-Scholes; and so the initial cost of the replicating portfolio is cheaper than for Black-Scholes. Taking this thought to the limit, the minimal replicating portfolio is achieved by using a delta which reaches zero when the spot price touches the barrier, i.e. a delta based on the true level b.

Appendix C: Net-delta arbitrage and credit limits

This appendix investigates the practicality of the net-delta arbitrage strategy highlighted in Section 6.2, a continuously varying long position given by:

$$\Delta = e^{-qT}\left\{1-\Phi\left(z_3\right)+\left(\frac{b}{S}\right)^{1+\theta}\Phi\left(z_4\right)\right\}. \tag{41}$$

I first show that the strategy is costly to scale if initiated when the spot price is well above the barrier. Figure 10 shows results from 1,000 simulations of the strategy over 25 years, initiated when $b/S = 0.5$ (i.e. barrier 50% below spot price, as envisaged in Thomas 2021), and other parameters as in Section 2.3. The left panel shows the distribution of the terminal gain. The right panel shows the lowest cash (i.e. largest borrowing) over the term. The terminal gains lie in a narrow positive band, centred on the difference of the initial forward contract replication costs, $F_B - {}^{S}F_B$, plus interest over the term (as expected). But the lowest cash over the term has a remarkable bi-modal distribution. It can be seen that strategy involves maximum borrowing of nearly 10x the terminal gain of 0.06 in around 25% of simulations.

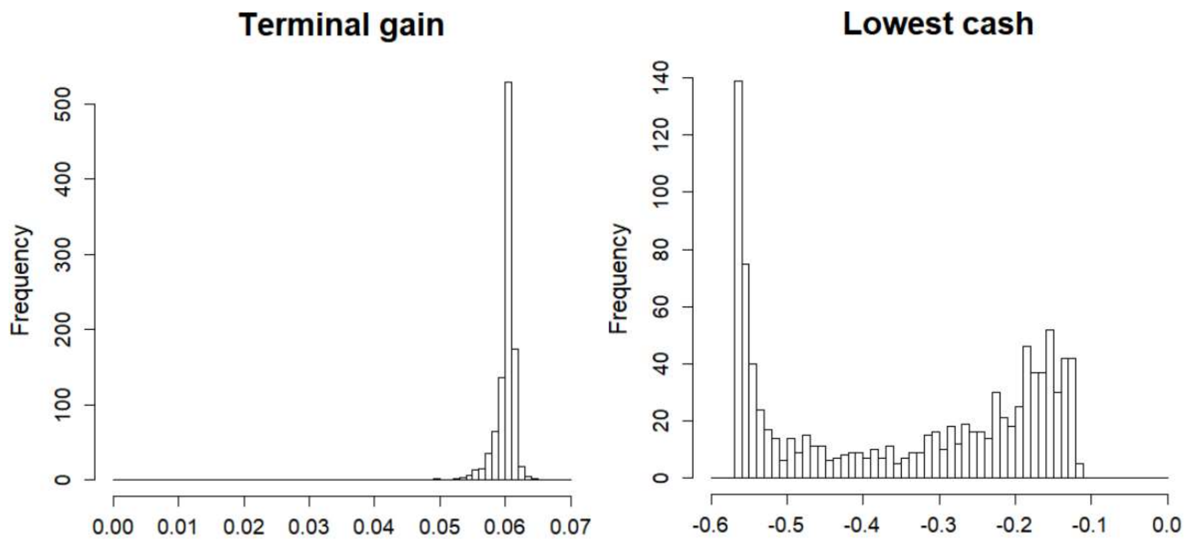

Figure 10. 25-year net-delta strategy, initiated when $b/S = 0.5$.

To be marketable to third-party investors, the strategy would actually need to be run with a much shorter time horizon (nobody will wait 25 years to see if it works). But when we try this, the drawdowns as a multiple of the terminal gain become much larger, as shown in Figure 11. For a five-year term (perhaps the longest lock-up a third-party investor might countenance), the maximum borrowing is around 600x the terminal gain, and minimum portfolio value (i.e. asset less borrowing) is around minus 100x the terminal gain.

