Metric Temporal Equilibrium Logic over Timed Traces

Supplementary Material

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Proof of Proposition 1.

$$\begin{array}{lr}
\quad \mathbf{M}, k \models \mathbf{I} & \\
\text{iff } \mathbf{M}, k \models \neg \bullet \top & \text{by Definition of } \mathbf{I} \\
\text{iff } \mathbf{M}, k \models \neg \bullet_{[0..\omega)} \top & \text{by Definition of } \bullet \\
\text{iff } \mathbf{M}, k \not\models \bullet_{[0..\omega)} \top & \text{by Proposition 4 and 3} \\
\text{iff } \mathbf{M}, k-1 \not\models \top \text{ or } k = 0 \text{ or } \tau(k) - \tau(k-1) \notin [0..\omega) & \text{by Definition 2(6)} \\
\text{iff } k = 0 \text{ or } \tau(k) - \tau(k-1) \notin [0..\omega) & \mathbf{M}, k \models \top \text{ for all } k \in [0..\lambda) \\
\text{iff } k = 0 & \text{since } \tau(k-1) \leq \tau(k)
\end{array}$$

$$\begin{array}{lr}
\quad \mathbf{M}, k \models \widehat{\bullet}_I \varphi & \\
\text{iff } \mathbf{M}, k \models \bullet_I \top \rightarrow \bullet_I \varphi & \text{by Definition of } \widehat{\bullet}_I \\
\text{iff } \mathbf{M}, k \not\models \bullet_I \top \text{ or } \mathbf{M}, k \models \bullet_I \varphi & \text{since } \bullet_I \top \text{ behaves classically} \\
\text{iff } k = 0 \text{ or } \tau(k) - \tau(k-1) \notin I \text{ or } \mathbf{M}, k \models \bullet_I \varphi & \text{by the satisfaction of } \bullet_I \top \\
\text{iff } k = 0 \text{ or } \tau(k) - \tau(k-1) \notin I \text{ or } \mathbf{M}, k-1 \models \varphi & \text{by the satisfaction of } \bullet_I \varphi \\
 & \text{and some propositional reasoning.}
\end{array}$$

$$
\begin{array}{ll}
\quad \mathbf{M}, k \models \blacklozenge_I \varphi & \\
\text{iff } \mathbf{M}, k \models \top \, \mathsf{S}_I \, \varphi & \text{by Definition of } \blacklozenge_I \\
\text{iff for some } i \in [0..k] \text{ with } \tau(k) - \tau(i) \in I & \\
\quad \text{we have } \mathbf{M}, i \models \varphi \text{ and } \mathbf{M}, j \models \top \text{ for all } j \in (i..k] & \text{by Definition 2(7)} \\
\text{iff } \mathbf{M}, i \models \varphi \text{ for some } i \in [0..k] \text{ with } \tau(k) - \tau(i) \in I & \mathbf{M}, k \models \top \text{ for all } k \in [0..\lambda)
\end{array}
$$

$$
\begin{array}{ll}
\quad \mathbf{M}, k \models \blacksquare_I \varphi & \\
\text{iff } \mathbf{M}, k \models \bot \, \mathsf{T}_I \, \varphi & \text{by Definition of } \blacksquare_I \\
\text{iff for all } i \in [0..k] \text{ with } \tau(k) - \tau(i) \in I, \text{we have } \mathbf{M}, i \models \varphi \text{ or} & \\
\quad \mathbf{M}, j \models \bot \text{ for some } j \in (i..k] & \text{by Definition 2(8)} \\
\text{iff } \mathbf{M}, i \models \varphi \text{ for all } i \in [0..k] \text{ with } \tau(k) - \tau(i) \in I & \mathbf{M}, k \not\models \bot \text{ for all } k \in [0..\lambda)
\end{array}
$$

For the resp. future cases 16-19 the same reasoning applies.

