Online Appendix for the paper:

An ASP approach for reasoning on neural networks under a finitely many-valued semantics for weighted conditional knowledge bases

Appendix A Existence of canonical φ-coherent/φ_n-coherent models

For canonical φ-coherent and φ_n-coherent models, we can prove the following result.

Proposition 4
A weighted $G_n\mathcal{LC}\mathbf{T}$ (Ł$_n\mathcal{LC}\mathbf{T}$) knowledge base K has a canonical φ-coherent (φ_n-coherent) model, if it has a φ-coherent (φ_n-coherent) model.

Proof(sketch)
We prove the result for φ-coherent models of a weighted $G_n\mathcal{LC}\mathbf{T}$ (Ł$_n\mathcal{LC}\mathbf{T}$) knowledge base K. The proof for φ_n-coherent models is the same.

Given a weighted $G_n\mathcal{LC}\mathbf{T}$ (Ł$_n\mathcal{LC}\mathbf{T}$) knowledge base $K = \langle \mathcal{T}, \mathcal{T}_{C_1}, \ldots, \mathcal{T}_{C_k}, \mathcal{A}\rangle$, let $I_0 = \langle \Delta_0, \cdot^{I_0}\rangle$ be a φ-coherent model of K. A canonical φ-coherent model for K can be constructed starting from the model I_0 as follows.

First, let $\mathcal{S}$ be the set of all concept names $B \in N_C$ occurring in K. The set $\mathcal{S}$ is finite. Considering the finitely many concept names B in $\mathcal{S}$ and the finitely many truth degrees in $\mathcal{C}_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, there are finitely many valuations e assigning a membership degree in $\mathcal{C}_n$ to each concept name B in $\mathcal{S}$, i.e., such that $e(B) \in \mathcal{C}_n$, for each $B \in \mathcal{S}$.

Let us call $e_1, \ldots, e_k$ all such possible valuations over $\mathcal{S}$. Starting from the φ-coherent model I_0 of K, we extend the domain Δ_0 by possibly introducing new domain elements x_i, one for each valuation e_i, provided valuation e_i is present in some φ-coherent model of K, but not in I_0.

We say that valuation e_i *is present in an interpretation* $I = \langle \Delta, \cdot^I\rangle$ of K if there is a domain element $x \in \Delta$ such that $B^I(x) = e_i(B)$, for all concept names $B \in \mathcal{S}$.

We say that a valuation e_i *is missing in* I_0 *for* K if it is present in some φ-coherent model I of K, but it is not present in I_0.

Let us define a new interpretation $I^* = \langle \Delta^*, \cdot^{I^*}\rangle$ with domain

$$\Delta^* = \Delta_0 \cup \{x_i \mid \text{ valuation } e_i \text{ is missing in } I_0 \}$$

Δ^* contains a new element x_i for each valuation e_i which is missing in I_0.

The interpretation of individual names in I^* remains the same as in I_0. The interpretation of concepts in I^* is defined as follows:
- $B^{I^*}(x) = B^{I_0}(x)$ for all $x \in \Delta_0$, for all $B \in N_I$;
- $B^{I^*}(x_i) = e_i(B)$, for all $B \in \mathcal{S}$;
- $B^{I^*}(x_i) = B^{I_0}(z)$, for all $B \in N_C$ s.t. $B \notin \mathcal{S}$,

where z is an arbitrarily chosen domain element in Δ_0. Informally, the interpretation of concepts in I^* is defined as in I_0 on the elements of Δ_0, while it is given by valuation e_i for the added domain element x_i, for the concept names B in $\mathcal{S}$. For the concept names B not occurring in K the interpretation of B in x_i is taken to be the same as the interpretation in I_0 of B in some domain element $z \in \Delta_0$.

We have to prove that I^* satisfies all $G_n\mathcal{LC}\mathbf{T}$ ($\text{Ł}_n\mathcal{LC}\mathbf{T}$) inclusions and assertions in K and that it is a φ-coherent model of K. I^* also satisfies condition (ii) in Definition 6 by construction, as all the finitely many possible valuations e_i over $\mathcal{S}$, which are present in some φ-coherent model of K, are considered.

