On the Foundations of Grounding in Answer Set Programming (Supplementary Material)

ROLAND KAMINSKI and TORSTEN SCHAUB
University of Potsdam
(*e-mail:* {kaminski,torsten}@cs.uni-potsdam.de)

Appendix A Proofs

We use the following reduct-based characterization of strong equivalence (Turner 2003).

Lemma 41. *Sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of infinitary formulas are strongly equivalent iff $\mathcal{H}_1^I$ and $\mathcal{H}_2^I$ are classically equivalent for all two-valued interpretations I.*

Proof. Let I and J be two-valued interpretations, and $\mathcal{H}$ be an infinitary formula. Clearly, $I \models \mathcal{H}^J$ iff $I \cap J \models \mathcal{H}^J$. Thus, we only need to consider interpretations such that $I \subseteq J$. By Lemma 1 due to Harrison et al. (2017), we have that $I \models \mathcal{H}^J$ iff (I, J) is an HT-model of $\mathcal{H}$. The proposition holds because by Theorem 3 Item (iii) due to Harrison et al. (2017), we have that $\mathcal{H}_1$ and $\mathcal{H}_2$ are strongly equivalent iff they have the same HT models. □

Proof of Proposition 2. Let $F = \{B(r) \to H(r) \mid r \in P\}^\wedge$.

First, we consider the case $I \models F$. By Lemma 1 due to Truszczyński (2012), this implies that P^I and F^I are classically equivalent and thus have the same minimal models.[10] Thus, I is a stable model of P iff I is a stable model of F.

Second, we consider the case $I \not\models F$ and show that I is neither a stable model of F nor P. Proposition 1 by Truszczyński (2012) states that I is a model of F iff I is a model of F^I. Thus, I is not a stable model of F. Furthermore, because $I \not\models F$, there is a rule $r \in P$ such that $I \not\models B(r) \to H(r)$. Consequently, we have $I \models B(r)$ and $I \not\models H(r)$. Using the above proposition again, we get $I \models B(r)^I$. Because $I \models B(r)^I$ and $I \not\models H(r)$, we get $I \not\models r^I$ and in turn $I \not\models P^I$. Thus, I is not a stable model of P either. □

Lemma 42. *Let F be a formula, and I and J be interpretations.*
If F is positive and $I \subseteq J$, then $I \models F$ implies $J \models F$.

Proof. This property can be shown by induction over the rank of the formula. □

The following two propositions shed some light on the two types of reducts.

Lemma 43. *Let F be a formula, and I and J be interpretations.*
Then,

(a) *if F is positive then F^I is positive,*
(b) *$I \models F$ iff $I \models F^I$,*
(c) *if F is strictly positive and $I \subseteq J$ then $I \models F$ iff $I \models F^J$.*

Proof.

Property (a). Because the reduct only replaces subformulas by $\bot$, the resulting formula is still positive.

Property (b). Corresponds to Proposition 1 by Truszczyński (2012).

Property (c). This property can be shown by induction over the rank of the formula. □

Lemma 44. *Let F be a formula, and I, J, and X be interpretations.*
Then,

(a) *F_I is positive,*

[10] To be precise, Lemma 1 by Truszczyński (2012) is stated for a set of formulas, which can be understood as an infinitary conjunction.

(b) *$I \models F$ iff $I \models F_I$,*
(c) *if F is positive then $F = F_I$, and*
(d) *if $I \subseteq J$ then $X \models F_J$ implies $X \models F_I$.*

Proof.

Property (a). Because the ID-reduct replaces all negative occurrences of atoms, the resulting formula is positive.

Property (b). This property holds because when the reduct replaces an atom a, it is replaced by either $\top$ or $\bot$ depending on whether $I \models a$ or $I \not\models a$. This does not change the satisfaction of the subformula w.r.t. I.

Property (c). Because a positive formula does not contain negative occurrences of atoms, it is not changed by the ID-reduct.

Property (d). We prove by induction over the rank of formula F that

$$X \models F_J \text{ implies } X \models F_I \text{ and} \tag{A1}$$

$$X \models F_{\overline{I}} \text{ implies } X \models F_{\overline{J}}. \tag{A2}$$

Base. We consider the case that F is a formula of rank 0.

$$F \text{ is a formula of rank 0 implies } F \text{ is an atom.}$$

First, we show (A1). We assume $X \models F_J$:

$$F \text{ is an atom implies } F_I = F_J = F.$$
$$F_I = F_J \text{ and } X \models F_J \text{ implies } X \models F_I.$$

Second, we show (A2). We assume $X \models F_{\overline{I}}$:

$$F \text{ is an atom and } X \models F_{\overline{I}} \text{ implies } F_{\overline{I}} = \top.$$
$$F_{\overline{I}} = \top \text{ implies } F \in I.$$
$$F \in I \text{ and } I \subseteq J \text{ implies } F \in J.$$
$$F \text{ is an atom and } F \in J \text{ implies } F_{\overline{J}} = \top.$$
$$F_{\overline{J}} = \top \text{ implies } X \models F_{\overline{J}}.$$

Hypothesis. We assume that (A1) and (A2) hold for formulas F of ranks smaller than i.

Step. We only show (A1) because (A2) can be shown in a similar way. We consider formulas F of rank i.

First, we consider the case that F is a conjunction of form $\mathcal{H}^\wedge$.

$$X \models \mathcal{H}^\wedge_J \text{ implies } X \models G_J \text{ for all } G \in \mathcal{H}.$$
$$X \models G_J \text{ implies } X \models G_I \text{ by hypothesis.}$$
$$X \models G_I \text{ implies } X \models \mathcal{H}^\wedge_I.$$

The case for disjunctions can be proven in a similar way.

Last, we consider the case that F is an implication of form $G \rightarrow H$. Observe that

$$F_I = G_{\overline{I}} \rightarrow H_I \text{ and}$$
$$F_J = G_{\overline{J}} \rightarrow H_J.$$

First, we consider the case $X \not\models G_{\overline{J}}$:

$$X \not\models G_{\overline{J}} \text{ implies } X \not\models G_{\overline{I}} \text{ by hypothesis.}$$
$$X \not\models G_{\overline{I}} \text{ implies } X \models F_I.$$

Second, we consider the case $X \models H_J$:

$$X \models H_J \text{ implies } X \models H_I \text{ by hypothesis.}$$
$$X \models H_I \text{ implies } X \models F_I. \qquad \square$$

Proof of Lemma 6. This lemma follows from Proposition 14 by Denecker et al. (2000) observing that the well-founded operator is a monotone symmetric operator. The proposition is actually a bit more general stating that the operator maps any consistent four-valued interpretation to a consistent four-valued interpretation. $\square$

Lemma 45. *Let O and O' be monotone operators over complete lattice $(L, \leq)$ with $O'(x) \leq O(x)$ for each $x \in L$.*

Then, we get $x' \leq x$ where x' and x are the least fixed points of O' and O, respectively.

Proof. Let y be a prefixed point of O. We have $O(y) \leq y$. Because $O'(y) \leq O(y)$, we get $O'(y) \leq y$. So each prefixed point of O is also a prefixed point of O'.

Let S' and S be the set of all prefixed points of O' and O, respectively. We obtain $S \subseteq S'$. By Theorem 1 (a), we get that x' is the greatest lower bound of S'. Observe that x' is a lower bound for S. By construction of S, we have $x \in S$. Hence, we get $x' \leq x$. $\square$

Lemma 46. *Let P and P' be $\mathcal{F}$-programs and I be an interpretation.*

Then, $P' \subseteq P$ implies $S_{P'}(I) \subseteq S_P(I)$.

Proof. This lemma is a direct consequence of Lemma 45 observing that the one-step provability operator derives fewer consequences for P'. $\square$

Lemma 47. *Let P be an $\mathcal{R}$-program, I be a two-valued interpretation, and $J = S_P(I)$.*

Then, X is a stable model of P, $I \subseteq X$, and $I \subseteq J$ implies $X \subseteq J$.

Proof. Because X is a stable model of P, it is the only minimal model of P^X. Furthermore, we have that J is a model of P_I. To show that $X \subseteq J$, we show that J is also a model of P^X. For this, it is enough to show that for each rule $r \in P$ we have $J \not\models B(r)_I$ implies $J \not\models B(r)^X$. We prove inductively over the rank of the formula $F = B(r)$ that $J \not\models F_I$ implies $J \not\models F^X$.

Base. We consider the case that F is a formula of rank 0.

If $X \not\models F$, we get $J \not\models F^X$ because $F^X = \bot$. Thus, we only have to consider the case $X \models F$:

$$F \text{ is a formula of rank 0 implies } F \text{ is an atom.}$$
$$F \text{ is an atom implies } F_I = F.$$
$$X \models F \text{ and } F \text{ is an atom implies } F^X = F.$$
$$F_I = F \text{ and } F^X = F \text{ implies } F_I = F^X.$$
$$J \not\models F_I \text{ and } F_I = F^X \text{ implies } J \not\models F^X.$$

Hypothesis. We assume that $J \not\models F_I$ implies $J \not\models F^X$ holds for formulas F of ranks smaller than i.

Step. We consider the case that F is a formula of rank i.

As in the base case, we only have to consider the case $X \models F$. Furthermore, we have to distinguish the cases that F is a conjunction, disjunction, or implication.

We first consider the case that F is a conjunction of form $\mathcal{F}^\wedge$:

$$\begin{aligned}
X \models F \text{ implies } F^X = \{G^X \mid G \in \mathcal{F}\}^\wedge.\\
J \not\models F_I \text{ and } F_I = \{G_I \mid G \in \mathcal{F}\}^\wedge \text{ implies } J \not\models G_I \text{ for some } G \in \mathcal{F}.\\
G \in \mathcal{F} \text{ and } F \text{ has rank } i \text{ implies } G \text{ has rank less than } i.\\
J \not\models G_I \text{ and } G \text{ has rank less than } i \text{ implies } J \not\models G^X \text{ by hypothesis.}\\
J \not\models G^X \text{ and } F^X = \{G^X \mid G \in F\}^\wedge \text{ implies } J \not\models F^X.
\end{aligned}$$

The case that F is a disjunction can be shown in a similar way to the case that F is a conjunction.

Last, we consider the case that F is an implication of form $G \to H$. Observe that G is positive because F has no occurrences of implications in its antecedent and, furthermore, given that F is a formula of rank i, H is a formula of rank less than i.

We show $I \models G$:

$$\begin{aligned}
J \not\models F_I \text{ and } F_I = G_{\overline{I}} \to H_I \text{ implies } J \models G_{\overline{I}}\\
J \models G_{\overline{I}} \text{ and } G \text{ is positive implies } I \models G \text{ because } G_{\overline{I}} \equiv \top.
\end{aligned}$$

We show $J \models G^X$:

$$\begin{aligned}
G \text{ is positive, } I \subseteq X, \text{ and } I \models G \text{ implies } X \models G \text{ by Lemma 42.}\\
X \models F,\ X \models G, \text{ and } F = G \to H \text{ implies } X \models H.\\
G \text{ is positive, } I \subseteq X, \text{ and } I \models G \text{ implies } I \models G^X \text{ by Lemma 43 (c).}\\
G \text{ is positive implies } G^X \text{ is positive by Lemma 43 (a).}\\
G^X \text{ is positive, } I \subseteq J, \text{ and } I \models G^X \text{ implies } J \models G^X \text{ by Lemma 42.}
\end{aligned}$$

We show $J \not\models H^X$:

$$\begin{aligned}
I \models G \text{ and } F_I = G_{\overline{I}} \to H_I \text{ implies } F_I \equiv H_I \text{ because } G_{\overline{I}} \equiv \top.\\
F_I \equiv H_I \text{ and } J \not\models F_I \text{ implies } J \not\models H_I.\\
J \not\models H_I \text{ and } H \text{ has rank less than } i \text{ implies } J \not\models H^X \text{ by hypothesis.}
\end{aligned}$$

Because $X \models F$, we have $F^X = G^X \to H^X$. Using $J \models G^X$ and $J \not\models H^X$, we get $J \not\models F^X$. □

Proof of Theorem 7. Let X be a stable model of P.