Proof of Proposition 2. For the complete definition of THT satisfaction, we refer the reader to (Aguado et al. 2023). Here, it suffices to observe that, when we use interval $I = [0..\omega)$ in all operators, all conditions $x \in I$ in Definition 2 (MHT satisfaction) become trivially true, so that the use of τ is irrelevant and the remaining conditions happen to coincide with THT satisfaction.

Proof of Proposition 3. The proof follows by structural induction on the formula φ. Note that universal quantification of $k \in [0..\lambda)$ is part of the induction hypothesis. In what follows, we denote $\mathbf{M} = (\langle \mathbf{H}, \mathbf{T} \rangle, \tau)$.

- If $\varphi = \bot$, the property holds trivially because $\mathbf{M}, k \not\models \bot$.
- If φ is an atom p, $\mathbf{M}, k \models p$ implies $p \in H_k \subseteq T_k$ and so $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \models p$
- For conjunction, disjunction and implication the proof follows the same steps as with persistence in (non-temporal) HT
- If $\varphi = \circ_I \alpha$ then $k + 1 < \lambda$, $\tau(k{+}1) - \tau(k) \in I$ and $\mathbf{M}, k{+}1 \models \alpha$. By induction, the latter implies $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k{+}1 \models \alpha$ so we get the conditions to conclude $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \models \circ_I \alpha$.
- If $\varphi = \alpha \mathbb{U}_I \beta$ then $\mathbf{M}, k \models \alpha \mathbb{U}_I \beta$ implies that for some $j \in [k..\lambda)$ with $\tau(j) - \tau(k) \in I$, we have $\mathbf{M}, j \models \beta$ and $\mathbf{M}, i \models \alpha$ for all $i \in [k..j)$. Since the induction hypothesis applies on any time point, we can apply it to subformulas β and α to conclude for some $j \in [k..\lambda)$ with $\tau(j) - \tau(k) \in I$, we have $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), j \models \beta$ and $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \alpha$ for all $i \in [k..j)$. But the latter amounts to $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \models \alpha \, \mathbb{U}_I \, \beta$.
- The proofs for $\bullet_I$ and S_I are completely analogous to the two previous steps, respectively.

Proof of Corollary 1. By referring to MTL_f-satisfiability as defined in (Koymans

1990), it is obvious that MHT_f-satisfiability for total traces collapes to MTL_f-satisfiability. Therefore we claim that a formula φ is satisfiable in MHT_f iff φ is satisfiable in MTL_f. This together with the decidability of MTL_f (Ouaknine and Worrell 2007) would imply that MHT_f is decidable.

The claim is proved as follows: from left to right, let us assume that φ is MHT_f-satisfiable. Therefore, there exists a MHT_f model $(\langle \mathbf{H}, \mathbf{T}\rangle, \tau)$ such that $(\langle \mathbf{H}, \mathbf{T}\rangle, \tau), 0 \models \varphi$. By Proposition 3, $(\langle \mathbf{T}, \mathbf{T}\rangle, \tau), 0 \models \varphi$. Therefore, φ is MTL_f-satisfiable.

Conversely, if φ is MTL_f-satisfiable then there exists a MTL_f model $(\mathbf{T}, \tau)$ such that $(\mathbf{T}, \tau), 0 \models \varphi$. $(\mathbf{T}, \tau)$ can be turned into the MHT_f model $(\langle \mathbf{T}, \mathbf{T}\rangle, \tau)$ satisfying φ at 0. Therefore, φ is MHT_f-satisfiable.

Proof of Proposition 4. Note that $(\langle \mathbf{H}, \mathbf{T}\rangle, \tau), k \models \neg\varphi$ amounts to $(\langle \mathbf{H}, \mathbf{T}\rangle, \tau), k \models \varphi \to \bot$ and the latter is equivalent to $\mathbf{M}, k \not\models \varphi$ or $\mathbf{M}, k \models \bot$, for both $\mathbf{M} = (\langle \mathbf{H}, \mathbf{T}\rangle, \tau)$ and $\mathbf{M} = (\langle \mathbf{T}, \mathbf{T}\rangle, \tau)$. Since $\mathbf{M}, k \models \bot$ never holds, we get that this condition is equivalent to both $(\langle \mathbf{H}, \mathbf{T}\rangle, \tau), k \not\models \varphi$ and $(\langle \mathbf{T}, \mathbf{T}\rangle, \tau), k \not\models \varphi$. However, by Proposition 3 (persistence), the latter implies the former, so we get that this is just equivalent to $(\langle \mathbf{T}, \mathbf{T}\rangle, \tau), k \not\models \varphi$.