To prove that I^* satisfies all assertions in $\mathcal{A}$, let $C(a)\ \theta\alpha$ be in K. Then all the concept names in C are in $\mathcal{S}$. By construction $a^{I^*} = a^{I_0}$. Furthermore, it can be proven that $(E(a))^{I^*} = E^{I^*}(a^{I_0}) = E^{I_0}(a^{I_0}) = (E(a))^{I_0}$ holds for all concepts E occurring in K (the proof is by induction on the structure of concept E). Hence, $(C(a))^{I^*} = (C(a))^{I_0}$. As $C(a)\ \theta\alpha$ is satisfied in I_0, then $(C(a))^{I_0}\ \theta\alpha$ holds, and $(C(a))^{I^*}\ \theta\alpha$ also holds.

To prove that I^* satisfies all concept inclusions in $\mathcal{T}$, let $C \sqsubseteq D\ \theta\alpha$ be in K. Then all the concept names in C and in D are in $\mathcal{S}$. We have to prove that, for all $x \in \Delta^*$, $C^{I^*}(x) \rhd D^{I^*}(x)\ \theta\alpha$. We prove it by cases.

For the case $x \in \Delta_0$. It can be proven that, for all $x \in \Delta_0$ $E^{I^*}(x) = E^{I_0}(x)$ holds for all concepts E occurring in K (the proof is by induction on the structure of concept E). Therefore, $C^{I^*}(x) = C^{I_0}(x)$ and $D^{I^*}(x) = D^{I_0}(x)$ hold. As axiom $C \sqsubseteq D\ \theta\alpha$ is satisfied in I_0, $C^{I_0}(x) \rhd D^{I_0}(x)\ \theta\alpha$ holds. Therefore, $C^{I^*}(x) \rhd D^{I^*}(x)\ \theta\alpha$ also holds.

For $x \notin \Delta_0$, $x = x_i$ for some i. By construction, as e_i is missing in I_0, e_i must be present in some interpretation $I' = \langle \Delta', \cdot^{I'} \rangle$ of K, i.e., there is a domain element $y \in \Delta'$ such that $B^{I'}(y) = e_i(B)$, for all concept names $B \in \mathcal{S}$. It can be proven that, $E^{I^*}(x_i) = E^{I'}(y)$ holds for all concepts E occurring in K. (the proof is by induction on the structure of concept C). Therefore, $C^{I^*}(x_i) = C^{I'}(y)$ and $D^{I^*}(x_i) = D^{I'}(y)$ hold. As axiom $C \sqsubseteq D\ \theta\alpha$ is satisfied in I', $C^{I'}(y) \rhd D^{I'}(y)\ \theta\alpha$ holds. Therefore, $C^{I^*}(x_i) \rhd D^{I^*}(x_i)\ \theta\alpha$ also holds.

The proof that I^* is a φ-coherent model of K is similar. □

Appendix B Proof of Proposition 2

Lemma 1
Given a weighted $G_n\mathcal{LC}\mathbf{T}$ ($\text{Ł}_n\mathcal{LC}\mathbf{T}$) knowledge base $K = \langle \mathcal{T}, \mathcal{T}_{C_1}, \ldots, \mathcal{T}_{C_k}, \mathcal{A} \rangle$ over the set of distinguished concepts $\mathcal{C} = \{C_1, \ldots, C_k\}$, and a subsumption $C \sqsubseteq D\theta\alpha$, we can prove the following:

(1) if there is an answer set S of the ASP program $\Pi(K, n, C, D, \theta, \alpha)$, such that $eval(E', aux_C, v) \in S$, for some concept E occurring in K, then there is a φ_n-coherent model $I = \langle \Delta, \cdot^I \rangle$ for K and an element $x \in \Delta$ such that $E^I(x) = \frac{v}{n}$.