We prove by transfinite induction over the sequence of postfixed points leading to the well-founded model:

$$\begin{aligned}(I_0, J_0) &= (\emptyset, \Sigma),\\ (I_{\alpha+1}, J_{\alpha+1}) &= W_P(I_\alpha, J_\alpha) \text{ for ordinals } \alpha, \text{ and}\\ (I_\beta, J_\beta) &= (\bigcup_{\alpha<\beta} I_\alpha, \bigcap_{\alpha<\beta} J_\alpha) \text{ for limit ordinals } \beta.\end{aligned}$$

We have that $\alpha < \beta$ implies $(I_\alpha, J_\alpha) \leq_p (I_\beta, J_\beta)$ for ordinals α and β, $I_\alpha \subseteq J_\alpha$ for ordinals α, and there is a least ordinal α such that $(I, J) = (I_\alpha, J_\alpha)$.

Base. We have $I_0 \subseteq X \subseteq J_0$.

Hypothesis. We assume $I_\beta \subseteq X \subseteq J_\beta$ for all ordinals $\beta < \alpha$.

Step. If $\alpha = \beta + 1$ is a successor ordinal we have

$$\begin{aligned}(I_\alpha, J_\alpha) &= W_P(I_\beta, J_\beta)\\ &= (S_P(J_\beta), S_P(I_\beta)).\end{aligned}$$

By the induction hypothesis we have $I_\beta \subseteq X \subseteq J_\beta$.

First, we show $I_\alpha \subseteq X$:

$$\begin{aligned}X \text{ is a (stable) model implies } & S_P(X) \subseteq X.\\ X \subseteq J_\beta \text{ implies } & S_P(J_\beta) \subseteq S_P(X).\\ S_P(X) \subseteq X \text{ and } S_P(J_\beta) \subseteq S_P(X) \text{ implies } & S_P(J_\beta) \subseteq X.\\ I_\alpha = S_P(J_\beta) \text{ and } S_P(J_\beta) \subseteq X \text{ implies } & I_\alpha \subseteq X.\end{aligned}$$

Second, we show $X \subseteq J_\alpha$:

$$\begin{aligned}\beta < \alpha \text{ implies } & (I_\beta, J_\beta) \leq_p (I_\alpha, J_\alpha)\\ (I_\beta, J_\beta) \leq_p (I_\alpha, J_\alpha) \text{ and } I_\alpha \subseteq J_\alpha \text{ implies } & I_\beta \subseteq J_\alpha\\ X \text{ is a stable model, } I_\beta \subseteq X, &\\ J_\alpha = S_P(I_\beta), \text{ and } I_\beta \subseteq J_\alpha \quad \text{ implies } & X \subseteq J_\alpha \text{ by Lemma 47.}\end{aligned}$$

We have shown $I_\alpha \subseteq X \subseteq J_\alpha$ for successor ordinals.

If α is a limit ordinal we have

$$(I_\alpha, J_\alpha) = (\bigcup_{\beta<\alpha} I_\beta, \bigcap_{\beta<\alpha} J_\beta).$$

Let $x \in I_\alpha$. There must be an ordinal $\beta < \alpha$ such that $x \in I_\beta$. Since $I_\beta \subseteq X$ by the hypothesis, we have $x \in X$. Thus, $I_\alpha \subseteq X$.

Let $x \in X$. For each ordinal $\beta < \alpha$ we have $x \in J_\beta$ because $X \subseteq J_\beta$ by the hypothesis. Thus, we get $x \in J_\alpha$. It follows that $X \subseteq J_\alpha$.

We have shown $I_\alpha \subseteq X \subseteq J_\alpha$ for limit ordinals. □

Lemma 48. *Let P be an $\mathcal{F}$-program and (I, J) be a four-valued interpretation. Then, we have $H(P^{I,J}) = T_{P_I}(J)$.*

Proof. The program $P^{I,J}$ contains all rules $r \in P$ such that $J \models B(r)_I$. This are exactly the rules whose heads are gathered by the T operator. □

Lemma 49. *Let P be an $\mathcal{F}$-program and (I, J) be the well-founded model of P. Then, we have*

(a) $S_{P^{I,J}}(I') = J$ *for all* $I' \subseteq I$, *and*
(b) $S_{P^{I,J}}(J') = S_P(J')$ *for all* $J \subseteq J'$.

Proof. Throughout the proof we use

$$\begin{aligned} S_P(J) &= I, \\ S_P(I) &= J, \\ P^{I,J} &\subseteq P, \text{ and} \\ I &\subseteq J \text{ because the well-founded model is consistent.} \end{aligned}$$

Property (a). We show $J = S_{P^{I,J}}(I)$. Let $\hat{J} = S_{P^{I,J}}(I)$ and $r \in P \setminus P^{I,J}$:

$$\begin{aligned} P^{I,J} &\subseteq P \text{ and} \\ \hat{J} &= S_{P^{I,J}}(I) \text{ and} \\ J = S_P(I) &\text{ implies } \hat{J} \subseteq J \text{ by Lemma 46.} \\ r \notin P^{I,J} &\text{ implies } J \not\models B(r)_I. \\ \hat{J} \subseteq J \text{ and } J \not\models B(r)_I &\text{ implies } \hat{J} \not\models B(r)_I \text{ by Lemma 42.} \\ \hat{J} = S_{P^{I,J}}(I) \text{ and } \hat{J} \not\models B(r)_I &\text{ implies } \hat{J} \models P_I. \\ \hat{J} \models P_I \text{ and } J = S_P(I) &\text{ implies } J \subseteq \hat{J}. \\ J \subseteq \hat{J} \text{ and } \hat{J} \subseteq J &\text{ implies } J = \hat{J}. \end{aligned}$$

Thus, we get that $S_{P^{I,J}}(I) = J$.

With this we can continue to prove $S_{P^{I,J}}(I') = J$. Let $r \in P^{I,J}$:

$$\begin{aligned} r \in P^{I,J} &\text{ implies } J \models B(r)_I. \\ r \in P^{I,J} \text{ and } P^{I,J} \subseteq P &\text{ implies } r \in P. \\ J \models B(r)_I,\ r \in P, \text{ and } S_P(I) = J &\text{ implies } H(r) \in J. \\ J \models B(r)_I \text{ and } I' \subseteq I &\text{ implies } J \models B(r)_{I'} \text{ by Lemma 44 (d).} \\ H(r) \in J \text{ and } J \models B(r)_{I'} &\text{ implies } S_{P^{I,J}}(I') \subseteq J. \\ I' \subseteq I \text{ and } J = S_{P^{I,J}}(I) &\text{ implies } J \subseteq S_{P^{I,J}}(I'). \end{aligned}$$

Thus, we get $S_{P^{I,J}}(I') = J$.

Property (b). Let $I' = S_{P^{I,J}}(J')$ and $r \in P \setminus P^{I,J}$:

$$\begin{aligned} r \notin P^{I,J} &\text{ implies } J \not\models B(r)_I. \\ I \subseteq J,\ J \subseteq J', \text{ and } J \not\models B(r)_I &\text{ implies } J \not\models B(r)_{J'}. \text{ by Lemma 44 (d)..} \\ I' \subseteq I,\ I \subseteq J, \text{ and } J \not\models B(r)_{J'} &\text{ implies } I' \not\models B(r)_{J'}. \text{ by Lemma 42.} \\ I' = S_{P^{I,J}}(J') \text{ and } I' \not\models B(r)_{J'} &\text{ implies } S_P(J') \subseteq S_{P^{I,J}}(J'). \\ P^{I,J} \subseteq P &\text{ implies } S_{P^{I,J}}(J') \subseteq S_P(J') \text{ by Lemma 46.} \end{aligned}$$

Thus, we get $S_{P^{I,J}}(J') = S_P(J')$. ∎

Proof of Theorem 8. By Lemma 49, we have $(I, J) = W_{P^{I,J}}(I, J)$. Furthermore, we let $(\widehat{I}, \widehat{J}) = WM(P^{I,J})$:

$$(\widehat{I}, \widehat{J}) = WM(P^{I,J}) \text{ and } (I, J) = W_{P^{I,J}}(I, J) \text{ implies } (\widehat{I}, \widehat{J}) \leq_p (I, J). \text{ by Theorem 1 (c).}$$

$$\widehat{I} \subseteq I \text{ implies } S_{P^{I,J}}(\widehat{I}) = S_{P^{I,J}}(I) \text{ by Lemma 49 (a).}$$

$$\hat{J} = S_{P^{I,J}}(\widehat{I}) = S_{P^{I,J}}(I) = J \text{ implies } \widehat{J} = J.$$

$$\widehat{I} = S_{P^{I,J}}(\widehat{J}),\ S_{P^{I,J}}(J) = I, \text{ and } \widehat{J} = J \text{ implies } \widehat{I} = I.$$

We obtain $(I, J) = (\widehat{I}, \widehat{J})$. □

Proof of Theorem 9. We first show that all rule bodies removed by the simplification are falsified by X. Let $r \in P \setminus P^{I,J}$ and assume $X \models B(r)$:

$$X \models B(r) \text{ implies } X \models B(r)_X \text{ by Lemma 44 (b).}$$

$$X \models B(r)_X \text{ and } I \subseteq X \text{ implies } X \models B(r)_I \text{ by Lemma 44 (d).}$$

$$X \models B(r)_I \text{ and } X \subseteq J \text{ implies } J \models B(r)_I \text{ by Lemma 42.}$$

This is a contradiction and, thus, $X \not\models B(r)$. We use the following consequence in the proof below:

$$X \not\models B(r) \text{ implies } (P \setminus P^{I,J})^X \equiv \emptyset.$$

To show the theorem, we show that P^X and $(P^{I,J})^X$ have the same minimal models. Clearly, we have $P^X = (P^{I,J})^X \cup (P \setminus P^{I,J})^X$. Using this and $(P \setminus P^{I,J})^X \equiv \emptyset$, we obtain that P^X and $(P^{I,J})^X$ have the same minimal models. □

Proof of Corollary 10. The result follows from Theorems 7 to 9. □

Proof of Theorem 11. By Lemma 46, we have $S_{P^{I,J}}(X) \subseteq S_Q(X) \subseteq S_P(X)$ for any two-valued interpretation X. Thus, by Theorem 8, we get $(I, J) = W_Q(I, J)$.

Let $(\widehat{I}, \widehat{J})$ be a prefixed point of W_Q with $(\widehat{I}, \widehat{J}) \leq_p (I, J)$. We have $(S_Q(\widehat{J}), S_Q(\widehat{I})) \leq_p (\widehat{I}, \widehat{J}) \leq_p (I, J)$.

$$J \subseteq \widehat{J} \text{ implies } S_{P^{I,J}}(\widehat{J}) = S_Q(\widehat{J}) = S_P(\widehat{J}) \text{ by Lemma 49 (b).}$$

$$S_Q(\widehat{J}) = S_P(\widehat{J}) \text{ and } S_Q(\widehat{J}) \subseteq \widehat{I} \text{ implies } S_P(\widehat{J}) \subseteq \widehat{I}.$$

$$\widehat{J} \subseteq S_Q(\widehat{I}) \text{ and } S_Q(\widehat{I}) \subseteq S_P(\widehat{I}) \text{ implies } \widehat{J} \subseteq S_P(\widehat{I}).$$

$$S_P(\widehat{J}) \subseteq \widehat{I} \text{ and } \widehat{J} \subseteq S_P(\widehat{I}) \text{ implies } W_P(\widehat{I}, \widehat{J}) \leq_p (\widehat{I}, \widehat{J}).$$

$$W_P(\widehat{I}, \widehat{J}) \leq_p (\widehat{I}, \widehat{J}) \text{ implies } (I, J) \leq_p (\widehat{I}, \widehat{J}) \text{ by Theorem 1 (a).}$$

$$(I, J) \leq_p (\widehat{I}, \widehat{J}) \text{ and } (\widehat{I}, \widehat{J}) \leq_p (I, J) \text{ implies } (I, J) = (\widehat{I}, \widehat{J}).$$

By Theorem 1 (a), we obtain that $WM(Q) = (I, J)$. □

Proof of Corollary 12. Observe that $P^{I,J} = Q^{I,J}$. With this, the corollary follows from Corollary 10 and Theorem 11. □

Alternatively, the incorporation of context atoms can also be seen as a form of partial evaluation applied to the underlying program.

Definition 22. Let IC be a two-valued interpretation.