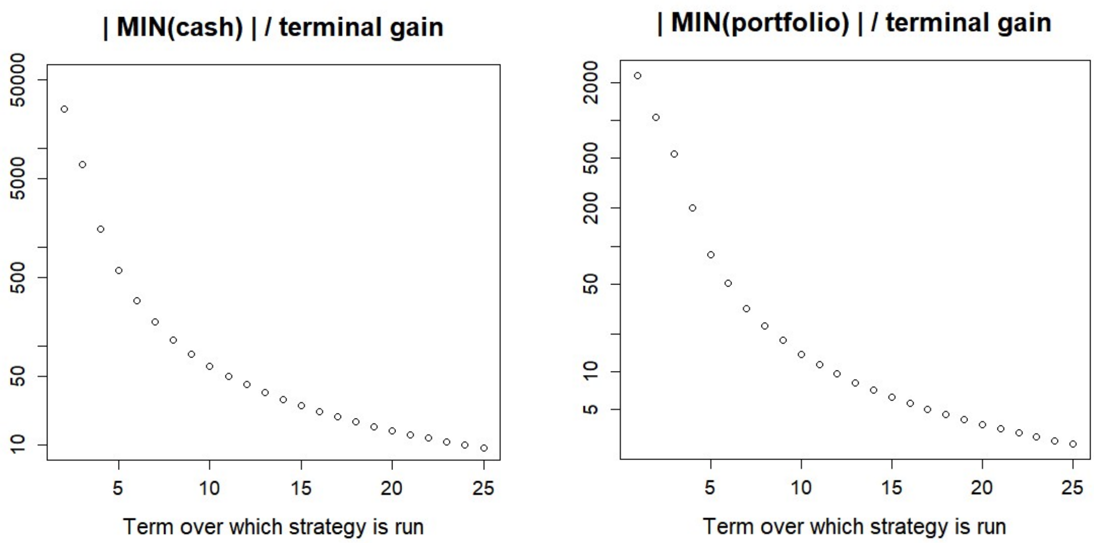

Figure 11. Net-delta strategy initiated when $b/S = 0.5$, for various terms.

The strategy becomes more attractive if initiated at a time when the spot price is already close to the barrier. Figure 12 shows 1,000 simulations of a one-year version of the strategy, initiated when b/S = 0.9 to 0.99, with other parameters as in Section 2.3. The maximum borrowing and portfolio value drawdowns as a multiple of terminal gain is now smaller, but still substantial: cash drawdown over 30x for $b/S = 0.9$, and 10x for $b/S = 0.99$.

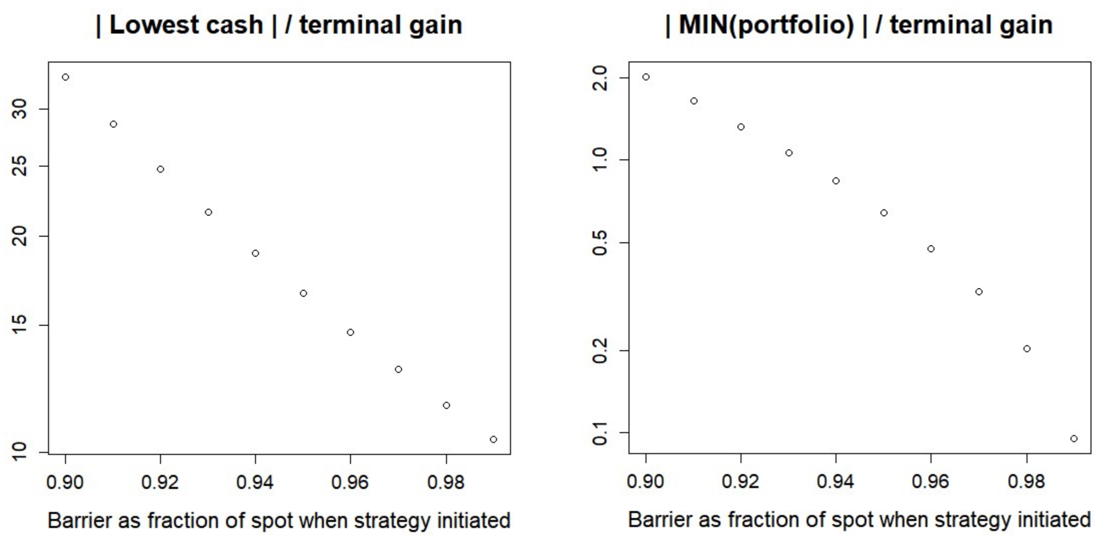

Figure 12. One-year net-delta strategy, initiated when b/S = 0.9 to 0.99.

The capital requirement of the strategy reduces to zero only in the limit, where it is initiated when the spot price is exactly at the barrier. But this is unlikely to be possible in any real market, and certainly not for housing.

There remains a more general point: even market participants who do not pursue explicit arbitrage strategies may nevertheless be more inclined to buy when the spot price is close to the barrier, thus potentially modifying the assumed price process in this region. One way of representing this is to draw the log increments for the asset price in a zone near the barrier from a skew-normal distribution, which can be constructed as a mixture of two normal distributions:

$$X_t = \sqrt{1-\alpha^2}\, W_{1,t} + \alpha \left| W_{2,t} \right|, \qquad -1 \le \alpha \le 1 \tag{42}$$

where $W_{1,t}$ and $W_{2,t}$ are independent Brownian motions, and $\alpha > 0$ gives a positive skew. Figure 13 illustrates this distribution for two values of the skew parameter, $\alpha = 0.3$ and $\alpha = 0.9$.