(2) if there is a φ_n-coherent model $I = \langle \Delta, \cdot^I \rangle$ for K and an element $x \in \Delta$ such that $E^I(x) = \frac{v}{n}$, for some concept E occurring in K and some $v \in \{0, \ldots, n\}$, then there is an answer set S of the ASP program $\Pi(K, n, C, D, \theta, \alpha)$, such that $eval(E', aux_C, v) \in S$.

Proof(sketch)
We prove the lemma for $G_n\mathcal{LC}\mathbf{T}$ (the proof for $\text{Ł}_n\mathcal{LC}\mathbf{T}$ is similar).

For part (1), given an answer set S of the program $\Pi(K, n, C, D, \theta, \alpha)$ such that $eval(E', aux_C,\ v) \in S$, for some concept E occurring in K, we construct a φ_n-coherent model $I = \langle \Delta, \cdot^I \rangle$ of K such that $E^I(x) = \frac{v}{n}$. Let N_I and N_C be the set of named individuals and named concepts in the language. We take as the domain Δ of I the set of constants including all the named individuals $d \in N_I$ occurring in K plus an auxiliary element z_C, i.e., $\Delta = \{e \mid e \in N_I$ and e occurs in $K\} \cup \{z_C\}$.

For each element $e \in \Delta$, we define a projection $\iota(e)$ to a corresponding ASP constant as follows:

- $\iota(z_C) = aux_C$;
- $\iota(e) = e$, if $e \in N_I$ and e occurs in K.

Note that, for all $e \in \Delta$, $nom(\iota(e)) \in S$ by construction of the program $\Pi(K, n, C, D, \theta, \alpha)$.

The interpretation of individual names in $e \in N_I$ over Δ is defined as follows:

- $e^I = e$, if e occurs in K;
- $e^I = a$, otherwise,

where a is an arbitrarily chosen element in Δ.

The interpretation of named concepts $A \in N_C$ is as follows:

- $A^I(e) = \frac{v}{n}$ iff $inst(\iota(e), A, v) \in S$, for all $e \in \Delta$, if A occurs in K;
- $A^I = B^I$, if A does not occur in K,

where B is an arbitrarily chosen concept name occurring in K.

This defines a $G_n\mathcal{LC}\mathbf{T}$ interpretation. Let us prove that $I = \langle \Delta, \cdot^I \rangle$ is a φ_n-coherent model of K.

Assume that the w_h^i are approximated to k decimal places. From the definition of the *eval* predicate, one can easily prove that the following statements hold, for all concepts C and distinguished concepts C_i occurring in K, and for all $e \in \Delta$:

- $C^I(e) = \frac{v}{n}$ if and only if $eval(C', \iota(e), v) \in S$;
- $weight(\iota(e), C_i', w) \in S$ if and only if $w = W_i(e) \times 10^k \times n$;
- $valphi(n, w, v) \in S$ if and only if $\frac{v}{n} = \varphi_n(w/(10^k \times n))$;
- $\varphi_n(\sum_h w_h^i \; D_{i,h}^I(e)) = \frac{v}{n}$ if and only if $weight(\iota(e), C_i', w) \in S$ and $valphi(n, w, v) \in S$;

where C' is the ASP encoding of concept C, and C_i' is the ASP encoding of concept C_i.

First we have to prove that I satisfies the $G_n\mathcal{LC}\mathbf{T}$ inclusions in TBox $\mathcal{T}$ and assertions in ABox $\mathcal{A}$. Suitable constraints in $\Pi(K, n, C, D, \theta, \alpha)$ guarantee that all assertions are satisfied. For instance, for assertion $C(a) \geq \alpha$, the constraint $\bot \leftarrow eval(C', a, V), V < \alpha n$, is included in the ASP program and we know that it is not the case that $eval(C', a, v) \in S$ and $v < \alpha n$ holds. By the equivalences above, it is not the case that $C^I(a^I) = \frac{v}{n}$ and $\frac{v}{n} < \alpha$ holds. Hence, $C^I(a^I) < \alpha$ does not hold.