We define the *partial evaluation* of an $\mathcal{F}$-formula w.r.t. IC as follows:

$$\begin{aligned}
pe_{IC}(a) &= \top \text{ if } a \in IC & pe_{\overline{IC}}(a) &= a \\
pe_{IC}(a) &= a \text{ if } a \notin IC & & \\
pe_{IC}(\mathcal{H}^\wedge) &= \{pe_{IC}(F) \mid F \in \mathcal{H}\}^\wedge & pe_{\overline{IC}}(\mathcal{H}^\wedge) &= \{pe_{\overline{IC}}(F) \mid F \in \mathcal{H}\}^\wedge \\
pe_{IC}(\mathcal{H}^\vee) &= \{pe_{IC}(F) \mid F \in \mathcal{H}\}^\vee & pe_{\overline{IC}}(\mathcal{H}^\vee) &= \{pe_{\overline{IC}}(F) \mid F \in \mathcal{H}\}^\vee \\
pe_{IC}(F \to G) &= pe_{\overline{IC}}(F) \to pe_{IC}(G) & pe_{\overline{IC}}(F \to G) &= pe_{IC}(F) \to pe_{\overline{IC}}(G)
\end{aligned}$$

where a is an atom, $\mathcal{H}$ a set of formulas, and F and G are formulas.

The partial evaluation of an $\mathcal{F}$-program P w.r.t. a two-valued interpretation IC is $pe_{IC}(P) = \{pe_{IC}(r) \mid r \in P\}$ where $pe_{IC}(r) = H(h) \leftarrow pe_{IC}(B(r))$. Accordingly, the partial evaluation of rules boils down to replacing satisfied positive occurrences of atoms in rule bodies by $\top$.

We observe the following relationship between the relative one-step operators and partial evaluations.

Observation 50. *Let P be a positive $\mathcal{F}$-program and IC be a two-valued interpretation. Then, we have for any two-valued interpretation I that*

$$T_P^{IC}(I) = T_{pe_{IC}(P)}(I).$$

Note that $pe_{IC}(P)_J = pe_{IC}(P_J)$.

Proof of Proposition 13. Clearly, $pe_{IC}(P)$ is positive whenever P is positive. Using Observation 50, we obtain that T_P^{IC} is monotone.

The second property directly follows from the monotonicity of the one-step provability operator. □

Lemma 51. *Let P be an $\mathcal{F}$-program and IC be a two-valued interpretation. For any two-valued interpretation J, we get*

$$S_P^{IC}(J) = LM(pe_{IC}(P)_J).$$

Proof. This lemma immediately follows from Observation 50. □

Proof of Proposition 14. Both properties can be shown by inspecting the reduced programs.

Property $J' \subseteq J$ implies $S_P^{IC}(J) \subseteq S_P^{IC}(J')$. Observe that we can use Lemma 51 to equivalently write $S_P^{IC}(J) = S_{pe_{IC}(P)}(J)$ and $S_P^{IC}(J') = S_{pe_{IC}(P)}(J')$. With this and Proposition 4, we see that the relative stable operator is antimonotone just as the stable operator.

Property $IC' \subseteq IC$ implies $S_P^{IC'}(J) \subseteq S_P^{IC}(J)$. Observe that $S_P^{IC}(J)$ is equal to the least fixed point of $T_{P_J}^{IC}$ and $S_P^{IC'}(J)$ is equal to the least fixed point of $T_{P_J}^{IC'}$. Furthermore, observe that $T_{P_J}^{IC'}(X) \subseteq T_{P_J}^{IC}(X)$ for any two-valued interpretation X because $IC' \subseteq IC$ and the underlying T operator is monotone. With this and Lemma 45, we have shown the property. □

Observation 52. *Let P be an $\mathcal{F}$-program, and IC and J be two-valued interpretations. We get the following properties:*

(a) $S_P^{\emptyset}(J) = S_P(J)$,
(b) $S_P^{IC}(J) \subseteq H(P)$, *and*
(c) $S_P^{IC}(J) = S_P^{IC \cap B(P)^+}(J \cap B(P)^-)$.

Proof of Proposition 15. Both properties can be shown by using the monotonicity of the underlying relative stable operator:

Property $(I', J') \leq_p (I, J)$ implies $W_P^{IC,JC}(I', J') \leq_p W_P^{IC,JC}(I, J)$. Given that S_P^{IC} is antimonotone and $J' \cup JC \subseteq J \cup JC$, we have $S_P^{IC}(J \cup JC) \subseteq S_P^{IC}(J' \cup JC)$. Analogously, we can show $S_P^{JC}(I' \cup IC) \subseteq S_P^{JC}(I \cup IC)$. We get $(S_P^{IC}(J \cup JC), S_P^{JC}(I \cup IC)) \leq_p (S_P^{IC}(J' \cup JC), S_P^{JC}(I' \cup IC))$.

Hence, $W_P^{IC,JC}$ is monotone.

Property $(IC', JC') \leq_p (IC, JC)$ implies $W_P^{IC',JC'}(I, J) \leq_p W_P^{IC,JC}(I, J)$. We have to show $(S_P^{IC'}(J \cup JC'), S_P^{JC'}(I \cup IC')) \leq_p (S_P^{IC}(J \cup JC), S_P^{JC}(I \cup IC))$.

Given that $IC' \subseteq IC$ and $J \cup JC \subseteq J \cup JC'$, we obtain $S_P^{IC'}(J \cup JC') \subseteq S_P^{IC}(J \cup JC)$ using Proposition 14. The same argument can be used for the possible atoms of the four-valued interpretations. Given that $JC \subseteq JC'$ and $I \cup IC' \subseteq I \cup IC$, we obtain $S_P^{JC}(I \cup IC) \subseteq S_P^{JC'}(I \cup IC')$ using Proposition 14.

Hence, we have shown $W_P^{IC',JC'}(I, J) \leq_p W_P^{IC,JC}(I, J)$. □

Observation 53. *Let P be an $\mathcal{F}$-program, and I, I' and IC be two-valued interpretations. We get the following properties:*

(a) *$I \models P$ and $IC \subseteq I$ implies $I \models pe_{IC}(P)$,*
(b) *$I \models pe_{IC}(P)$ and $I' \cap B(P)^+ \subseteq IC$ implies $I \cup I' \models pe_{IC}(P)$, and*
(c) *$I \models pe_{IC}(P)$ implies $I \models P$.*

Lemma 54. *Let PB and PT be $\mathcal{F}$-programs, IC and J be two-valued interpretations, $I = S_{PB \cup PT}^{IC}(J)$, $IE = I \cap (B(PB)^+ \cap H(PT))$, $IB = S_{PB}^{IC \cup IE}(J)$, and $IT = S_{PT}^{IC \cup IB}(J)$. Then, we have $I = IB \cup IT$.*

Proof. Let $\widetilde{I} = IB \cup IT$. Furthermore, we use the following programs:

$$\widehat{PB} = pe_{IC}(PB_J) \qquad \widetilde{PB} = pe_{IC \cup IE}(PB_J) = pe_{IE}(\widehat{PB})$$
$$\widehat{PT} = pe_{IC}(PT_J) \qquad \widetilde{PT} = pe_{IC \cup IB}(PT_J) = pe_{IB}(\widehat{PT})$$

Observe that

$$I = S_{PB \cup PT}^{IC}(J) = LM(\widehat{PB} \cup \widehat{PT}),$$
$$IB = S_{PB}^{IC \cup IE}(J) = LM(\widetilde{PB}), \text{ and}$$
$$IT = S_{PT}^{IC \cup IB}(J) = LM(\widetilde{PT}).$$

To show that $\widetilde{I} \subseteq I$, we show that I is a model of both $\widetilde{PB}$ and $\widetilde{PT}$. To show that $I \subseteq \widetilde{I}$, we show that $\widetilde{I}$ is a model of both $\widehat{PB}$ and $\widehat{PT}$.

Property $I \models \widetilde{PB}$.

$$
\begin{aligned}
I = LM(\widehat{PB} \cup \widehat{PT}) &\text{ implies } I \models \widehat{PB}.\\
I \models \widehat{PB} \text{ and } IE \subseteq I &\text{ implies } I \models \widetilde{PB}\\
&\text{ by Observation 53 (a).}
\end{aligned}
$$

Property $I \models \widetilde{PT}$.

$$
\begin{aligned}
I = LM(\widehat{PB} \cup \widehat{PT}) &\text{ implies } I \models \widehat{PT}.\\
I \models \widetilde{PB} \text{ and } IB = LM(\widetilde{PB}) &\text{ implies } IB \subseteq I.\\
I \models \widehat{PT} \text{ and } IB \subseteq I &\text{ implies } I \models \widetilde{PT}.\\
&\text{ by Observation 53 (a).}
\end{aligned}
$$

Property $\widetilde{I} \models \widehat{PB}$. Let $E = B(PB)^+ \cap H(PT)$:

$$
\begin{aligned}
\widetilde{I} \subseteq I \text{ and } IE = I \cap E &\text{ implies } IT \cap E \subseteq IE.\\
IT = LM(\widetilde{PT}) &\text{ implies } IT \subseteq H(\widetilde{PT}).\\
IT \cap E \subseteq IE \text{ and } IT \subseteq H(\widetilde{PT}) &\text{ implies } IT \cap B(PB)^+ \subseteq IE.\\
IT \cap B(PB)^+ \subseteq IE &\text{ implies } IT \cap B(\widehat{PB})^+ \subseteq IE.\\
IT \cap B(\widehat{PB})^+ \subseteq IE \text{ and } IB = LM(\widetilde{PB}) &\text{ implies } \widetilde{I} \models \widetilde{PB}\\
&\text{ by Observation 53 (b).}\\
\widetilde{I} \models \widetilde{PB} &\text{ implies } \widetilde{I} \models \widehat{PB}\\
&\text{ by Observation 53 (c).}
\end{aligned}
$$

Property $\widetilde{I} \models \widehat{PT}$.

$$
\begin{aligned}
IT = LM(\widetilde{PT}) &\text{ implies } \widetilde{I} \models \widetilde{PT}\\
&\text{ by Observation 53 (b).}\\
\widetilde{I} \models \widetilde{PT} &\text{ implies } \widetilde{I} \models \widehat{PT}\\
&\text{ by Observation 53 (c).}
\end{aligned}
$$

□

Proof of Theorem 16. Let $P = PB \cup PT$ and $E = B(PB)^{\pm} \cap H(PT)$. We begin by evaluating P, PB and PT w.r.t. (I, J) and obtain

$$
\begin{aligned}
(I, J) &= W_P^{IC,JC}(I, J)\\
&= (S_P^{IC}(JC \cup J), S_P^{JC}(IC \cup I)),\\
(\widehat{IB}, \widehat{JB}) &= W_{PB}^{(IC,JC)\sqcup(IE,JE)}(I, J)\\
&= (S_{PB}^{IC \cup IE}(JC \cup JE \cup J), S_{PB}^{JC \cup JE}(IC \cup IE \cup I)), \text{ and}\\
(\widehat{IT}, \widehat{JT}) &= W_{PT}^{(IC,JC)\sqcup(\widehat{IB},\widehat{JB})}(I, J)\\
&= (S_{PT}^{IC \cup \widehat{IB}}(JC \cup \widehat{JB} \cup J), S_{PT}^{JC \cup \widehat{JB}}(IC \cup \widehat{IB} \cup I)).
\end{aligned}
$$

Using $(IE, JE) \sqsubseteq (I, J)$, we get

$$(\widehat{IB}, \widehat{JB}) = (S_{PB}^{IC \cup IE}(JC \cup J), S_{PB}^{JC \cup JE}(IC \cup I)).$$

By Lemma 54 and Observation 52 (c), we get

$$\begin{aligned}
(\widehat{IB}, \widehat{JB}) &\sqsubseteq (I, J),\\
(\widehat{IT}, \widehat{JT}) &= (S_{PT}^{IC \cup IE}(JC \cup J), S_{PT}^{JC \cup JE}(IC \cup I)), \text{ and}\\
(I, J) &= (\widehat{IB}, \widehat{JB}) \sqcup (\widehat{IT}, \widehat{JT}).
\end{aligned}$$

We first show $(IB, JB) = (\widehat{IB}, \widehat{JB})$ and then $(IT, JT) = (\widehat{IT}, \widehat{JT})$.