Proof of Proposition 5. From left to right, assume by contradiction that $\mathbf{H} \neq \mathbf{T}$. By construction of an HT-trace, $H_j \subseteq T_j$ for all $0 \leq j < \lambda$, but as $\mathbf{H} \neq \mathbf{T}$, the subset relation must be strict $H_i \subset T_i$ for some $0 \leq i < \lambda$. This means that there exists $p \in \mathcal{A}$ such that $p \in T_i \setminus H_i$. Therefore, $(\langle \mathbf{H}, \mathbf{T}\rangle, \tau) \not\models p \vee \neg p$. Since $i \geq 0$ and, clearly, $\tau(i) - \tau(0) \in [0..\omega)$, we obtain that $(\langle \mathbf{H}, \mathbf{T}\rangle, \tau), 0 \not\models \Box\,(p \vee \neg p)$. As a consequence we get $(\langle \mathbf{H}, \mathbf{T}\rangle, \tau), 0 \not\models \mathrm{EM}(\mathcal{A})$: a contradiction. Conversely, assume by contradiction that $(\langle \mathbf{H}, \mathbf{T}\rangle, \tau), 0 \not\models \mathrm{EM}(\mathcal{A})$. Therefore, there exists $0 \leq i < \lambda$ such that $\tau(i) - \tau(0) \in [0..\omega)$ and $(\langle \mathbf{H}, \mathbf{T}\rangle, \tau), i \not\models p \vee \neg p$. This means that $p \in T_i \setminus H_i$ so $H_i \subset T_i$. As a consequence, $\mathbf{H} \neq \mathbf{T}$: a contradiction.

Proof of Proposition 6. The proof follows similar steps to Proposition 10 in (Aguado et al. 2023) for the non-metric case (and LTL instead of MTL). For a proof sketch, note that if no implication or negation is involved, the evaluation of the formula is exclusively performed on trace $\mathbf{H}$, while the there-component $\mathbf{T}$ is never used, becoming irrelevant (we are free to choose any trace $\mathbf{T} \geq \mathbf{H}$). Thus, checking the equivalence on total traces $(\langle \mathbf{H}, \mathbf{H}\rangle, \tau)$ does not lose generality, whereas total traces exactly correspond to MTL satisfaction.

Proof of Lemma 1. The proof follows similar steps to Lemma 2 in (Aguado et al. 2023) for the non-metric case. Again, we define $\varrho(\mathbf{M})$ as the timed trace $(\langle \mathbf{H}', \mathbf{T}'\rangle, \tau')$ where $H'_i = H_{\lambda-1-i}$ and $T'_i = T_{\lambda-1-i}$ for all $i \in [0..\lambda)$. The only difference here is that we must also "reverse" the time function τ defining $\tau'(i) = \tau(\lambda{-}1) - \tau(\lambda{-}1{-}i)$ to keep the same relative distances but in reversed order. Then, the proof follows from the complete temporal symmetry of satisfaction of operators (when the trace is finite).

Proof of Theorem 1. The proof follows similar steps to Theorem 3 in (Aguado et al. 2023) for the non-metric case but relying here on Lemma 1 instead.