For a $G_n\mathcal{LC}\mathbf{T}$ concept inclusion of the form $E \sqsubseteq D \geq \alpha$, the following constraint

$$\bot \leftarrow eval(E', X, V1), eval(D', X, V2), V1 > V2, V2 < \alpha n.$$

holds for X instantiated with any constant a such that $nom(a) \in S$. Hence, it is not the case that, for any such an a, $eval(E', a, v_1), eval(D', a, v_2)$ belong to S and that $v_1 > v_2$ and $v_2 < \alpha n$ hold. Therefore, it is not the case that for some $d \in \Delta$ $E^I(d) = \frac{v_1}{n}$, $D^I(d) = \frac{v_2}{n}$ and that $\frac{v_1}{n} > \frac{v_2}{n}$, $\frac{v_2}{n} < \alpha$ hold. That is, $E \sqsubseteq D \geq \alpha$ is satisfied in I. Similarly, for other concept inclusions in $\mathcal{T}$.

The interpretation I represents a φ_n-coherent model of K if

$$C_i^I(e) = \varphi_n(\sum_h w_h^i \; D_{i,h}^I(x))$$

holds for all $e \in \Delta$ and for all distinguished concepts C_i. We prove that this condition holds for I. In fact, all ground instances of the following constraint

$$\bot \leftarrow nom(x), dcls(Ci), eval(Ci, x, V), weight(x, Ci, W), valphi(n, W, V1), V! = V1.$$

must be statisfied in S. Hence, there cannot be a distinguished concept C_i and a constant a with $nom(a) \in S$, such that $eval(C'_i, a, v), weight(a, C'_i, w)$ and $valphi(n, w, v_1)$ belong to S, and $v_1 \neq v$. Thus, it is not the case that, for some $e \in \Delta$, $C_i^I(e) = \frac{v}{n}$, $\varphi_n(\sum_h w_h^i \, D_{i,h}^I(e)) = \frac{v_1}{n}$, and $v \neq v_1$.

By construction of the φ_n-coherent model $I = \langle \Delta, \cdot^I \rangle$ of K, if $eval(E', aux_C, v) \in S$, for some concept E occurring in K, as $aux_C = \iota(z_C)$, it follows that $E^I(z_C) = \frac{v}{n}$ holds in I.

For part (2), assume that there is a φ_n-coherent model $I = \langle \Delta, \cdot^I \rangle$ for K and an element $x \in \Delta$ such that $E^I(x) = \frac{v}{n}$, for some concept E occurring in K. We can construct an answer set S of the ASP program $\Pi(K, n, C, D, \theta, \alpha)$, such that $eval(E', aux_C, v) \in S$.

Let us define a set of atoms S_0 by letting:

$inst(a, A, v) \in S_0$ if $A^I(a^I) = \frac{v}{n}$ in the model I, and
$inst(aux_C, A, v) \in S_0$ if $A^I(x) = \frac{v}{n}$ in the model I,

for all concept names A occurring in K, and for all $a \in N_I$ such that $nom(a)$ is in $\Pi(K, n, C, D, \theta, \alpha)$. Nothing else is in S_0.

Let Π_1 be the set of ground instances of all definite clauses and facts in $\Pi(K, n, C, D, \theta, \alpha)$, i.e., the grounding of all rules in $\Pi(K, n, C, D, \theta, \alpha)$ with the exception of rule (r1), of the constraints and of the rule for *notok*.

Let S be the set of all ground facts which are derivable from program $\Pi_1 \cup S_0$ plus, in addition, *notok* in case *ok* is not derivable. It can be proven that, for all constants $a \in N_I$ such that $nom(a)$ is in $\Pi(K, n, C, D, \theta, \alpha)$ and for all concepts E occurring in K (including subconcepts):

$eval(E', a, v) \in S$ if and only if $E^I(a^I) = \frac{v}{n}$
$eval(E', aux_C, v) \in S$ if and only if $E^I(x) = \frac{v}{n}$

where E' is the ASP encoding of concept E. Furthermore, for all distinguished concepts C_i.:

$\varphi_n(\sum_h w_h^i \, D_{i,h}^I(a^I)) = \frac{v}{n}$ if and only if $weight(a, C'_i, w)$ and $valphi(n, w, v)$ are in S;

where C'_i is the ASP encoding of concept C_i.

