Web Appendix of
"How much do means-tested benefits reduce the demand for annuities?"

MONIKA BÜTLER
SEW–HSG Universität St. Gallen, CES*Ifo* & Netspar
KIM PEIJNENBURG
Bocconi University, IGIER, & Netspar
STEFAN STAUBLI
University of Calgary, CEPR & IZA

August 20, 2015

1 Summary of the Life-Cycle Problem

We optimize over consumption and asset allocation dynamically. The exogenous state variables are the risk free rate and inflation, the endogenous state variable is wealth. Agents receive means-tested benefits and the amount depends on wealth and income. The optimization problem is solved via dynamic programming and we proceed backwards to find the optimal investment and consumption strategy. In the last period the individual consumes all remaining wealth, hence we exactly know the utility from terminal wealth. Specifically the value at time T is equal to

$$J_T(W_T, R_T^f, \pi_T) = \frac{(W_T + Y_T^I + Y_T^{II} + M_T)^{1-\gamma}}{1-\gamma} \tag{1}$$

We solve the following Bellman equation:

$$V_t(W_t, R_t^f, \pi_t) = \max_{w_t, C_t} \left(\frac{C_t^{1-\gamma}}{1-\gamma} + \beta p_{t+1} E_t(V_{t+1}(W_{t+1}, R_{t+1}^f, \pi_{t+1})) \right) \tag{2}$$

subject to

$$W_{t+1} = (W_t + Y_t^I + Y_t^{II} + M_t - \overline{C_t})(1 + R_t^f + (R_{t+1} - R_t^f)w_t), . \tag{3}$$

If the agent receives means-tested benefits, his consumption is always at least as high as the guaranteed income level, $\tilde{M}_t$.

The individual faces a number of constraints on the consumption and investment decisions. First, we assume that the retiree faces borrowing and short-sales constraints

$$w_t \geq 0 \text{ and } w_t \leq 1. \tag{4}$$

Second, we impose that the investor is borrowing constrained

$$\overline{C_t} \leq W_t, \tag{5}$$

which implies that the individual cannot borrow against future annuity income to increase consumption today.

We define the portfolio return as:

$$R^P_{t+1} = 1 + R^f_t + (R_{t+1} - R^f_t)w_t \tag{6}$$

Furthermore we denote the wealth level after annuity income, consumption, and means-tested benefits as:

$$A_t = W_t + Y^I_t + Y^{II}_t - \overline{C_t} + \max(0, M_t) \tag{7}$$

2 Optimality conditions

In order to find the optimal consumption and investment decisions we derive the Euler conditions. The optimal asset allocation follows from

$$\frac{\partial V_t}{\partial w_t} = E_t\left(\frac{1}{\Pi_{t+1}} C^{*-\gamma}_{t+1}(R_{t+1} - R^f_t)\right) = 0. \tag{8}$$

To obtain the consumption policies we take the first order condition with respect to C_t

$$\frac{\partial V_t}{\partial C_t} = C^{*-\gamma}_t - \beta p_{t+1} E_t\left(\frac{\partial V_{t+1}}{\partial W_{t+1}} \Pi_t R^{P*}_{t+1}\right) = 0. \tag{9}$$

Using the envelope theorem, we find

$$\frac{\partial V_t}{\partial W_t} = \beta p_{t+1} E_t\left(\frac{\partial V_{t+1}}{\partial W_{t+1}} R^{P*}_{t+1}\right)$$
$$\text{if } \tilde{M}_t - Y^I_t - Y^{II}_t - rW_t - gW_t \leq 0 \tag{10}$$
$$\frac{\partial V_t}{\partial W_t} = \beta p_{t+1}(1-r-g) E_t\left(\frac{\partial V_{t+1}}{\partial W_{t+1}} R^{P*}_{t+1}\right)$$
$$\text{if } \tilde{M}_t - Y^I_t - Y^{II}_t - rW_t - gW_t > 0. \tag{11}$$

To solve for the optimal consumption, substitute (10) and (11) into (9) to get the following Euler condition

$$C^{*-\gamma}_t = \beta p_{t+1} E_t\left(\frac{\Pi_t}{\Pi_{t+1}} C^{*-\gamma}_{t+1} R^{P*}_{t+1}\right)$$
$$\text{if } \tilde{M}_t - Y^I_t - Y^{II}_t - rW_t - gW_t \leq 0 \tag{12}$$
$$C^{*-\gamma}_t = \beta p_{t+1}(1-r-g) E_t\left(\frac{\Pi_t}{\Pi_{t+1}} C^{*-\gamma}_{t+1} R^{P*}_{t+1}\right)$$
$$\text{if } \tilde{M}_t - Y^I_t - Y^{II}_t - rW_t - gW_t > 0 \tag{13}$$

3 Optimization procedure for optimal asset weights

Due to the richness and complexity of the model it cannot be solved analytically hence we employ numerical techniques instead. We use the method proposed by Brandt et al. (2005) and Carroll (2006) with several extensions added by Koijen et al. (2010). Brandt et al. (2005) adopt a simulation-based method which can deal with many exogenous state variables. In our case $X_t = (R^f_t, \pi_t)$ is the relevant exogenous state variable. Wealth acts as an endogenous state variable. For this reason, following Carroll (2006), we specify a grid for wealth *after* (annuity) income and consumption. As a result, it is not required to do numerical rootfinding to find the optimal consumption decision.