Proof of Proposition 7.

$$\begin{array}{lr}
\quad \mathbf{M}, k \models \circ_I (\varphi \vee \psi) & \\
\text{iff } \mathbf{M}, k+1 \models \varphi \vee \psi \text{ and } \tau(k+1) - \tau(k) \in I & \text{by Definition 2(9)} \\
\text{iff } (\mathbf{M}, k+1 \models \varphi \text{ or } \mathbf{M}, k+1 \models \psi) \text{ and } \tau(k+1) - \tau(k) \in I & \text{by Definition 2(4)} \\
\text{iff } (\mathbf{M}, k+1 \models \varphi \text{ and } \tau(k+1) - \tau(k) \in I) & \text{by Distributivity} \\
\quad \text{or } (\mathbf{M}, k+1 \models \psi \text{ and } \tau(k+1) - \tau(k) \in I) & \\
\text{iff } \mathbf{M}, k \models \circ_I \varphi \vee \circ_I \psi & \text{by Definition 2(9)}
\end{array}$$

$$\begin{array}{lr}
\quad \mathbf{M}, k \models \circ_I (\varphi \wedge \psi) & \\
\text{iff } \mathbf{M}, k+1 \models \varphi \wedge \psi \text{ and } \tau(k+1) - \tau(k) \in I & \text{by Definition 2(9)} \\
\text{iff } (\mathbf{M}, k+1 \models \varphi \text{ and } \mathbf{M}, k+1 \models \psi) \text{ and } \tau(k+1) - \tau(k) \in I & \text{by Definition 2(3)} \\
\text{iff } (\mathbf{M}, k+1 \models \varphi \text{ and } \tau(k+1) - \tau(k) \in I) & \text{by Distributivity} \\
\quad \text{and } (\mathbf{M}, k+1 \models \psi \text{ and } \tau(k+1) - \tau(k) \in I) & \\
\text{iff } \mathbf{M}, k \models \circ_I \varphi \wedge \circ_I \psi & \text{by Definition 2(9)}
\end{array}$$

$$\begin{array}{lr}
\quad \mathbf{M}, k \models \widehat{\circ}_I (\varphi \vee \psi) & \\
\text{iff } k+1 = \lambda \text{ or } \mathbf{M}, k+1 \models \varphi \vee \psi \text{ or } \tau(k+1) - \tau(k) \notin I & \text{by Proposition 1(17)} \\
\text{iff } k+1 = \lambda \text{ or } (\mathbf{M}, k+1 \models \varphi \text{ or } \mathbf{M}, k+1 \models \psi) \text{ or } \tau(k+1) - \tau(k) \notin I & \text{by Definition 2(4)} \\
\text{iff } (k+1 = \lambda \text{ or } \mathbf{M}, k+1 \models \varphi \text{ or } \tau(k+1) - \tau(k) \notin I) & \text{by Distributivity} \\
\quad \text{or } (k+1 = \lambda \text{ or } \mathbf{M}, k+1 \models \psi \text{ or } \tau(k+1) - \tau(k) \notin I) & \\
\text{iff } \mathbf{M}, k \models \widehat{\circ}_I \varphi \vee \widehat{\circ}_I \psi & \text{by Proposition 1(17)}
\end{array}$$

$$\begin{array}{lr}
\quad \mathbf{M}, k \models \widehat{\circ}_I (\varphi \wedge \psi) & \\
\text{iff } k+1 = \lambda \text{ or } \mathbf{M}, k+1 \models \varphi \wedge \psi \text{ or } \tau(k+1) - \tau(k) \notin I & \text{by Proposition 1(17)} \\
\text{iff } k+1 = \lambda \text{ or } (\mathbf{M}, k+1 \models \varphi \text{ and } \mathbf{M}, k+1 \models \psi) \text{ or } \tau(k+1) - \tau(k) \notin I & \text{by Definition 2(3)} \\
\text{iff } (k+1 = \lambda \text{ or } \mathbf{M}, k+1 \models \varphi \text{ or } \tau(k+1) - \tau(k) \notin I) & \text{by Distributivity} \\
\quad \text{and } (k+1 = \lambda \text{ or } \mathbf{M}, k+1 \models \psi \text{ or } \tau(k+1) - \tau(k) \notin I) & \\
\text{iff } \mathbf{M}, k \models \widehat{\circ}_I \varphi \wedge \widehat{\circ}_I \psi & \text{by Proposition 1(17)}
\end{array}$$