S is a consistent set of ground atoms, i.e., $\bot \notin S$. Notice that our ASP encoding does not make use of explicit negation and S cannot contain complementary literals. It can be proven that all constraints in $\Pi(K, n, C, D, \theta, \alpha)$ are satisfied by S. Consider, for instance the constraint $\bot \leftarrow eval(C', a, V), V < \alpha n$, associated to an assertion $C(a)\theta\alpha$ in K. As the assertion $C(a)\theta\alpha$ is in K, it must be satisfied in the model I and, for some v, $C^I(a^I) = \frac{v}{n}$ and $\frac{v}{n}\theta\alpha$. Hence, atom $eval(C', a, v)$ is in S and $v\theta\alpha n$ holds, so that the constraint associated to assertion $C(a)\theta\alpha$ in $\Pi(K, n, C, D, \theta, \alpha)$ is satisfied in S.

Similarly, we can prove that all other constraints, those associated to the inclusion axioms and those that encode the φ_n-coherence condition are as well satisfied in S, as the interpretation I from which we have built the set S is a φ_n-coherent model of K, and satisfies all inclusion axioms in $\mathcal{T}$.

We can further prove that all ground instances of the rules in $\Pi(K, C, D, n, \theta, \alpha)$ are satisfied in S. This is obviously true for all the definite clauses which have been used deductively to determine S starting from S_0 by forward chaining. This is also true for the choice rule (r1),

$1\{inst(x, A, V) : val(V)\}1 \ \leftarrow cls(A), nom(x).$

as the choice of atoms $inst(a, A, v)$ we have included in S_0 is one of the possible choices allowed by rule (r1). We have already seen that all constraints in $\Pi(K, n, C, D, \theta, \alpha)$ are satisfied in S. Finally, also rule $notok \leftarrow not\ ok$. is satisfied in S, as we have added $notok$ in S in case $ok \notin S$.

We have proven that S is a consistent set of ground atoms and all ground instances of the rules in $\Pi(K, n, C, D, \theta, \alpha)$ are satisfied in S. To see that S is an answer set of $\Pi(K, n, C, D, \theta, \alpha)$, it has to be proven that all literals in S are supported in S. Informally, observe that, all literals (facts) in S can be obtained as follows: first by applying the choice rule (r1), which supports the choice of the atoms $inst(a, A, v)$ in S_0 (and in S); then by exhaustively applying all ground definite clauses in $\Pi(K, n, C, D, \theta, \alpha)$ (by forward chaining) and, finally, by applying rule $notok \leftarrow not\ ok.$, to conclude $nottok$ if $ok \notin S$.

From the hypothesis, for element $x \in \Delta$ it holds that $E^I(x) = \frac{v_1}{n}$. Then, we can conclude that $eval(E', aux_C, v) \in S$, which concludes the proof. □

Proposition 2
Given a weighted $G_n\mathcal{LC}\mathbf{T}$ ($Ł_n\mathcal{LC}\mathbf{T}$) knowledge base K, a query $\mathbf{T}(C) \sqsubseteq D\theta\alpha$ is falsified in some canonical φ_n-coherent model of K if and only if there is a preferred answer set S of the program $\Pi(K, C, D, n, \theta, \alpha)$ such that $eval(D', aux_C, v)$ is in S and $v\theta\alpha n$ does not hold.

Proof(sketch)
Let $K = \langle \mathcal{T}, \mathcal{T}_{C_1}, \ldots, \mathcal{T}_{C_k}, \mathcal{A}\rangle$ be a $G_n\mathcal{LC}\mathbf{T}$ knowledge base over the set of distinguished concepts $\mathcal{C} = \{C_1, \ldots, C_k\}$. We prove the two directions:

(1) if there is a canonical φ_n-coherent model $I = (\Delta, \cdot^I)$ of K that falsifies $\mathbf{T}(C) \sqsubseteq D\theta\alpha$, then there is a preferred answer set S of $\Pi(K, n, C, D, \theta, \alpha)$ such that, for some v, $eval(D', aux_C, v) \in S$ and $v\theta\alpha n$ does not hold.