Property $(IB, JB) \leq_p (\widehat{IB}, \widehat{JB})$.

$$\begin{aligned}
&(\widehat{IB}, \widehat{JB}) \sqsubseteq (I, J) \text{ and}\\
&(IE, JE) = (I, J) \sqcap E \text{ implies } (\widehat{IB}, \widehat{JB}) \sqcup (IE, JE) \sqsubseteq (I, J).\\
&(I, J) = (\widehat{IB}, \widehat{JB}) \sqcup (\widehat{IT}, \widehat{JT}) \text{ implies } (\widehat{IT}, \widehat{JT}) \sqsubseteq (I, J).\\
&(\widehat{IT}, \widehat{JT}) \sqsubseteq H(PT) \text{ implies } (\widehat{IT}, \widehat{JT}) \sqcap B(PB)^{\pm} \sqsubseteq (\widehat{IT}, \widehat{JT}) \sqcap E.\\
&(\widehat{IT}, \widehat{JT}) \sqcap B(PB)^{\pm} \sqsubseteq (\widehat{IT}, \widehat{JT}) \sqcap E \text{ and}\\
&(\widehat{IT}, \widehat{JT}) \sqsubseteq (I, J) \text{ implies } (\widehat{IT}, \widehat{JT}) \sqcap B(PB)^{\pm} \sqsubseteq (IE, JE).\\
&(\widehat{IT}, \widehat{JT}) \sqcap B(PB)^{\pm} \sqsubseteq (IE, JE) \text{ and}\\
&(I, J) = (\widehat{IB}, \widehat{JB}) \sqcup (\widehat{IT}, \widehat{JT}) \text{ implies } (I, J) \sqcap B(PB)^{\pm} \sqsubseteq (\widehat{IB}, \widehat{JB}) \sqcup (IE, JE).
\end{aligned}$$

With the above, we use Observation 52 (c) to show that $(\widehat{IB}, \widehat{JB})$ is a fixed point of $W_{PB}^{(IC,JC) \sqcup (JC,JE)}$:

$$\begin{aligned}
(\widehat{IB}, \widehat{JB}) &= W_{PB}^{(IC,JC) \sqcup (IE,JE)}(I, J)\\
&= W_{PB}^{(IC,JC) \sqcup (IE,JE)}((I, J) \sqcap B(PB)^{\pm})\\
&= W_{PB}^{(IC,JC) \sqcup (IE,JE)}((\widehat{IB}, \widehat{JB}) \sqcup (IE, JE))\\
&= W_{PB}^{(IC,JC) \sqcup (IE,JE)}(\widehat{IB}, \widehat{JB})
\end{aligned}$$

Thus, by Theorem 1 (c), $(IB, JB) \leq_p (\widehat{IB}, \widehat{JB})$.

Property $(IB, JB) = (\widehat{IB}, \widehat{JB})$. To show the property, let

$$\begin{aligned}
(\widetilde{I}, \widetilde{J}) &= (IB, JB) \sqcup (IE, JE) \sqcup (\widehat{IT}, \widehat{JT}),\\
(\widetilde{IE}, \widetilde{JE}) &= W_P^{IC,JC}(\widetilde{I}, \widetilde{J}) \sqcap E,\\
(\widetilde{IB}, \widetilde{JB}) &= (S_{PB}^{IC \cup \widetilde{IE}}(JC \cup \widetilde{J}), S_{PB}^{JC \cup \widetilde{JE}}(IC \cup \widetilde{I})),\\
(\widetilde{IT}, \widetilde{JT}) &= (S_{PT}^{IC \cup \widetilde{IB}}(JC \cup \widetilde{J}), S_{PT}^{JC \cup \widetilde{JB}}(IC \cup \widetilde{I})), \text{ and}\\
W_P^{IC,JC}(\widetilde{I}, \widetilde{J}) &= (\widetilde{IB}, \widetilde{JB}) \sqcup (\widetilde{IT}, \widetilde{JT}) \text{ by Lemma 54.}
\end{aligned}$$

We get:

$$\begin{aligned}
&(IB, JB) \leq_p (\widehat{IB}, \widehat{JB}) \text{ implies } (\widetilde{I}, \widetilde{J}) \leq_p (I, J).\\
&(\widetilde{I}, \widetilde{J}) \leq_p (I, J) \text{ implies } W_P^{IC,JC}(\widetilde{I}, \widetilde{J}) \leq_p (I, J).
\end{aligned}$$

$$
\begin{aligned}
W_P^{IC,JC}(\widetilde{I},\widetilde{J}) \leq_p (I,J) &\text{ implies } (\widetilde{IE},\widetilde{JE}) \leq_p (IE,JE).\\
(\widehat{IT},\widehat{JT}) \sqcap B(PB)^{\pm} \sqsubseteq (IE,JE) &\text{ implies } (\widetilde{I},\widetilde{J}) \sqcap B(PB)^{\pm} \sqsubseteq (\widehat{IB},\widehat{JB}) \sqcup (IE,JE).\\
(\widetilde{I},\widetilde{J}) \sqcap B(PB)^{\pm} \sqsubseteq (\widehat{IB},\widehat{JB}) \sqcup (IE,JE) &\text{ implies } \widetilde{IB} = S_{PB}^{IC\cup\widetilde{IE}}(JC \cup JE \cup JB)\\
&\text{ and } \quad \widetilde{JB} = S_{PB}^{JC\cup\widetilde{JE}}(IC \cup IE \cup IB).\\
&\text{ by Observation 52 (c).}\\
\widetilde{IB} = S_{PB}^{IC\cup\widetilde{IE}}(JC \cup JE \cup JB) &\text{ and}\\
\widetilde{JB} = S_{PB}^{JC\cup\widetilde{JE}}(IC \cup IE \cup IB) &\text{ and}\\
IB = S_{PB}^{IC\cup IE}(JC \cup JE \cup JB) &\text{ and}\\
JB = S_{PB}^{JC\cup JE}(IC \cup IE \cup IB) &\text{ and}\\
(\widetilde{IE},\widetilde{JE}) \leq_p (IE,JE) &\text{ implies } (\widetilde{IB},\widetilde{JB}) \leq_p (IB,JB)\\
&\text{ by Proposition 15.}\\
(\widetilde{IB},\widetilde{JB}) \leq_p (IB,JB) &\text{ and}\\
(IB,JB) \leq_p (\widehat{IB},\widehat{JB}) &\text{ implies } (\widetilde{IB},\widetilde{JB}) \leq_p (\widehat{IB},\widehat{JB}).\\
\widehat{IT} = S_{PT}^{IC\cup\widehat{IB}}(JC \cup J) &\text{ and}\\
\widehat{JT} = S_{PT}^{JC\cup\widehat{JB}}(IC \cup I) &\text{ and}\\
\widetilde{IT} = S_{PT}^{IC\cup\widetilde{IB}}(JC \cup \widetilde{J}) &\text{ and}\\
\widetilde{JT} = S_{PT}^{JC\cup\widetilde{JB}}(IC \cup \widetilde{I}) &\text{ and}\\
(\widetilde{IB},\widetilde{JB}) \leq_p (\widehat{IB},\widehat{JB}) &\text{ and}\\
(\widetilde{I},\widetilde{J}) \leq_p (I,J) &\text{ implies } (\widetilde{IT},\widetilde{JT}) \leq_p (\widehat{IT},\widehat{JT})\\
&\text{ by Proposition 14.}\\
W_P^{IC,JC}(\widetilde{I},\widetilde{J}) = (\widetilde{IB},\widetilde{JB}) \sqcup (\widetilde{IT},\widetilde{JT}) &\text{ and}\\
(\widetilde{IB},\widetilde{JB}) \leq_p (IB,JB) &\text{ and}\\
(\widetilde{IT},\widetilde{JT}) \leq_p (\widehat{IT},\widehat{JT}) &\text{ implies } W_P^{IC,JC}(\widetilde{I},\widetilde{J}) \leq_p (IB,JB) \sqcup (\widehat{IT},\widehat{JT}).\\
(\widetilde{IE},\widetilde{JE}) \leq_p (IE,JE) &\text{ and}\\
(\widetilde{IE},\widetilde{JE}) \sqsubseteq W_P^{IC,JC}(\widetilde{I},\widetilde{J}) &\text{ and}\\
W_P^{IC,JC}(\widetilde{I},\widetilde{J}) \leq_p (IB,JB) \sqcup (\widehat{IT},\widehat{JT}) &\text{ and}\\
(\widetilde{I},\widetilde{J}) = (IB,JB) \sqcup (IE,JE) \sqcup (\widehat{IT},\widehat{JT}) &\text{ implies } W_P^{IC,JC}(\widetilde{I},\widetilde{J}) \leq_p (\widetilde{I},\widetilde{J}).\\
WM^{IC,JC}(P) = (I,J) &\text{ and}\\
W_P^{IC,JC}(\widetilde{I},\widetilde{J}) \leq_p (\widetilde{I},\widetilde{J}) &\text{ implies } (I,J) \leq_p (\widetilde{I},\widetilde{J})\\
&\text{ by Theorem 1 (a).}\\
(\widetilde{I},\widetilde{J}) \leq_p (I,J) \text{ and } (I,J) \leq_p (\widetilde{I},\widetilde{J}) &\text{ implies } (I,J) = (\widetilde{I},\widetilde{J}).\\
(\widetilde{I},\widetilde{J}) = (IB,JB) \sqcup (IE,JE) \sqcup (\widehat{IT},\widehat{JT}) &\text{ implies } (IB,JB) \sqcup (IE,JE) \sqsubseteq (\widetilde{I},\widetilde{J}).\\
(\widetilde{I},\widetilde{J}) = (IB,JB) \sqcup (IE,JE) \sqcup (\widehat{IT},\widehat{JT}) &\text{ and}
\end{aligned}
$$

$$\begin{aligned}
(\widehat{IT},\widehat{JT}) \sqcap B(PB)^{\pm} \sqsubseteq (IE,JE) &\text{ implies } (\widetilde{I},\widetilde{J}) \sqcap B(PB)^{\pm} \sqsubseteq (IB,JB) \sqcup (IE,JE).\\
(I,J) = (\widetilde{I},\widetilde{J}) &\text{ and}\\
(IB,JB) \sqcup (IE,JE) \sqsubseteq (\widetilde{I},\widetilde{J}) &\text{ and}\\
(\widetilde{I},\widetilde{J}) \sqcap B(PB)^{\pm} \sqsubseteq (IB,JB) \sqcup (IE,JE) &\text{ and}\\
(IB,JB) = W_{PB}^{(IC,JC)\sqcup(IE,JE)}(IB,JB) &\text{ and}\\
(\widehat{IB},\widehat{JB}) = W_{PB}^{(IC,JC)\sqcup(IE,JE)}(I,J) &\text{ implies } (IB,JB) = (\widehat{IB},\widehat{JB})\\
&\text{by Observation 52 (c).}
\end{aligned}$$

Property $(\widehat{IT},\widehat{JT}) = (IT,JT)$. Observe that the lemma can be applied with PB and PT exchanged. Let

$$\begin{aligned}
\widetilde{E} &= B(PT)^{\pm} \cap H(PB),\\
(\widetilde{IE},\widetilde{JE}) &= (I,J) \cap \widetilde{E},\\
(\widetilde{IT},\widetilde{JT}) &= WM^{(IC,JC)\sqcup(\widetilde{IE},\widetilde{JE})}(PT), \text{ and}\\
(\widetilde{IB},\widetilde{JB}) &= W_{PB}^{(IC,JC)\sqcup(\widetilde{IT},\widetilde{JT})}(I,J).
\end{aligned}$$

Using the properties shown so far, we obtain

$$(I,J) = (\widetilde{IT},\widetilde{JT}) \sqcup (\widetilde{IB},\widetilde{JB}).$$