$$
\begin{array}{lr}
\quad \mathbf{M}, k \models \Diamond_I (\varphi \vee \psi) & \\
\text{iff } \mathbf{M}, i \models \varphi \vee \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I & \text{by Definition 2(18)} \\
\text{iff } (\mathbf{M}, i \models \varphi \text{ or } \mathbf{M}, i \models \psi) \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I & \text{by Definition 2(4)} \\
\text{iff } (\mathbf{M}, i \models \varphi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I) & \text{by Distributivity} \\
\quad \text{or } (\mathbf{M}, i \models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I) & \\
\text{iff } \mathbf{M}, k \models \Diamond_I \varphi \vee \Diamond_I \psi & \text{by Definition 2(18)}
\end{array}
$$

$$
\begin{array}{lr}
\quad \mathbf{M}, k \models \Box_I (\varphi \wedge \psi) & \\
\text{iff } \mathbf{M}, i \models \varphi \wedge \psi \text{ for all } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I & \text{by Definition 2(19)} \\
\text{iff } (\mathbf{M}, i \models \varphi \text{ and } \mathbf{M}, i \models \psi) \text{ for all } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I & \text{by Definition 2(3)} \\
\text{iff } (\mathbf{M}, i \models \varphi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I) & \text{by Distributivity} \\
\quad \text{and } (\mathbf{M}, i \models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I) & \\
\text{iff } \mathbf{M}, k \models \Box_I \varphi \wedge \Box_I \psi & \text{by Definition 2(19)}
\end{array}
$$

$$
\begin{array}{lr}
\quad \mathbf{M}, k \models \varphi \, \mathbb{U}_I \, (\chi \vee \psi) & \\
\text{iff } \mathbf{M}, i \models \chi \vee \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I & \\
\quad \text{and } \mathbf{M}, j \models \varphi \text{ for all } j \in [k..i) & \text{by Definition 2(10)} \\
\text{iff } (\mathbf{M}, i \models \chi \text{ or } \mathbf{M}, i \models \psi) \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I & \\
\quad \text{and } \mathbf{M}, j \models \varphi \text{ for all } j \in [k..i) & \text{by Definition 2(3)} \\
\text{iff } \mathbf{M}, i \models \chi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I \text{ or} & \text{by Distributivity} \\
\quad \mathbf{M}, i \models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I & \\
\quad \text{and } \mathbf{M}, j \models \varphi \text{ for all } j \in [k..i) & \\
\text{iff } \mathbf{M}, k \models (\varphi \, \mathbb{U}_I \, \chi) \vee (\varphi \, \mathbb{U}_I \, \psi) & \text{by Definition 2(10)}
\end{array}
$$