(2) if there is a preferred answer set S of $\Pi(K, n, C, D, \theta, \alpha)$ such that, for some v, $eval(D', aux_C, v)$ is in S and $v\theta\alpha n$ does not hold, then there is a canonical φ_n-coherent model $I = (\Delta, \cdot^I)$ of K that falsifies $\mathbf{T}(C) \sqsubseteq D\theta\alpha$.

We prove (1) and (2) for $G_n\mathcal{LC}\mathbf{T}$ (the proof for $Ł_n\mathcal{LC}\mathbf{T}$ is similar).

For part (1), assume that there is a canonical φ_n-coherent model $I = (\Delta, \cdot^I)$ of K that falsifies $\mathbf{T}(C) \sqsubseteq D\theta\alpha$. Then, there is some $x \in \Delta$, such that $x \in min_{<_C}(C^I_{>0})$, $D^I(x) = \frac{v}{n}$ and it does not hold that $\frac{v}{n}\theta\alpha$.

By Lemma 1, part (2), we know that there is an answer set S of the ASP program $\Pi(K, n, C, D, \theta, \alpha)$ such that $eval(D', aux_C, v) \in S$. Clearly, $v\theta\alpha n$ does not hold. We have to prove that S is a preferred answer set of $\Pi(K, n, C, D, \theta, \alpha)$.

By construction of S, for all constants $a \in N_I$ such that $nom(a)$ is in $\Pi(K, n, C, D, \theta, \alpha)$, we have:

$inst(a, A, v) \in S$ if $A^I(a^I) = \frac{v}{n}$ in model I,
$inst(aux_C, A, v) \in S$ if $A^I(x) = \frac{v}{n}$ in model I,

for all concept names A occurring in K.

Suppose by absurd that S is not preferred among the answer sets of $\Pi(K, n, C, D, \theta, \alpha)$. Then there is another answer set S' which is preferred to S. This means that if $eval(C', aux_C, v_1) \in S$ and $eval(C', aux_C, v_2) \in S'$, then $v_2 > v_1$.

By construction of S (see Lemma 1, part (2)), from $eval(C', aux_C, v_1) \in S$ it follows that $C^I(x) = \frac{v_1}{n}$ in the φ_n-coherent model I of K.

As S' is also an answer set of $\Pi(K, n, C, D, \theta, \alpha)$, by Lemma 1, part (1), from S' we can build a φ_n-coherent model $I' = \langle \Delta', \cdot^{I'} \rangle$ of K such that $C^{I'}(z_C) = \frac{v_2}{n}$, for $z_C \in \Delta'$.

As I is a canonical model, there must be an element $y \in \Delta$ such that $B^I(y) = B^{I'}(z_C)$, for all concepts B. Therefore, $C^I(y) = C^{I'}(z_C) = \frac{v_2}{n}$. As $\frac{v_2}{n} > \frac{v_1}{n}$, this contradicts the hypothesis that $x \in min_{<_C}(C^I_{>0})$. Then, S must be preferred among the answer sets of $\Pi(K, n, C, D, \theta, \alpha)$.

For part (2), let us assume that there is a preferred answer set S of $\Pi(K, n, C, D, \theta, \alpha)$ such that, $eval(C', aux_C, v_1)$ $eval(D', aux_C, v_2)$ are in S and $v_2 \theta \alpha n$ does not hold. By Lemma 1, part (1), from the answer set S we can construct a φ_n-coherent model $I^* = (\Delta^*, \cdot^{I^*})$ of K in which $C^{I^*}(z_C) = \frac{v_1}{n}$ and $D^{I^*}(z_C) = \frac{v_2}{n}$, for domain element z_C.