With this we get:

$$\begin{aligned}
(IB,JB) = (\widehat{IB},\widehat{JB}) &\text{ and}\\
(I,J) = (\widehat{IB},\widehat{JB}) \sqcup (\widehat{IT},\widehat{JT}) &\text{ and}\\
(IB,JB) \sqsubseteq H(PB) &\text{ and}\\
(\widetilde{IE},\widetilde{JE}) = (I,J) \sqcap \widetilde{E} &\text{ implies } (IB,JB) \sqcap B(PT)^{\pm} \sqsubseteq (\widetilde{IE},\widetilde{JE}).\\
(I,J) = (\widetilde{IT},\widetilde{JT}) \sqcup (\widetilde{IB},\widetilde{JB}) &\text{ and}\\
(\widetilde{IB},\widetilde{JB}) \sqsubseteq H(PB) &\text{ and}\\
(\widetilde{IE},\widetilde{JE}) = (I,J) \sqcap \widetilde{E} &\text{ implies } (\widetilde{IB},\widetilde{JB}) \sqcap B(PT)^{\pm} \sqsubseteq (\widetilde{IE},\widetilde{JE}).\\
(\widetilde{IB},\widetilde{JB}) \sqcap B(PT)^{\pm} \sqsubseteq (\widetilde{IE},\widetilde{JE}) &\text{ and}\\
(I,J) = (\widetilde{IT},\widetilde{JT}) \sqcup (\widetilde{IB},\widetilde{JB}) &\text{ implies } (I,J) \sqcap B(PB)^{\pm} \sqsubseteq (\widetilde{IT},\widetilde{JT}) \sqcup (\widetilde{IE},\widetilde{JE}).\\
(IB,JB) \sqcap B(PT)^{\pm} \sqsubseteq (\widetilde{IE},\widetilde{JE}) &\text{ and}\\
(I,J) \sqcap B(PB)^{\pm} \sqsubseteq (\widetilde{IT},\widetilde{JT}) \sqcup (\widetilde{IE},\widetilde{JE}) &\text{ and}\\
(\widetilde{IT},\widetilde{JT}) = W_{PT}^{(IC,JC)\sqcup(\widetilde{IE},\widetilde{JE})}(\widetilde{IT},\widetilde{JT}) &\text{ and}\\
(\widehat{IT},\widehat{JT}) = W_{PT}^{(IC,JC)\sqcup(IB,JB)}(I,J) &\text{ implies } (\widetilde{IT},\widetilde{JT}) = (\widehat{IT},\widehat{JT})\\
&\text{by Observation 52 (c).}\\
(IB,JB) \sqcap B(PT)^{\pm} \sqsubseteq (\widetilde{IE},\widetilde{JE}) &\text{ and}\\
(\widetilde{IT},\widetilde{JT}) = WM^{(IC,JC)\sqcup(\widetilde{IE},\widetilde{JE})}(PT) &\text{ and}\\
(IT,JT) = WM^{(IC,JC)\sqcup(IB,JB)}(PT) &\text{ implies } (\widetilde{IT},\widetilde{JT}) = (IT,JT)
\end{aligned}$$

by Observation 52 (c).

Thus, we get $(\widehat{IT}, \widehat{JT}) = (IT, JT)$. □

Proof of Theorem 17. The theorem can be shown by transfinite induction over the sequence indices. We do not give the full induction proof here but focus on the key idea. Let (I'_i, J'_i) be the intermediate interpretations as in (5) when computing the well-founded model of the sequence. Furthermore, let

$$(I_i, J_i) = WM^{(IC_i, JC_i) \sqcup (IE_i, JE_i)}(P_i)$$

be the intermediate interpretations where (IC_i, JC_i) is the union of the intermediate interpretations as in (4) and

$$(IE_i, JE_i) = (I, J) \cap E_i$$

with E_i as in (3).

Observe that with Theorem 16, we have $WM(\bigcup_{i \in \mathbb{I}} P_i) = \bigcup_{i \in \mathbb{I}} (I_i, J_i)$. By Proposition 15, we have $(I'_i, J'_i) \leq_p (I_i, J_i)$ and, thus, we obtain that $WM((P_i)_{i \in \mathbb{I}}) \leq_p WM(\bigcup_{i \in \mathbb{I}} P_i)$. □

Proof of Theorem 18. By Theorem 17, we have

$$\bigsqcup_{i \in \mathbb{I}} (I_i, J_i) \leq_p (I, J).$$

We get $\bigcup_{i \in \mathbb{I}} I_i \subseteq I$ and, thus,

$$\bigcup_{i \leq k} I_i = IC_k \cup I_k \subseteq I.$$

Using $J \subseteq \bigcup_{i \in \mathbb{I}} J_i$ and $J_i \subseteq H(P_i)$, we get

$$\begin{aligned} J \cap B(P_k)^{\pm} &\subseteq (\bigcup_{i \leq k} J_i \cup \bigcup_{k < i} H(P_i)) \cap B(P_k)^{\pm} \\ &\subseteq (\bigcup_{i \leq k} J_i \cup E_k) \cap B(P_k)^{\pm} \\ &\subseteq (JC_k \cup J_k \cup E_k) \cap B(P_k)^{\pm}. \end{aligned}$$

Using both results, we obtain

$$((IC_k, JC_k) \sqcup (\emptyset, E_k) \sqcup (I_k, J_k)) \sqcap B(P_k)^{\pm} \leq_p (I, J) \sqcap B(P_k)^{\pm}.$$

Because the body literals determine the simplification, we get

$$P_k^{I,J} \subseteq P_k^{(IC_k, JC_k) \sqcup (\emptyset, E_k) \sqcup (I_k, J_k)}.$$ □

Lemma 55. *Let $(P_i)_{i \in \mathbb{I}}$ be a sequence of $\mathcal{R}$-programs, and (I, J) be the well-founded model of $\bigcup_{i \in \mathbb{I}} P_i$.*

Then, $\bigcup_{i \in \mathbb{I}} P_i$ and $\bigcup_{i \in \mathbb{I}} Q_i$ with $P_i^{I,J} \subseteq Q_i \subseteq P_i$ have the same well-founded and stable models.

Proof. This lemma is a direct consequence of Theorems 11 and 18 and Corollary 12. □

Proof of Corollary 19. This corollary is a direct consequence of Theorem 18 and Lemma 55. □

Proof of Corollary 20. This can be proven in the same way as Theorem 17 but note that because E_i is empty, we get $(I_i', J_i') = (I_i, J_i)$. □

Proof of Corollary 21. This can be proven in the same way as Theorem 18 but note that because E_i is empty, all $\leq_p$ and most $\subseteq$ relations can be replaced with equivalences. □

Whenever head atoms do not interfere with negative body literals, the relative well founded-model of a program can be calculated with just two applications of the relative stable operator.

Lemma 56. *Let P be an $\mathcal{F}$-program such that $B(P)^- \cap H(P) = \emptyset$ and (IC, JC) be a four-valued interpretation.*

Then, $WM^{IC,JC}(P) = (S_P^{IC}(JC), S_P^{JC}(IC))$.

Proof. Let $(I, J) = WM^{IC,JC}(P)$.

We have $J = S_P^{JC}(IC \cup I)$. By Observation 52 (b), we get $J \subseteq H(P)$. With this and $B(P)^- \cap H(P) = \emptyset$, we get $B(P)^- \cap J = \emptyset$. Thus, $S_P^{IC}(JC \cup J) = S_P^{IC}(JC)$ by Observation 52 (c).

The same arguments apply to show $S_P^{JC}(IC \cup I) = S_P^{JC}(IC)$. □

Any sequence as in Corollary 20 in which each P_i additionally satisfies the precondition of Lemma 56 has a total well-founded model. Furthermore, the well-founded model of such a sequence can be calculated with just two (independent) applications of the relative stable operator per program P_i in the sequence.

Proof of Proposition 23. We use Lemma 41 to show that both formulas are strongly equivalent.

Property $I \models \pi(a)^J$ implies $I \models \tau(a)^J$ for arbitrary interpretations I. The formulas $\pi(a)$ and $\tau(a)$ only differ in the consequents of their implications. Observe that the consequents in $\pi(a)$ are stronger than the ones in $\tau(a)$. Thus, it follows that $\pi(a)$ is stronger than $\tau(a)$. Furthermore, observe that the same holds for their reducts.

Property $I \not\models \pi(a)^J$ implies $I \not\models \tau(a)^J$ for arbitrary interpretations I. Let G be the set of all instance of the aggregate elements of a. Because $I \not\models \pi(a)^J$, there must be a set $D \subseteq G$ such that $D \not\triangleright a$, $I \models (\tau(D)^\wedge)^J$, and $I \not\models (\pi_a(D)^\vee)^J$. With this, we construct the set

$$\widehat{D} = D \cup \{e \in G \setminus D \mid I \models \tau(e)^J\}.$$

The construction of $\widehat{D}$ and

$$D \not\succcurlyeq a \text{ and}$$

$$I \not\models (\pi_a(D)^\vee)^J \text{ implies } \widehat{D} \not\succcurlyeq a \text{ and } I \not\models (\tau_a(\widehat{D})^\vee)^J.$$

The construction of $\widehat{D}$ and

$$I \models (\tau(D)^\wedge)^J \text{ implies } I \models (\tau(\widehat{D})^\wedge)^J.$$

$$\widehat{D} \not\succcurlyeq a \text{ and}$$

$$I \not\models (\tau_a(\widehat{D})^\vee)^J \text{ and}$$

$$I \models (\tau(\widehat{D})^\wedge)^J \text{ implies } I \not\models \tau(a)^J.$$

□

Proof of Proposition 24. Let G be the set of ground instances of the aggregate elements of a. Furthermore, observe that a monotone aggregate a is either constantly true or not justified by the empty set.

In case that $\pi(a) \equiv \top$, we get $\pi(a)_I \equiv \top$ and the lemma holds.

Next, we consider the case that the empty set does not justify the aggregate. Observe that $\pi_a(\emptyset)$ is stronger than $\pi_a(D)$ for any $D \subseteq G$. And, we have that $\pi(a)$ contains the implication $\top \rightarrow \pi_a(\emptyset)$. Because of this, we have $\pi(a) \equiv \pi_a(\emptyset)$. Furthermore, all consequents in $\pi(a)$ are positive formulas and, thus, not modified by the reduct. Thus, the reduct $(\top \rightarrow \pi_a(\emptyset))_I$ is equal to $\top \rightarrow \pi_a(\emptyset)$. And as before, it is stronger than all other implications in $\pi(a)_I$. Hence, we get $\pi(a)_I \equiv \pi_a(\emptyset)$. □

Proof of Theorem 25. Remember that the translation $\pi(a)$ is a conjunction of implications. The antecedents of the implications are conjunctions of aggregate elements and the consequents are disjunctions of conjunctions of aggregate elements.

Property (a). If the conjunction in an antecedent contains an element not in J, then the conjunction is not satisfied by X and the implication does not affect the satisfiability of $\pi(a)$. If a conjunction in a consequent contains an element not in J, then X does not satisfy the conjunction and the conjunction does not affect the satisfiability of the encompassing disjunction. Observe that both cases correspond exactly to those subformulas omitted in $\pi_J(a)$.

The remaining two properties follow for similar reasons. □

The next observation summarizes how dependencies transfer from non-ground aggregate programs to the corresponding ground $\mathcal{R}$-programs.

Observation 57. *Let P_1 and P_2 be aggregate programs, and $G_1 = \pi(P_1)$ and $G_2 = \pi(P_2)$. Then,*

(a) *P_1 does not depend on P_2 implies $B(G_1)^\pm \cap H(G_2) = \emptyset$,*
(b) *P_1 does not positively depend on P_2 implies $B(G_1)^+ \cap H(G_2) = \emptyset$,*
(c) *P_1 does not negatively depend on P_2 implies $B(G_1)^- \cap H(G_2) = \emptyset$.*

The next two lemmas pin down important properties of instantiation sequences. First of all, there are no external atoms in the components of instantiation sequences.

Lemma 58. *Let P be an aggregate program and $(P_i)_{i\in\mathbb{I}}$ be an instantiation sequence for P.*

Then, for the sequence $(G_i)_{i\in\mathbb{I}}$ with $G_i = \pi(P_i)$, we have $E_i = \emptyset$ for each $i \in \mathbb{I}$ where E_i is defined as in (3)*.*

Proof. This lemma is a direct consequence of Observation 57 (a) and the anti-symmetry of the dependency relation between components. □

Proof of Theorem 26. This theorem is a direct consequence of Lemma 58 and Corollary 20. □

Moreover, for each stratified component in an instantiation sequence, we obtain a total well-founded model.

Lemma 59. *Let P be an aggregate program and $(P_i)_{i\in\mathbb{I}}$ be an instantiation sequence for P.*

Then, for the sequence $(G_i)_{i\in\mathbb{I}}$ with $G_i = \pi(P_i)$, we have $I_i = J_i = S_{G_i}^{IC_i}(IC_i)$ for each stratified component P_i where (IC_i, JC_i) and (I_i, J_i) are defined as in (4) *and* (5) *in the construction of the well-founded model of $(G_i)_{i\in\mathbb{I}}$ in Definition 6.*

Proof. In the following, we use E_i and (IC_i, JC_i) for the sequence $(G_i)_{i\in\mathbb{I}}$ as defined in (3) and (4). Note that, by Lemma 58, we have $E_i = \emptyset$.