$$
\begin{aligned}
&\mathbf{M}, k \models (\varphi \wedge \chi)\, \mathbb{U}_I\, \psi \\
&\text{iff } \mathbf{M}, i \models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I \\
&\quad \text{and } \mathbf{M}, j \models \varphi \wedge \psi \text{ for all } j \in [k..i) && \text{by Definition 2(10)} \\
&\text{iff } \mathbf{M}, i \models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I && \text{by Definition 2(3)} \\
&\quad \text{and } \mathbf{M}, j \models \varphi \text{ and } \mathbf{M}, j \models \psi \text{ for all } j \in [k..i) \\
&\text{iff } \mathbf{M}, i \models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I && \text{by Distributivity} \\
&\quad \text{and } \mathbf{M}, j \models \varphi \text{ for all } j \in [k..i) \text{ and} \\
&\quad \mathbf{M}, i \models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I \\
&\quad \text{and } \mathbf{M}, j \models \chi \text{ for all } j \in [k..i) \\
&\text{iff } \mathbf{M}, k \models (\varphi\, \mathbb{U}_I\, \psi) \text{ and } \mathbf{M}, k \models (\chi\, \mathbb{U}_I\, \psi) && \text{by Definition 2(10)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{M}, k \models \varphi\, \mathbb{R}_I\, (\chi \wedge \psi) \\
&\text{iff for all } i \text{ with } \tau(i) - \tau(k) \in \mathcal{I}, \text{ we have} && \text{by Definition 2(11)} \\
&\quad \mathbf{M}, i \models \chi \wedge \psi \text{ or } \mathbf{M}, j \models \varphi \text{ for some } j \in [k..i) \\
&\text{iff for all } i \text{ with } \tau(i) - \tau(k) \in \mathcal{I}, \text{ we have} \\
&\quad \mathbf{M}, i \models \chi \text{ and } \mathbf{M}, i \models \psi \text{ or } \mathbf{M}, j \models \varphi \text{ for some } j \in [k..i) && \text{by Definition 2(3)} \\
&\text{iff for all } i \text{ with } \tau(i) - \tau(k) \in \mathcal{I}, \text{ we have} \\
&\quad \mathbf{M}, i \models \chi \text{ or } \mathbf{M}, j \models \varphi \text{ for some } j \in [k..i) \\
&\quad \text{and iff for all } i \text{ with } \tau(i) - \tau(k) \in \mathcal{I}, \text{ we have} \\
&\quad \mathbf{M}, i \models \psi \text{ or } \mathbf{M}, j \models \varphi \text{ for some } j \in [k..i) && \text{by Distributivity} \\
&\text{iff } \mathbf{M}, k \models (\varphi\, \mathbb{R}_I\, \chi) \text{ and } (\varphi\, \mathbb{R}_I\, \psi) && \text{by Definition 2(11)} \\
&\text{iff } \mathbf{M}, k \models (\varphi\, \mathbb{R}_I\, \chi) \wedge (\varphi\, \mathbb{R}_I\, \psi) && \text{by Definition 2(3)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{M}, k \models (\varphi \vee \chi)\, \mathbb{R}_I\, \psi \\
&\text{iff for all } i \text{ with } \tau(i) - \tau(k) \in \mathcal{I}, \text{ we have} && \text{by Definition 2(11)} \\
&\quad \mathbf{M}, i \models \psi \text{ or } \mathbf{M}, j \models \varphi \vee \chi \text{ for some } j \in [k..i) \\
&\text{iff for all } i \text{ with } \tau(i) - \tau(k) \in \mathcal{I}, \text{ we have} && \text{by Definition 2(4)} \\
&\quad \mathbf{M}, i \models \psi \text{ or } \mathbf{M}, j \models \varphi \text{ or } \mathbf{M}, j \models \chi \text{ for some } j \in [k..i) \\
&\text{iff for all } i \text{ with } \tau(i) - \tau(k) \in \mathcal{I}, \text{ we have} && \text{by Distributivity} \\
&\quad \mathbf{M}, i \models \psi \text{ or } \mathbf{M}, j \models \varphi \text{ for some } j \in [k..i) \text{ or} \\
&\quad \text{iff for all } i \text{ with } \tau(i) - \tau(k) \in \mathcal{I}, \text{ we have} \\
&\quad \mathbf{M}, i \models \psi \text{ or } \mathbf{M}, j \models \chi \text{ for some } j \in [k..i) \\
&\text{iff } \mathbf{M}, k \models (\varphi\, \mathbb{R}_I\, \psi) \text{ or } (\chi\, \mathbb{R}_I\, \psi) && \text{by Definition 2(11)} \\
&\text{iff } \mathbf{M}, k \models (\varphi\, \mathbb{R}_I\, \psi) \vee (\chi\, \mathbb{R}_I\, \psi) && \text{by Definition 2(4)}
\end{aligned}
$$

For the resp. past cases 11-20 the same reasoning applies.

Proof of Proposition 8. We consider the first equivalence. From left to right, assume towards a contradiction that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \not\models \neg\varphi \, \mathbb{R}_I \, \neg\psi$. Therefore, there exists $j \in [i..\lambda)$ such that $\tau(j) - \tau(i) \in I$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \not\models \neg\psi$ and for all $k \in [i..j)$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \not\models \neg\varphi$. By Proposition 4, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), j \models \psi$ and $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \models \varphi$ for all $k \in [i..j)$. By the semantics of the until operator we obtain that $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \varphi \, \mathbb{U}_I \, \psi$. By Proposition 4 it follows that $\langle \mathbf{H}, \mathbf{T} \rangle, i \not\models \neg \left(\varphi \, \mathbb{U}_I \, \psi\right)$: a contradiction.