In each period we find the optimal asset weights by considering the following first order condition

$$E_t(C^{*-\gamma}_{t+1}(R_{t+1} - R^f_t)/\Pi_{t+1}) = 0, \tag{14}$$

where C^*_{t+1} denotes the optimal real consumption level. Because we solve the optimization problem via backwards recursion we know C^*_{t+1} at time $t+1$. Furthermore we simulate the exogenous state variables for N trajectories and T time periods hence we can calculate the realizations of $C^{*-\gamma}_{t+1}(R_{t+1} - R^f_t)/\Pi_{t+1}$. We regress these realizations on a polynomial expansion in the state variables to obtain an approximation of the conditional expectation of the Euler condition

$$E\left(C_{t+1}^{*-\gamma}(R_{t+1}-R_t^f)/\Pi_{t+1}\right) \simeq \tilde{X}_p'\theta_h. \tag{15}$$

In addition we employ a further extension introduced in Koijen et al. (2010). They found that the regression coefficients θ_h are smooth functions of the asset weights and consequently we approximate the regression coefficients θ_h by projecting them further on polynomial expansion in the asset weights:

$$\theta_h' \simeq g(w)\psi. \tag{16}$$

The Euler condition must be set to zero to find the optimal asset weights

$$\tilde{X}_p'\psi g(w)' = 0. \tag{17}$$

Due to the maximization function in the budget constraint the Euler condition can be described by two equations, one for when the agent *does* receive means-tested benefits and a second for when the agent *does not* receive means-tested benefits:

$$C_t^{*-\gamma} = \beta p_{t+1} E_t\left(\frac{\Pi_t}{\Pi_{t+1}} C_{t+1}^{*-\gamma} R_{t+1}^{P*}\right) \text{ if } M_t = 0, \tag{18}$$

$$C_t^{*-\gamma} = \beta p_{t+1}(1-r-g) E_t\left(\frac{\Pi_t}{\Pi_{t+1}} C_{t+1}^{*-\gamma} R_{t+1}^{P*}\right) \text{ if } M_t > 0. \tag{19}$$

4 Optimization procedure for optimal consumption

Similarly to the case of the optimal asset weights, we regress the different realizations of the argument of the Euler condition on a polynomial expansion in the state variables to obtain an approximation of the conditional expectation of the Euler condition. However, now we calculate two potential optimal consumption levels, one for each Euler condition (12) and (13), corresponding to whether or not the agent receives means-tested benefits. Note that $C_t^{*mtb} > C_t^{*nomtb}$, where C_t^{*mtb} is the optimal consumption if an agent receives means-tested benefits and C_t^{*nomtb} is the optimal consumption if the agent does not receive means-tested benefits. This can be seen from (12) and (13) that the optimal consumption with means-tested benefits derived from the maximization procedure is always higher due to the additional factor $(1-r-g)^{-(1/\gamma)}$, which is always higher than

1. The means-tested benefits can be calculated once we know the optimal consumption levels:

$$M_t^{mtb} = \frac{\tilde{M}_t - Y_t^I - Y_t^{II} - (r+g)(A_t + C_t^{*mtb} - Y_t^I - Y_t^{II})}{1-r-g} \tag{20}$$

$$M_t^{nomtb} = \tilde{M}_t - Y_t^I - Y_t^{II} - (r+g)(A_t + C_t^{*nomtb} - Y_t^I - Y_t^{II}). \tag{21}$$

Hence for every time period and every trajectory we find a set of consumption and means-tested benefits: (C_t^{*mtb}, M_t^{mtb}) and $(C_t^{*nomtb}, M_t^{nomtb})$. However, we need to determine which set is the optimal set. First, note that if the income level is higher than the guaranteed consumption level, then an agent does not receive means-tested benefits and the optimal consumption level is C_t^{*nomtb}. On the other hand, when $Y_t < \tilde{M}_t$, then the optimal consumption results from applying the following rules:

$$\text{If } M_t^{mtb} > 0 \text{ and } M_t^{nomtb} > 0 \text{ then } C_t^{*mtb} \text{ is optimal} \tag{22}$$

$$\text{If } M_t^{mtb} > 0 \text{ and } M_t^{nomtb} < 0 \text{ then } C_t^{*mtb} \text{ is optimal} \tag{23}$$

$$\text{If } M_t^{mtb} \leq 0 \text{ and } M_t^{nomtb} < 0 \text{ then } C_t^{*nomtb} \text{ is optimal} \tag{24}$$

$$\begin{aligned} &\text{If } M_t^{mtb} \leq 0 \text{ and } M_t^{nomtb} > 0 \text{ and } |M_t^{nomtb}| < |M_t^{mtb}| \\ &\qquad \text{then } C_t^{*nomtb} \text{ is optimal otherwise } C_t^{*mtb} \text{ is optimal}. \end{aligned} \tag{25}$$

These rules are based on whether the implied means-tested benefits derived from the optimal consumption level are viable and/or optimal.

References

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