From the existence of a φ_n-coherent model I^* of K it follows, by Proposition 4, that a canonical φ_n-coherent model $I = (\Delta, \cdot^I)$ of K exists. As I is canonical, there must be an element $y \in \Delta$ such that $B^I(y) = B^{I^*}(z_C)$, for all concept names B occurring in K. Therefore, $B^I(y) = \frac{v'}{n}$ iff $eval(B', aux_C, v') \in S$, for all concept names B occurring in K. In particular, $C^I(y) = \frac{v_1}{n}$ and $D^I(y) = \frac{v_2}{n}$. Hence, there is a canonical φ_n-coherent model of K such that $D^I(y) = \frac{v_2}{n}$ and $\frac{v_2}{n} \theta \alpha$ does not hold.

To conclude that I falsifies $\mathbf{T}(C) \sqsubseteq D\theta\alpha$, we have still to prove that y is $<_C$-minimal in I with respect to all domain elements in Δ, i.e., $y \in min_{<_C}(C^I_{>0})$. If y were not in $min_{<_C}(C^I_{>0})$, there would be a $z \in \Delta$ such that $z <_C y$, that is, $C^I(z) > C^I(y)$. This leads to a contradiction. Assume $C^I(z) = \frac{v_3}{n} > \frac{v_1}{n}$, by Lemma 1, part (2), there is an answer set S' of $\Pi(K, n, C, D, \theta, \alpha)$ such that $eval(C', aux_C, v_3)$. However, this would contradict the hypothesis that S is a preferred answer set of $\Pi(K, n, C, D, \theta, \alpha)$, as $v_3 > v_1$. □

Proposition 3

φ_n-coherent entailment from a weighted $G_n\mathcal{LC}\mathbf{T}$ (Ł$_n\mathcal{LC}\mathbf{T}$) knowledge base is in Π^p_2.

Proof

Let K be a weighted $G_n\mathcal{LC}\mathbf{T}$ knowledge base K (the proof for Ł$_n\mathcal{LC}\mathbf{T}$ is similar). We consider the complementary problem, that is, the problem of deciding whether $\mathbf{T}(C) \sqsubseteq D\theta\alpha$ is not entailed by K in the φ_n-coherent semantics. It requires to determine whether there is a canonical φ_n-coherent model of K falsifying $\mathbf{T}(C) \sqsubseteq D\theta\alpha$ or, equivalently (by Proposition 2), whether there is a preferred answer set S of $\Pi(K, n, C, D, \theta, \alpha)$ such that $eval(D', aux_C, v)$ belongs to S and $v\theta\alpha n$ does not hold.

This problem can be solved by an algorithm that non-deterministically guesses a ground interpretation S over the language of $\Pi(K, n, C, D, \theta, \alpha)$, of polynomial size (in the size of $\Pi(K, n, C, D, \theta, \alpha)$) and, then, verifies that S satisfies all rules in $\Pi(K, n, C, D, \theta, \alpha)$ and is supported in S (i.e., it is an answer set of $\Pi(K, n, C, D, \theta, \alpha)$), that $eval(D', aux_C, v)$ is in S, that $v\theta\alpha n$ does not hold, and that S is preferred among the answer sets of $\Pi(K, n, C, D, \theta, \alpha)$. The last point can be verified using an NP-oracle which answers "yes" when S is a preferred answer set of $\Pi(K, n, C, D, \theta, \alpha)$, and "no" otherwise.

The oracle checks if there is an answer set S' of $\Pi(K, n, C, D, \theta, \alpha)$ which is preferred to S, by non-deterministically guessing a ground polynomial interpretation S' over the language of $\Pi(K, n, C, D, \theta, \alpha)$, and verifying that S satisfies all rules and is supported in S' (i.e., S' is an answer set of $\Pi(K, C, D, \theta\alpha)$), and that S' is preferred to S. These checks can be done in polynomial time.

Hence, deciding whether $\mathbf{T}(C) \sqsubseteq D\theta\alpha$ is not entailed by K in the φ_n-coherent semantics is in Σ^p_2, and the complementary problem of deciding φ_n-coherent entailment is in Π^p_2. $\square$