We prove by induction.

Base. Let P_i be a stratified component that does not depend on any other component. Because P_i does not depend on any other component, we have $\bigcup_{j<i} H(G_j) \cap B(G_i)^{\pm} = \emptyset$. Thus, by Observation 52 (b), we get $IC_i \cap B(G_i)^{\pm} = JC_i \cap B(G_i)^{\pm} = \emptyset$. By Observation 52 (c), we get $(I_i, J_i) = WM^{IC_i,JC_i}(G_i) = WM^{IC_i,IC_i}(G_i)$. Because P_i is stratified, we have $B(G_i)^{-} \cap H(G_i) = \emptyset$. We then use Lemma 56 to obtain $I_i = J_i = S_{G_i}^{IC_i}(IC_i)$.

Hypothesis. We assume that the theorem holds for any component P_j with $j < i$.

Step. Let P_i be a stratified component. For any $j < i$, component P_i either depends on P_j or not. If P_i depends on P_j, then P_j is stratified and we get $I_j = J_j$ by the induction hypothesis. If P_i does not depend on P_j, then $I_j \cap B(G_i)^{\pm} = J_j \cap B(G_i)^{\pm} = \emptyset$. By Observation 52 (c), we get $(I_i, J_i) = WM^{IC_i,JC_i}(G_i) = WM^{IC_i,IC_i}(G_i)$. Just as in the base case, by Lemma 56, we get $I_i = J_i = S_{G_i}^{IC_i}(IC_i)$. □

Lemma 60. *Let P be an aggregate program and $(P_{i,j})_{(i,j)\in\mathbb{J}}$ be a refined instantiation sequence for P.*

Then, for the sequence $(G_{i,j})_{(i,j)\in\mathbb{J}}$ with $G_{i,j} = \pi(P_{i,j})$, we have $E_{i,j} \cap B(G_{i,j})^{+} = \emptyset$ for each $(i,j) \in \mathbb{J}$ where $E_{i,j}$ is defined as in (3)*.*

Proof. The same arguments as in the proof of Lemma 58 can be used but using Observation 57 (b) instead. □

Proof of Theorem 27. Let $G = \pi(P)$, $G' = \pi(P')$ with P' as in Definition 13, $(I, J) = AM_E^{IC,JC}(P)$, and $(I', J') = WM^{IC,JC\cup EC}(G)$.

We first show $I \subseteq I'$, or equivalently

$$S_{G'}^{IC}(JC) \subseteq S_G^{IC}(JC \cup EC \cup J').$$

Because $G' \subseteq G$, we get

$$S_{G'}^{IC}(JC \cup EC \cup J') \subseteq S_G^{IC}(JC \cup EC \cup J').$$

Because $\mathrm{pred}(B(P')^-) \cap E = \emptyset$, all rules $r \in G'$ satisfy $B(r)^- \cap EC \neq \emptyset$ and we obtain

$$S_{G'}^{IC}(JC \cup J') \subseteq S_{G'}^{IC}(JC \cup EC \cup J').$$

Because $\mathrm{pred}(H(P)) \cap \mathrm{pred}(B(P)^-) \subseteq E$ and $\mathrm{pred}(B(P')^-) \cap \mathcal{E} = \emptyset$, all rules $r \in G'$ satisfy $B(r)^- \cap J' = \emptyset$ and we obtain

$$\begin{aligned} S_{G'}^{IC}(JC) &= S_{G'}^{IC}(JC \cup J') \\ &\subseteq S_G^{IC}(JC \cup EC \cup J'). \end{aligned}$$

To show $J' \subseteq J$, we use $I \subseteq I'$ and Proposition 14:

$$S_G^{JC}(IC \cup I') \subseteq S_G^{JC}(IC \cup I). \qquad \square$$

Proof of Theorem 28. We begin by showing $AM((P_j)_{j \in \mathbb{J}}) \leq_p WM(\pi(P))$ and then show $AM((P_i)_{i \in \mathbb{I}}) \leq_p AM((P_j)_{j \in \mathbb{J}})$.

Property ($AM((P_j)_{j \in \mathbb{J}}) \leq_p WM(\pi(P))$). Let $\mathcal{E}_j$, (IC_j, JC_j), and (I_j, J_j) be defined as in (14) to (16) for the sequence $(P_j)_{j \in \mathbb{J}}$. Similarly, let E'_j, (IC'_j, JC'_j), and (I'_j, J'_j) be defined as in (3) to (5) for the sequence $(G_j)_{j \in \mathbb{J}}$ with $G_j = \pi(P_j)$. Furthermore, let EC_j be the set of all ground atoms over atoms in $\mathcal{E}_j$.

We first show $E'_j \subseteq EC_j$ for each $j \in \mathbb{J}$ by showing that $E'_j \subseteq EC_j$. By Lemma 60, only negative body literals have to be taken into account:

$$\begin{aligned} E'_j &= B(G_j)^{\pm} \cap \bigcup_{j<k} H(G_k) \\ &= B(G_j)^{-} \cap \bigcup_{j<k} H(G_k). \end{aligned}$$

Observe that $\mathrm{pred}(B(G_j)^- \subseteq \mathrm{pred}(B(P_j)^-))$ and $\mathrm{pred}(H(G_j)) \subseteq \mathrm{pred}(H(P_j))$. Thus, we get

$$\begin{aligned} \mathrm{pred}(E'_j) &= \mathrm{pred}(B(G_j)^- \cap \bigcup_{j<k} H(G_k)) \\ &\subseteq \mathrm{pred}(B(P_j)^-) \cap \mathrm{pred}(\bigcup_{j<k} H(P_k)) \\ &\subseteq \mathrm{pred}(B(P_j)^-) \cap \mathrm{pred}(\bigcup_{j\leq k} H(P_k)) \\ &= \mathcal{E}_j. \end{aligned}$$

It follows that $E'_j \subseteq EC_j$.

By Theorem 17, we have $\bigsqcup_{j \in \mathbb{J}}(I'_j, J'_j) \leq_p WM(\pi(G))$. To show the theorem, we show $(I_j, J_j) \leq_p (I'_j, J'_j)$. We omit the full induction proof and focus on the key idea: Using

Theorem 27, whose precondition holds by construction of $\mathcal{E}_j$, and Proposition 15, we get

$$\begin{aligned} AM^{IC_j,JC_j}_{\mathcal{E}_j}(P_j) &\leq_p WM^{(IC_j,JC_j)\sqcup(\emptyset,EC_j)}(G_j) \\ &\leq_p WM^{(IC'_j,JC'_j)\sqcup(\emptyset,E'_j)}(G_j). \end{aligned}$$

Property ($AM((P_i)_{i\in\mathbb{I}}) \leq_p AM((P_{i,j})_{(i,j)\in\mathbb{J}})$). We omit a full induction proof for this property because it would be very technical. Instead, we focus on the key idea why the approximate model of a refined instantiation sequence is at least as precise as the one of an instantiation sequence.

Let $\mathcal{E}_i$ and $\mathcal{E}_{i,j}$ be defined as in (14) for the instantiation and refined instantiation sequence, respectively. Clearly, we have $\mathcal{E}_{i,j} \subseteq \mathcal{E}_i$ for each $(i,j) \in \mathbb{J}$. Observe, that (due to rule dependencies and Observation 52 (c)) calculating the approximate model of the refined sequence, using $\mathcal{E}_i$ instead of $\mathcal{E}_{i,j}$ in (16), would result in the same approximate model as for the instantiation sequence. With this, the property simply follows from the monotonicity of the stable operator. □

Proof of Theorem 29. Let $\mathcal{E}_i$, (IC_i, JC_i), and (I_i, J_i) be defined as in (14) to (16) for the sequence $(P_i)_{i\in\mathbb{I}}$. Similarly, let E'_i, (IC'_i, JC'_i), and (I'_i, J'_i) be defined as in (3) to (5) for the sequence $(G_i)_{i\in\mathbb{I}}$ with $G_i = \pi(P_i)$. Furthermore, we assume w.l.o.g. that $\mathbb{I} = \{1,\dots,n\}$.

We have already seen in the proof of Theorem 28 that the atoms E'_i are a subset of the ground atoms over predicates $\mathcal{E}_i$ and that $(I_i, J_i) \leq_p (I'_i, J'_i)$. Observing that ground atoms over predicates $\mathcal{E}_i$ can only appear negatively in rule bodies, we obtain that $G_i^{(IC'_i,JC'_i)\cup(I'_i,J'_i)\cup(\emptyset,E'_i)} = G_i^{(IC'_i,JC'_i)\cup(I'_i,J'_i)}$. By Theorem 18 and Lemma 55, we obtain that $\bigcup_{i\in\mathbb{I}} G_i^{(IC_i,JC_i)\sqcup(I_i,J_i)}$ and $\pi(P)$ have the same well-founded and stable models. To shorten the notation, we let

$$\begin{aligned} F_i &= \pi(P_i)^{(IC_i,JC_i)\sqcup(I_i,J_i)}, & H_i &= \pi_{JC_i\cup J_i}(P_i)^{(IC_i,JC_i)\sqcup(I_i,J_i)}, \\ F &= \bigcup_{i\in\mathbb{I}} F_i, \text{ and} & H &= \bigcup_{i\in\mathbb{I}} H_i. \end{aligned}$$

With this, it remains to show that programs F and H have the same well-founded and stable models.

We let $J = \bigcup_{i\in\mathbb{I}} J_i$. Furthermore, we let $\pi(a)$ be a subformula in F_i and $\pi_{JC_i\cup J_i}(a)$ be a subformula in H_i where both subformulas originate from the translation of the closed aggregate a. (We see below that existence of one implies the existence of the other because both formulas are identical in their context.)

Because an aggregate always depends positively on the predicates occurring in its elements, the intersection between $\bigcup_{i<k} H(F_k) = \bigcup_{i<k} J_k$ and the atoms occurring in $\pi(a)$ is empty. Thus the two formulas $\pi_{JC_i\cup J_i}(a)$ and $\pi_J(a)$ are identical. Observe that each stable model of either F and H is a subset of J. By Theorem 25, satisfiability of the aggregates formulas as well as their reducts is the same for subsets of J. Thus, both formulas have the same stable models. Similarly, the well-founded model of both formulas must be more-precise than $(\emptyset, J)$. By Theorem 25, satisfiability of the aggregate formulas as well as their ID-reducts is the same. Thus, both formulas have the same well-founded model. □

Proof of Theorem 30. Clearly, we have $\mathcal{E}_i = \emptyset$ if all components are stratified. With this, the theorem follows from Lemma 59. □

We can characterize the result of Algorithm 1 as follows.

Lemma 61. *Let r be a safe normal rule, (I, J) be a finite four-valued interpretation, $f \in \{\mathbf{t}, \mathbf{f}\}$, and J' be a finite two-valued interpretation.*

Then, a call to $\texttt{GroundRule}^{I,J}_{r,f,J'}(\iota, B(r))$ *returns the finite set of instances g of r satisfying*

$$J \models \tau(B(g))^{\wedge}_{I} \text{ and } (f = \mathbf{t} \text{ or } B(g)^{+} \not\subseteq J'). \tag{A3}$$

Proof. Observe that the algorithm does not modify f, r, (I, J), and J'. To shorten the notation below, let $G_{\sigma,L} = \texttt{GroundRule}^{I,J}_{r,f,J'}(\sigma, L)$.

Calling $G_{\iota,B(r)}$, the algorithm maintains the following invariants in subsequent calls $G_{\sigma,L}$:

$$(B(r) \setminus L)\sigma^{+} \subseteq J, \tag{I1}$$

$$(B(r) \setminus L)\sigma^{-} \cap I = \emptyset, \text{ and} \tag{I2}$$

$$\text{each comparison in } (B(r) \setminus L)\sigma \text{ holds.} \tag{I3}$$

We only prove the first invariant because the latter two can be shown in a similar way. We prove by induction.

Base. For the call $G_{\iota,B(r)}$, the invariant holds because the set difference $B(r) \setminus L$ is empty for $L = B(r)$.