From right to left, if $\langle \mathbf{H}, \mathbf{T} \rangle, i \not\models \neg \left(\varphi \, \mathbb{U}_I \, \psi\right)$ then, by Proposition 4, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \models \varphi \, \mathbb{U}_I \, \psi$. Therefore there exists $j \in [i..\lambda)$ such that $\tau(j) - \tau(i) \in I$, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), j \models \psi$ and for all $k \in [i..j)$, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \models \varphi$. Since $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau)$ satisfies the law of excluded middle, it follows that $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), j \not\models \neg\psi$ and for all $k \in [i..j)$, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), k \not\models \neg\varphi$. By the semantics, $(\langle \mathbf{T}, \mathbf{T} \rangle, \tau), i \not\models \neg\varphi \, \mathbb{R}_I \, \neg\psi$. By persistency, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau) \not\models \neg\varphi \, \mathbb{R}_I \, \neg\psi$.

The remaining equivalences can be verified in a similar way.

Proof of Proposition 9.

$$
\begin{aligned}
\mathbf{M}, k \models (\varphi \, \mathbb{U}_I \, \psi) \text{ iff } & \mathbf{M}, i \models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I \\
& \quad \text{and } \mathbf{M}, j \models \varphi \text{ for all } j \in [k..i) && \text{by Definition 2(10)} \\
\text{implies } & \mathbf{M}, i \models \psi \text{ for some } i \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in J \\
& \quad \text{and } \mathbf{M}, j \models \varphi \text{ for all } j \in [k..i) && \text{since } I \subseteq J \\
\text{iff } & \mathbf{M}, k \models (\varphi \, \mathbb{U}_J \, \psi) && \text{by Definition 2(10)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}, k \models (\varphi \, \mathbb{R}_J \, \psi) \text{ iff } & \text{for all } j \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in J \\
& \quad \text{we have } \mathbf{M}, i \models \psi \text{ or } \mathbf{M}, j \models \varphi \text{ for some } j \in [k..i) && \text{by Definition 2(11)} \\
\text{implies } & \text{for all } j \in [k..\lambda) \text{ with } \tau(i) - \tau(k) \in I \\
& \quad \text{we have } \mathbf{M}, i \models \psi \text{ or } \mathbf{M}, j \models \varphi \text{ for some } j \in [k..i) && \text{since } I \subseteq J \\
\text{iff } & \mathbf{M}, k \models (\varphi \, \mathbb{R}_I \, \psi) && \text{by Definition 2(11)}
\end{aligned}
$$

The cases 2 and 4 work analogously

Proof of equivalences (4)-(6).

- Equivalence (4): Take any $i \in [0, \lambda)$. $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \, \mathbb{U}_0 \, \psi$ iff there exists $j \in [i, \lambda)$ such that $\tau(j) - \tau(i) = 0$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \models \psi$ and for all $i \leq k < j$, $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$. From $\tau(j) - \tau(i) = 0$ it follows that $\tau(j) = \tau(i)$. Under strict semantics, it follows $j = i$. From this we get the iff $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi$. Furthermore,
  $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \varphi \, \mathbb{R}_0 \, \psi$ iff for all $j \in [i, \lambda)$ if $\tau(j) - \tau(i) = 0$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), j \not\models \psi$ then there exists $i \leq k < j$ such that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), k \models \varphi$.
  From $\tau(j) - \tau(i) = 0$ it follows that $\tau(j) = \tau(i)$. Under strict semantics, it follows $j = i$. From this we get the iff $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \psi$.
- Equivalence (5): For the case of $\circ_0\varphi$ we have that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \circ_0\varphi$ iff $i+1 < \lambda$, $\tau(i+1) - \tau(i) = 0$ and $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i+1 \models \varphi$. Since we are considering strict semantics, we get that $\tau(i+1) - \tau(i) \neq 0$ and we can derive $\bot$. We can follow a similar reasoning for the case of $\bullet_0\varphi$.
- Equivalence (6): For the case of $\widehat{\circ}_0\varphi$ we have that $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i \models \widehat{\circ}_0\varphi$ iff if $i+1 < \lambda$ and $\tau(i+1) - \tau(i) = 0$ then $(\langle \mathbf{H}, \mathbf{T} \rangle, \tau), i+1 \models \varphi$. Since we are considering strict