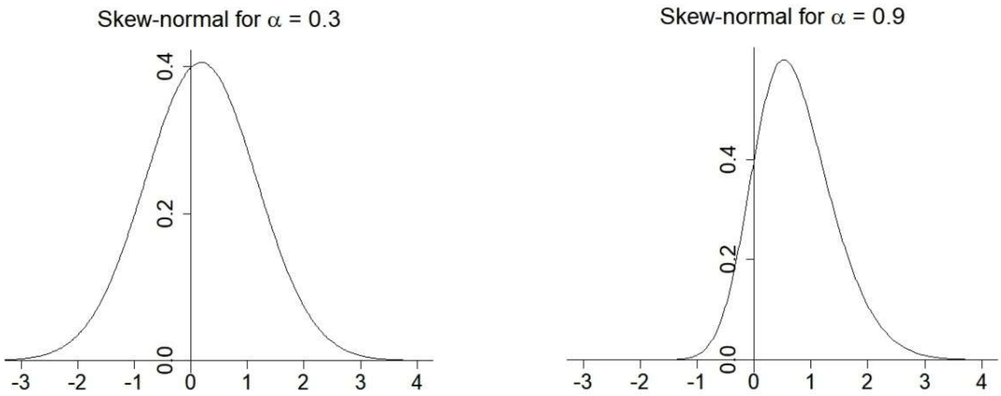

Figure 13. Skew-normal distribution.

To represent an increasing preponderance of buying with increasing proximity to the barrier, we can draw increments with a skew parameter $\alpha = 0$ for $S_{t-1} > 1.09b$, $\alpha = 0.1$ for $1.08b < S_{t-1} \le 1.09b$ and then increase α by a factor of 1.3161 for every 1%-of-b slice below this, reaching a maximum skew of α = 0.9 for $S_{t-1} \le 1.01b$. When we simulate this, the mean Monte Carlo put payoff is reduced (as expected). For parameters as in Section 2.3, (in particular b = 0.5, K = 1), the reduction is about 6%; but along each path, the lower payoff is almost exactly tracked by our put replication scheme. If we anticipated the skew, we could possibly replicate the payoff for a slightly lower initial cost; to that extent, our formula gives a prudent valuation for a put.

Appendix D: Modified upper bound for equity release mortgage valuations

For certain regulatory purposes, the Prudential Regulation Authority (PRA) has promulgated four principles (labelled I to IV) for valuation of an equity release mortgage (ERM) and the NNEG embedded within it. These principles have a common-sense appeal beyond their immediate regulatory context. However, under the assumption of a lower reflecting barrier, part of Principle II may be inappropriate, at least in the form usually stated.

Principle II states[19]:

> *"(II) The economic value of ERM cash flows cannot be greater than either the value of an equivalent loan without an NNEG or the present value of deferred possession of the property providing collateral.*
> *…[This principle] is derived from the following considerations:*
>
> *(i) Given the choice between an ERM and an equivalent loan without an NNEG, a market participant would choose the latter, since either the guarantee is not exercised, in which case the ERM and the loan have the same payoff, or it is, in which case the ERM pays less.*
> *(ii) Similarly, a market participant would prefer future possession of the property on exit to an ERM, given that the property will be of greater value than the ERM if the guarantee is not exercised, or the same value if it is."*

The verbal presentation of the principle obscures the underlying rationale of the dual limits as an application of put-call parity, which can be seen as follows. Define:

K_T Rolled-up loan at maturity time T (the strike price of the NNEG)

H_T House price observed at maturity time T

Then the potential payoff of an ERM in period T (the T-period 'ERM-let') is

$$\min\left(K_T, H_T\right) = K_T - \max\left(K_T - H_T, 0\right) \tag{43}$$

and the present value of this is

$$\text{Present value of } T\text{-period 'ERM-let'} = e^{-rT} K_T - put \tag{44}$$

Noting that *put* is always positive then gives

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19 PRA (2020), Paragraph 3.15.

$$\text{Present value of } T\text{-period 'ERM-let'} \leq e^{-rT} K_T \qquad (45)$$

as the first leg of Principle II.

In the absence of a reflecting barrier, applying the lower bound $put \geq \max\left(K_T e^{-rT} - Se^{-qT}, 0\right)$ in Equation 44 then gives

$$\text{Present value of } T\text{-period 'ERM-let'} \leq Se^{-qT} \qquad (46)$$

as the second leg of Principle II.

But in the presence of a reflecting barrier, the standard lower bound for a put is not sensible, as explained in Section 6.5. So the step at Equation 46 becomes invalid, and hence the second leg of Principle II (i.e. the prepaid forward price Se^{-qT} as an upper limit for ERM) does not apply.

The second leg of Principle II is more likely to be relevant when the term *T* is large, so that the putative upper bound for ERM given by Se^{-qT} becomes small. This is unchanged by the barrier, and so still becomes small for long terms. But this does not matter to the ERM writer, because the presence of the barrier drives a wedge between the pricing of forwards (including prepaid forwards) and options, and in particular, puts become cheaper to hedge than in the absence of a barrier. By dynamically hedging the put it has written, the ERM writer can be sure of receiving a maturity payment with present value $e^{-rT} K_T - put$ (where the *put* valuation allows for the barrier); the ERM writer does not need to be concerned with the (lower) prepaid forward price.

A generalised form of Principle II to encompass the barrier case might be stated as:

> "The economic value of ERM cash flows cannot be greater than either the value of an equivalent loan without an NNEG, *or any related limit derived from put-call parity*."