Hypothesis. We assume the invariant holds for call $G_{\sigma,L}$ and show that it is maintained in subsequent calls.

Step. Observe that there are only further calls if L is non-empty. In Line 3, a body literal l is selected from L. Observe that it is always possible to select such a literal. In case that there are positive literals in L, we can select any one of them. In case that there are no positive literals in L, σ replaces all variables in the positive body of r. Because r is safe, all literals in $L\sigma$ are ground and we can select any one of them.

In case that l is a positive literal, all substitutions σ', obtained by calling $\texttt{Matches}^{I,J}_{l}(\sigma)$ in the following line, ensure

$$l\sigma' \in J.$$

Furthermore, σ is more general than σ'. Thus, we have

$$\begin{aligned}(B(r) \setminus L)\sigma'^{+} &= (B(r) \setminus L)\sigma^{+} \\ &\subseteq J.\end{aligned}$$

In Line 5, the algorithm calls $G_{\sigma',L'}$ with $L' = L \setminus \{l\}$. We obtain

$$\begin{aligned}(B(r) \setminus L')\sigma'^{+} &= (B(r) \setminus L)\sigma'^{+} \cup \{l\sigma'\} \\ &\subseteq J.\end{aligned}$$

In case that l is a comparison or negative literal, we get $(B(r) \setminus L)^{+} = (B(r) \setminus L \setminus \{l\})^{+}$. Furthermore, the substitution σ is either not changed or is discarded altogether. Thus, the invariant is maintained in subsequent calls to `GroundRule`.

We prove by induction over subsets L of $B(r)$ with corresponding substitution σ satisfying invariants (I1)–(I3) that $G_{L,\sigma}$ is finite and that $g \in G_{L,\sigma}$ iff g is a ground instance of $r\sigma$ that satisfies (A3).

Base. We show the base case for $L = \emptyset$. Using invariant (I1), we only have to consider substitutions σ with $B(r)^+\sigma \subseteq J$. Because r is safe and σ replaces all variables in its positive body, σ also replaces all variables in its head and negative body. Thus, $r\sigma$ is ground and the remainder of the algorithm just filters the set $\{r\sigma\}$ while the invariants (I1)–(I3) ensure that $J \models \tau(B(r\sigma))^\wedge_I$. The condition in Line 2 cannot apply because $L = \emptyset$. The condition in Line 9 discards rules $r\sigma$ not satisfying $f = \mathbf{t}$ or $B(r\sigma)^+ \nsubseteq J'$.

Hypothesis. We show that the property holds for $L \neq \emptyset$ assuming that it holds for subsets $L' \subset L$ with corresponding substitutions σ'.

Step. Because $L \neq \emptyset$ we only have to consider the case in Line 2.

First, the algorithm selects an element $l \in L$. We have already seen that it is always possible to select such an element. Let $L' = L \setminus \{l\}$. The algorithm then loops over the set

$$\Sigma = \mathtt{Matches}_l^{I,J}(\sigma)$$

and, in Lines 4 to 5, computes the union

$$G_{\sigma,L} = \bigcup_{\sigma' \in \Sigma} G_{\sigma',L'}.$$

First, we show that the set $G_{\sigma,L}$ is finite. In case l is not a positive literal, the set Σ has at most one element. In case l is a positive literal, observe that there is a one-to-one correspondence between Σ and the set $\{l\sigma' \mid \sigma' \in \Sigma\}$. We obtain that Σ is finite because $\{l\sigma' \mid \sigma' \in \Sigma\} \subseteq J$ and J is finite. Furthermore, using the induction hypothesis, each set $G_{\sigma',L'}$ in the union $G_{\sigma,L}$ is finite. Hence, the set $G_{\sigma,L}$ returned by the algorithm is finite.

Second, we show $g \in G_{\sigma,L}$ implies that g is a ground instance of $r\sigma$ satisfying (A3). We have that g is a member of some $G_{\sigma',L'}$. By the induction hypothesis, g is a ground instance of $r\sigma'$ satisfying (A3). Observe that g is also a ground instance of $r\sigma$ because σ is more general than σ'.

Third, we show that each ground instance g of $r\sigma$ satisfying (A3) is also contained in $G_{\sigma,L}$. Because g is a ground instance of $r\sigma$, there is a substitution θ more specific than σ such that $g = r\theta$. In case that the selected literal $l \in L$ is a positive literal, we have $l\theta \in J$. Then, there is also a substitution θ' such that $\theta' \in \mathtt{match}(l\sigma, l\theta)$. Let $\sigma' = \sigma \circ \theta'$. By Definition 15, we have $\sigma' \in \Sigma$. It follows that $g \in G_{\sigma,L}$ because $g \in G_{\sigma',L'}$ by the induction hypothesis and $G_{\sigma',L'} \subseteq G_{\sigma,L}$. In the case that l is not a positive literal, we have $\sigma \in \Sigma$ and can apply a similar argument.

Hence, we have shown that the proposition holds for $G_{\iota,B(r)}$. □

In terms of the program simplification in Definition 1, the first condition in Lemma 61 amounts to checking whether $H(g) \leftarrow \tau(B(g))^\wedge$ is in $\tau(P)^{I,J}$, which is the simplification of the (ground) $\mathcal{R}$-program $\tau(P)$ preserving all stable models between I and J. The two last conditions are meant to avoid duplicates from a previous invocation. Since r is a normal rule, translation τ is sufficient.

Proof of Proposition 31. The first property directly follows from Lemma 61 and the definition of $\mathrm{Inst}^{I,J}(\{r\})$.

It remains to show the second property. Let G be the set of all ground instances of r

and

$$G_f^{X,Y} = \{g \in G \mid Y \models \tau(B(g))_I^\wedge, (f = \mathbf{t} \text{ or } B(g)^+ \nsubseteq X)\}.$$

By Lemma 61 and the first property, we can reformulate the second property of the proposition as $G_{\mathbf{t}}^{\emptyset,J} = G_{\mathbf{t}}^{\emptyset,J'} \cup G_{\mathbf{f}}^{J',J}$. We have

$$\begin{aligned} G_{\mathbf{t}}^{\emptyset,J} &= \{g \in G \mid J \models \tau(B(g))_I^\wedge\} \\ G_{\mathbf{t}}^{\emptyset,J'} &= \{g \in G \mid J' \models \tau(B(g))_I^\wedge\}, \text{ and} \\ G_{\mathbf{f}}^{J',J} &= \{g \in G \mid J \models \tau(B(g))_I^\wedge, B(g)^+ \nsubseteq J'\}. \end{aligned}$$

Observe that, given $J' \subseteq J$, we can equivalently write $G_{\mathbf{t}}^{\emptyset,J'}$ as

$$G_{\mathbf{t}}^{\emptyset,J'} = \{g \in G \mid J \models \tau(B(g))_I^\wedge, B(g)^+ \subseteq J'\}.$$

Because $B(g)^+ \subseteq J'$ and $B(g)^+ \nsubseteq J'$ cancel each other, we get

$$G_{\emptyset,J} = G_{\emptyset,J'} \cup G_{J',J}. \qquad \square$$

Proof of Proposition 32. We first show Property (a) and then (b).

Property (a). For a rule $r \in P$, we use r^α to refer to the corresponding rule with replaced aggregate occurrences in P^α. Similarly, for a ground instance g of r, we use g^α to to refer to the corresponding instance of r^α. Observe that $\pi_J(P)^{I,J} = \pi_J(\text{Inst}^{I,J}(P))$. We show that $g \in \text{Inst}^{I,J}(P)$ iff $g^\alpha \in \text{Inst}^{I,J \cup JA}(P^\alpha)$. In the following, because the rule bodies of g and g^α only differ regarding aggregates and their replacement atoms, we only consider rules with aggregates in their bodies.

Case $g \in \text{Inst}^{I,J}(P)$. Let r be a rule in P containing aggregate a, α be the replacement atom of form (20) for a, and σ be a ground substitution such that $r\sigma = g$. We show that for each aggregate $a\sigma \in B(g)$, we have $\epsilon_{r,a}(G^\epsilon, \sigma) \cup G \neq \emptyset$ and $J \models \pi_G(a\sigma)_I$ with $G = \eta_{r,a}(G^\eta, \sigma)$ and in turn $JA \models \alpha\sigma$. Because $J \models \pi(B(g))_I^\wedge$, we get $J \models \pi_J(a\sigma)_I$. It remains to show that $\epsilon_{r,a}(G^\epsilon, \sigma) \cup G \neq \emptyset$ and $\pi_J(a\sigma) = \pi_G(a\sigma)$. Observe that $\pi_J(a\sigma) = \pi_G(a\sigma)$ because the set G obtained from rules in G^η contains all instances of elements of $a\sigma$ whose conditions are satisfied by J while the remaining literals of these rules are contained in the body of g. Furthermore, observe that if no aggregate element is satisfied by J, we get $\epsilon_{r,a}(G^\epsilon, \sigma) \neq \emptyset$ because the corresponding ground instance of (21) is satisfied.

Case $g^\alpha \in \text{Inst}^{I,J \cup JA}(P^\alpha)$. Let r^α be a rule in P^α containing replacement atom α of form (20) for aggregate a and σ be a ground substitution such that $r^\alpha\sigma = g^\alpha$. Because $J \cup JA \models \tau(B(g^\alpha))_I^\wedge$, we have $\alpha\sigma \in JA$. Thus, we get that $J \models \pi_G(a\sigma)_I$ with $G = \eta_{r,a}(G^\eta, \sigma)$. We have already seen in the previous case that $\pi_J(a\sigma) = \pi_G(a\sigma)$. Thus, $J \models \pi_J(a\sigma)_I$. Observing that $g = r\sigma$ and $a\sigma \in B(g)$, we get $g \in \text{Inst}^{I,J}(P)$.

Property (b). This property follows from Property (a), Theorem 25, and Lemma 48. $\square$

Proof of Proposition 33. We prove Properties (a) and (b) by showing that the function calculates the stable model by iteratively calling the T operator until a fixed-point is reached.

Property (a) *and* (b). At each iteration i of the loop starting with 1, let JA_i be the value of $\texttt{Propagate}_P^{I,J}(G^\epsilon, G^\eta)$ in Line 6, G_i^ϵ, G_i^η, and G_i^α be the values on the right-hand-side of the assignments in Lines 4, 5 and 7, and $J_i = H(G_i^\alpha)$. Furthermore, let $J_0 = \emptyset$.

By Proposition 31, we get

$$\begin{aligned} G_i^\epsilon &= \mathrm{Inst}^{IC, JC \cup J_{i-1}}(P^\epsilon), \\ G_i^\eta &= \mathrm{Inst}^{IC, JC \cup J_{i-1}}(P^\eta), \\ JA_i &= \texttt{Propagate}_P^{I,J}(G_i^\epsilon, G_i^\eta), \text{ and} \\ G_i^\alpha &= \mathrm{Inst}^{IC, JC \cup JA_i \cup J_{i-1}}(P^\alpha). \end{aligned}$$

Using Proposition 32 (b) and observing the one-to-one correspondence between G_i^α and $\pi(P)^{IC, JC \cup JC_i}$, we get

$$\begin{aligned} J_i &= H(G_i^\alpha) \\ &= T_{\pi(P)_{IC}}(JC \cup J_{i-1}). \end{aligned}$$

Observe that, if the loop exits, then the algorithm computes the fixed point of $T^{JC}_{\pi(P)_{IC}}$, i.e., $J = S^{IC}_{\pi(P)}(JC)$. Furthermore, observe that this fixed point calculation terminates whenever $S^{IC}_{\pi(P)}(JC)$ is finite. Finally, we obtain $\texttt{GroundComponent}(P, IC, JC) = \pi_{JC \cup J}(P)^{IC, JC \cup J}$ using Proposition 32 (a).

Property (c). We have seen above that the interpretation J is a fixed point of $T^{JC}_{\pi(P)_{IC}}$. By Proposition 32 (b) and observing that function `Assemble` only modifies rule bodies, we get $H(\texttt{GroundComponent}(P, IC, JC)) = J$. □

Proof of Theorem 34. Since the program is finite, its instantiation sequences are finite, too. We assume w.l.o.g. that $\mathbb{I} = \{1, \dots, n\}$ for some $n \geq 0$. We let FC_i and GC_i be the values of variables F and G at iteration i at the beginning of the loop in Lines 4 to 7, and F_i and G_i be the results of the calls to `GroundComponent` in Lines 6 and 7 at iteration i.