semantics, we get that $\tau(i+1) - \tau(i) \neq 0$ and we can derive $\top$. We can follow a similar reasoning for the case of $\widehat{\bullet}_0\varphi$.

Proof of Proposition 10.

- Equivalence (15): In case $m \geq n$ then $[m \cdots n)$ is empty so $\bigvee_{i=m}^{n-1} \circ_i\varphi \equiv \bot$. It follows that $\mathbf{M}, k \not\models \circ_{[\,.\,.[m\cdots n)]}\varphi$ and $\mathbf{M}, k \not\models \bot$. Otherwise, from left to right, if $\mathbf{M}, k \not\models \circ_{[\,.\,.[m\cdots n)]}\varphi$ then $\mathbf{M}, k+1 \models \varphi$ and $\tau(k+1) - \tau(k) \in [m \cdots n)$. This means that there exists $t \in [m..n)$ such that $\tau(k+1) - \tau(k) = t$. From this and $\mathbf{M}, k+1 \models \varphi$ we get that $\mathbf{M}, k \models \circ_t\varphi$ so $\mathbf{M}, k \models \bigvee_{i=m}^{n-1} \circ_i\varphi$. Conversely, if $\mathbf{M}, k \models \bigvee_{i=m}^{n-1} \circ_i\varphi$ then there exists $t \in [m..n)$ such that $\mathbf{M}, k \models \circ_t$. Therefore, $\mathbf{M}, k+1 \models \varphi$ and $\tau(k+1) - \tau(k) = t$. Since $t \in [m..n)$ and $\tau(k+1) - \tau(k) = t$, $\tau(k+1) - \tau(k) \in [m \cdots n)$ so $\mathbf{M}, k \models \circ_{[m\cdots n)}\varphi$.
- Equivalence (16): In case $m \geq n$ then $[m \cdots n)$ is empty so $\bigwedge_{i=m}^{n-1} \widehat{\circ}_i\varphi \equiv \top$. It follows that $\mathbf{M}, k \models \widehat{\circ}_{[\,.\,.[m\cdots n)]}\varphi$ and $\mathbf{M}, i \models \top$. Otherwise, from left to right, if $\mathbf{M}, k \not\models \bigwedge_{i=m}^{n-1} \widehat{\circ}_i\varphi$ then there exists $t \in [m..n)$ such that $\mathbf{M}, k \not\models \widehat{\circ}_t$. Therefore, $\mathbf{M}, k+1 \not\models \varphi$ and $\tau(k+1) - \tau(k) = t$. Since $t \in [m..n)$ and $\tau(k+1) - \tau(k) = t$, $\tau(k+1) - \tau(k) \in [m \cdots n)$ so $\mathbf{M}, k \not\models \widehat{\circ}_{[m\cdots n)}\varphi$: a contradiction. Conversely, if $\mathbf{M}, k \not\models \widehat{\circ}_{[\,.\,.[m\cdots n)]}\varphi$ then $\mathbf{M}, k+1 \not\models \varphi$ and $\tau(k+1) - \tau(k) \in [m \cdots n)$. This means that there exists $t \in [m..n)$ such that $\tau(k+1) - \tau(k) = t$. From this and $\mathbf{M}, k+1 \not\models \varphi$ we get that $\mathbf{M}, k \not\models \widehat{\circ}_t\varphi$. Therefore, $\mathbf{M}, k \not\models \bigwedge_{i=m}^{n-1} \widehat{\circ}_i\varphi$: a contradiction.

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