By Proposition 33, we get that Lines 5 to 7 correspond to an application of the approximate model operator as given in Definition 13. For each iteration i, we get

$$\begin{aligned} (FC_i, GC_i) &= \bigsqcup_{j<i} (F_i, G_i), \\ (IC_i, JC_i) &= (H(FC_i), H(GC_i)), \\ (I_i, J_i) &= (H(F_i), H(G_i)), \text{ and} \\ G_i &= \pi_{JC_i \cup J_i}(P_i)^{(IC_i, JC_i) \sqcup (I_i, J_i)} \end{aligned}$$

whenever (I_i, J_i) is finite. In case that each (I_i, J_i) is finite, the algorithm returns in Line 8 the program

$$\begin{aligned} GC_n \cup G_n &= \bigcup_{i \in \mathbb{I}} G_i \\ &= \bigcup_{i \in \mathbb{I}} \pi_{JC_i \cup J_i}(P_i)^{(IC_i, JC_i) \sqcup (I_i, J_i)}. \end{aligned}$$

Thus, the algorithm terminates iff each call to `GroundComponent` is finite, which is exactly the case when $AM((P_i)_{i \in \mathbb{I}})$ is finite. □

Proof of Corollary 35. This is a direct consequence of Theorems 29 and 34. □

Proof of Proposition 36. Let G be the set of ground instances of the aggregate elements of a and $D \subseteq G$ be a set such that $D \not\triangleright a$.

Due to the antimonotonicity of the aggregate, we get $\pi_a(D) = \bot$. Thus, the reduct is constant because all consequents in $\pi(a)$ as well as $\pi(a)_I$ are equal to $\bot$ and the antecedents in $\pi(a)$ are completely evaluated by the reduct. Hence, the lemma follows by Lemma 44 (b). □

Proof of Proposition 37. We only show Property (a) because the proof of Property (b) is symmetric.

Let $a = \#\text{sum}\{E\} \succ b$ and $a_+ = \#\text{sum}^+\{E\} \succ b'$. Given an arbitrary two-valued interpretation J, we consider the following two cases:

Case $J \not\models \pi(a)_I$. There is a set $D \subseteq G$ such that $D \not\triangleright a$, $I \models \tau(D)^\wedge$, and $J \not\models \pi_a(D)^\vee$. Let $\widehat{D} = D \cup \{e \in G \mid I \models \tau(e), w(H(e)) < 0\}$.

Clearly, $\widehat{D} \not\triangleright a$ and $I \models \tau(\widehat{D})^\wedge$. Furthermore, $J \not\models \pi_a(\widehat{D})^\vee$ because we constructed $\widehat{D}$ so that $\pi_a(\widehat{D})^\vee$ is stronger than $\pi_a(D)^\vee$ because more elements with negative weights have to be taken into considerations.

Next, observe that $\widehat{D} \not\triangleright a_+$ holds because we have $\#\text{sum}^+(H(\widehat{D})) = \#\text{sum}^+(H(D))$ and $\#\text{sum}^-(H(\widehat{D})) = \#\text{sum}^-(T)$, which corresponds to the value subtracted from the bound of a_+. To show that $J \not\models \pi_{a_+}(\widehat{D})^\vee$, we show $\pi_{a_+}(\widehat{D})^\vee$ is stronger than $\pi_a(\widehat{D})^\vee$. Let $C \subseteq G \setminus \widehat{D}$ be a set of elements such that $\widehat{D} \cup C \triangleright a_+$. Because the justification of a_+ is independent of elements with negative weights, each clause in $\pi_{a_+}(\widehat{D})^\vee$ involving an element with a negative weight is subsumed by another clause without that element. Thus, we only consider sets C containing elements with positive weights. Observe that $\widehat{D} \cup C \triangleright a$ holds because we have $\#\text{sum}(H(\widehat{D} \cup C)) = \#\text{sum}^+(H(\widehat{D} \cup C)) + \#\text{sum}^-(T)$. Hence, we get $J \not\models \pi_{a_+}(\widehat{D})^\vee$.

Case $J \not\models \pi(a_+)_I$. There is a set $D \subseteq G$ such that $D \not\triangleright a_+$, $I \models \tau(D)^\wedge$, and $J \not\models \pi_{a_+}(D)^\vee$. Let $\widehat{D} = D \cup \{e \in G \mid I \models \tau(e), w(H(e)) < 0\}$.

Observe that $\widehat{D} \not\triangleright a_+$, $I \models \tau(\widehat{D})^\wedge$, and $J \not\models \pi_{a_+}(\widehat{D})^\vee$. As in the previous case, we can show that $\pi_a(\widehat{D})^\vee$ is stronger than $\pi_{a_+}(\widehat{D})^\vee$ because clauses in $\pi_a(\widehat{D})^\vee$ involving elements with negative weights are subsumed. Hence, we get $J \not\models \pi_a(\widehat{D})^\vee$. □

Proof of Proposition 38. Let G be the set of ground instances of E, $a_\prec = f\{E\} \prec b$ for aggregate relation $\prec$, and J be a two-valued interpretation.

Property (a)*.* We show that $J \models \pi(a_<)_I \vee \pi(a_>)_I$ implies $J \models \pi(a_\neq)_I$.

Case $J \models \pi(a_<)_I$. Observe that $\pi(a_\neq)$ is conjunction of implications of form $\tau(D)^\wedge \to \pi_{a_\neq}(D)^\vee$ with $D \subseteq G$ and $D \not\triangleright a_\neq$. Furthermore, note that $D \not\triangleright a_\neq$ implies $D \not\triangleright a_<$. Thus, $\pi(a_<)$ contains the implication $\tau(D)^\wedge \to \pi_{a_<}(D)^\vee$. Because $J \models \pi(a_<)_I$, we get $I \not\models \tau(D)^\wedge$ or $J \models \pi_{a_<}(D)^\vee$. Hence, the property holds in this case because $J \models \pi_{a_<}(D)^\vee$ implies $J \models \pi_{a_\neq}(D)^\vee$.

Case $J \models \pi(a_>)_I$. The property can be shown analogously for this case.

Property (a)*.* This property can be shown in a similar way as the previous one. We show by contraposition that $J \models \pi(a_=)_I$ implies $J \models \pi(a_\leq)_I \wedge \pi(a_\geq)_I$.

Case $J \not\models \pi(a_{\leq})_I$. Observe that $\pi(a_{\leq})$ is conjunction of implications of form $\tau(D)^{\wedge} \rightarrow \pi_{a_{\leq}}(D)^{\vee}$ with $D \subseteq G$ and $D \not\vdash a_{\leq}$. Furthermore, note that $D \not\vdash a_{\leq}$ implies $D \not\vdash a_{=}$. Thus, $\pi(a_{=})$ contains the implication $\tau(D)^{\wedge} \rightarrow \pi_{a_{=}}(D)^{\vee}$. Because $J \not\models \pi(a_{\leq})_I$, we get $I \models \tau(D)^{\wedge}$ and $J \not\models \pi_{a_{<}}(D)^{\vee}$ for some $D \subseteq G$ with $D \not\vdash a_{\leq}$. Hence, the property holds in this case because $J \not\models \pi_{a_{\leq}}(D)^{\vee}$ implies $J \not\models \pi_{a_{=}}(D)^{\vee}$.

Case $J \not\models \pi(a_{\geq})_I$. The property can be shown analogously for this case. □

Proof of Proposition 39. We only consider the case that f is the #sum function because the other ones are special cases of this function. Furthermore, we only consider the only if directions because we have already established the other directions in Proposition 38.

Let G be the set of ground instances of E, $T_I = H(\{g \in G \mid I \models B(g)\})$, and $T_J = H(\{g \in G \mid J \models B(g)\})$.

Property (a). Because $I \subseteq J$, we get $\#\text{sum}^+(T_I) \leq \#\text{sum}^+(T_J)$ and $\#\text{sum}^-(T_J) \leq \#\text{sum}^-(T_I)$. We prove by contraposition.

Case $J \not\models \pi(a_{<})_I$ *and* $J \not\models \pi(a_{>})_I$. We use Propositions 24 and 37 to get the following two inequalities:

$$\begin{aligned}\#\text{sum}^-(T_J) &\geq b - \#\text{sum}^+(T_I) \text{ because } J \not\models \pi(a_{<})_I \text{ and}\\ \#\text{sum}^+(T_J) &\leq b - \#\text{sum}^-(T_I) \text{ because } J \not\models \pi(a_{>})_I.\end{aligned}$$

Using $\#\text{sum}^-(T_J) \leq \#\text{sum}^-(T_I)$, we can rearrange as

$$\begin{aligned}b - \#\text{sum}^+(T_I) &\leq \#\text{sum}^-(T_J)\\ &\leq \#\text{sum}^-(T_I)\\ &\leq b - \#\text{sum}^+(T_J).\end{aligned}$$

Using $\#\text{sum}^+(T_I) \leq \#\text{sum}^+(T_J)$, we obtain

$$\#\text{sum}^+(T_I) = \#\text{sum}^+(T_J).$$

Using $\#\text{sum}^+(T_I) = \#\text{sum}^+(T_J)$, we get

$$\begin{aligned}b - \#\text{sum}^+(T_I) &\leq \#\text{sum}^-(T_J)\\ &\leq \#\text{sum}^-(T_I)\\ &\leq b - \#\text{sum}^+(T_I)\end{aligned}$$

and, thus, obtain

$$\begin{aligned}\#\text{sum}^-(T_I) &= \#\text{sum}^-(T_J) \text{ and}\\ b &= \#\text{sum}(T_I)\\ &= \#\text{sum}(T_J).\end{aligned}$$

Observe that this gives rise to an implication in $\pi(a_{\neq})_I$ that is not satisfied by J. Hence, we get $J \not\models \pi(a_{\neq})_I$.

Property (b). Because $J \subseteq I$, we get $\#\text{sum}^+(T_J) \leq \#\text{sum}^+(T_I)$ and $\#\text{sum}^-(T_I) \leq \#\text{sum}^-(T_J)$.

Case $J \models \pi(a_{\leq})_I$ and $J \models \pi(a_{\geq})_I$. Using Propositions 24 and 37, we get

$$\#\text{sum}^+(T_J) \geq b - \#\text{sum}^-(T_I) \text{ because } J \models \pi(a_{\geq})_I \text{ and}$$
$$\#\text{sum}^-(T_J) \leq b - \#\text{sum}^+(T_I) \text{ because } J \models \pi(a_{\leq})_I.$$

Observe that we can proceed as in the proof of the previous property because the relation symbols are just flipped. We obtain

$$\begin{aligned}\#\text{sum}^-(T_I) &= \#\text{sum}^-(T_J) \text{ and}\\ b &= \#\text{sum}(T_I)\\ &= \#\text{sum}(T_J).\end{aligned}$$

We get $J \models \pi(a_{=})_I$ because for any subset of tuples in T_I that do not satisfy the aggregate, we have additional tuples in T_J that satisfy the aggregate. □

Proof of Proposition 40. Let $a_{\prec} = f\{E\} \prec b$ for $\prec \in \{=, \neq\}$.

Property (a). We prove by contraposition that $J \models \pi(a_{\neq})_I$ implies that there is no set $X \subseteq T_I$ such that $f(X \cup T_J) = b$.

Case there is a set $X \subseteq T_I$ such that $f(X \cup T_J) = b$. Let $D = \{e \in G \mid I \models B(e), H(e) \in X \cup T_J\}$. Because $T_J \subseteq T_I$ $D \not\triangleright a_{\neq}$. Furthermore, we have $I \models \tau(D)^{\wedge}$. Observe that D contains all elements with conditions satisfied by J. Hence, we get $J \not\models \pi_{a_{\neq}}(D)^{\vee}$ and, in turn, $J \not\models \pi(a_{\neq})_I$.

We prove the remaining direction, again, by contraposition.

Case $J \not\models \pi(a_{\neq})_I$. There is a set $D \subseteq G$ such that $I \models \tau(D)^{\wedge}$ and $J \not\models \pi_{a_{\neq}}(D)^{\vee}$. Let $X = H(D)$. Because $J \not\models \pi_{a_{\neq}}(D)^{\vee}$, we get $f(X \cup T_J) = b$.

Property (b). This property can be shown in a similar way as the previous one. □