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Analyzing Semantics of Aggregate Answer Set Programming Using Approximation Fixpoint Theory: Technical proofs

Proposition 1
$\mathcal{H}^3$ is regular iff $\models_3$ is a lower- and $\models_3^{\uparrow}$ an upper-regular *TSR*.

Proof
Let $\mathcal{H}^2$ denote the two-valued truth assignment.

1. Assume $\mathcal{H}^3$ is regular; we show that $\models_3$ is a lower-regular ternary satisfaction relation:
   - $\mathcal{H}^2_I(\psi) = v$ with $v \in \{\mathbf{f}, \mathbf{t}\}$, iff $\mathcal{H}^3_{(I,I)}(\psi) = v$. This entails that $\mathcal{H}^3_{(I,I)}(\psi) = \mathbf{t}$ iff $\mathcal{H}^2_I(\psi) = \mathbf{t}$. So, $(I, I) \models_3 \psi$ iff $I \models_2 \psi$ and $\models_3$ extends $\models_2$.
   - If $(I, J) \leq_p (I', J')$ then for every ψ : $\mathcal{H}^3_{(I,J)}(\psi) \leq_p \mathcal{H}^3_{(I',J')}(\psi)$. So, if $\mathcal{H}^3_{(I,J)}(\psi) = \mathbf{t} = (\mathbf{t}, \mathbf{t})$, the truth-value according to $\mathcal{H}^3_{(I',J')}(\psi)$ should also be equal to $\mathbf{t}$. Consequently $(I, J) \models_3 \psi$ implies $(I', J') \models_3 \psi$ (monotone).
2. Assume $\mathcal{H}^3$ is regular; we show that $\models_3^{\uparrow}$ is an upper-regular ternary satisfaction relation:
   - $\mathcal{H}^2_I(\psi) = v$ with $v \in \{\mathbf{f}, \mathbf{t}\}$, iff $\mathcal{H}^3_{(I,I)}(\psi) = v$. This entails that $\mathcal{H}^3_{(I,I)}(\psi) = \mathbf{f}$ iff $\mathcal{H}^2_I(\psi) = \mathbf{f}$. In other words, $\mathcal{H}^3_{(I,I)}(\psi) \neq \mathbf{f}$ iff $\mathcal{H}^2_I(\psi) \neq \mathbf{f}$. Based on the derivation of the upper satisfaction relation from the truth assignment, it is clear that $\mathcal{H}^3_{(I,I)}(\psi) \neq \mathbf{f}$ iff $(I, I) \models_3^{\uparrow} \psi$. On the other hand, for the two-valued truth-assignment, it holds that $\mathcal{H}^2_I(\psi) \neq \mathbf{f}$ iff $\mathcal{H}^2_I(\psi) = \mathbf{t}$ iff $I \models_2 \psi$. So, $(I, I) \models_3^{\uparrow} \psi$ iff $I \models_2 \psi$ and $\models_3^{\uparrow}$ extends $\models_2$.
   - If $(I, J) \leq_p (I', J')$ then for every ψ : $\mathcal{H}^3_{(I,J)}(\psi) \leq_p \mathcal{H}^3_{(I',J')}(\psi)$. So, if $\mathcal{H}^3_{(I,J)}(\psi) = \mathbf{f} = (\mathbf{f}, \mathbf{f})$, the truth-value according to $\mathcal{H}_{(I',J')^3}(\psi)$ should also be equal to $\mathbf{f}$. Consequently $(I, J) \not\models_3^{\uparrow} \psi$ implies $(I', J') \not\models_3^{\uparrow} \psi$. Therefore, $(I', J') \models_3^{\uparrow} \psi$ implies $(I, J) \models_3^{\uparrow} \psi$ (anti-monotone).
3. Assume $\models_3$ is a lower- and $\models_3^{\uparrow}$ an upper-regular ternary satisfaction relation; we show that $\mathcal{H}^3$ is regular.
   - $(I, I) \models_3 \psi$ iff $I \models_2 \psi$ iff $(I, I) \models_3^{\uparrow} \psi$. Therefore $\mathcal{H}^3_{(I,I)}(\psi) = \mathbf{t}$ iff $\mathcal{H}^2_I(\psi) = \mathbf{t}$ iff $\mathcal{H}^3_{(I,I)}(\psi) \in \{\mathbf{t}, \mathbf{u}\}$. Consequently, $\mathcal{H}^3_{(I,I)}(\psi) = \mathbf{f}$ iff $\mathcal{H}^2_I(\psi) = \mathbf{f}$. Hence $\mathcal{H}^3$ coincides with $\mathcal{H}^2$ for two-valued interpretations.
   - If $(I, J) \leq_p (I', J')$, then $(I, J) \models_3 \psi$ implies $(I', J') \models_3 \psi$. Therefore $\mathcal{H}^3_{(I,J)}(\psi) = \mathbf{t}$ implies $\mathcal{H}^3_{(I',J')}(\psi) = \mathbf{t}$. At the same time, $(I', J') \models_3^{\uparrow} \psi$ implies that $(I, J) \models_3^{\uparrow} \psi$. Or equivalently, $(I, J) \not\models_3^{\uparrow} \psi$ implies $(I', J') \not\models_3^{\uparrow} \psi$. Consequently $\mathcal{H}^3_{(I,J)}(\psi) = \mathbf{f}$ implies $\mathcal{H}^3_{(I',J')}(\psi) = \mathbf{f}$. If $\mathcal{H}^3_{(I,J)}(\psi) = \mathbf{u}$ no re-

strictions are imposed on $\mathcal{H}^3_{(I',J')}(\psi) = \mathbf{u}$. In all three scenarios it holds that if $(I,J) \leq_p (I',J')$, then $\mathcal{H}^3_{(I,J)}(\psi) \leq_p \mathcal{H}^3_{(I',J')}(\psi)$.

□

Proposition 2
If $\models_3$ and $\models_3^\uparrow$ are lower- and upper-regular TSRs, then A_P is an L^c approximator. Moreover it is isomorphic to the three-valued Φ_P induced by the three-valued truth assignment $\mathcal{H}^3$ combining $\models_3$ and $\models_3^\uparrow$.

Proof
We have to proof that $A_P = (A_P^{\models_3}(I,J), A_P^{\models_3^\uparrow}(I,J)$ is an approximating operator for T_P if $\models_3$ and $\models_3^\uparrow$ are lower- and upper-regular respectively.

1. $\leq_p$-monotonicity, i.e., $\forall I,J,I',J' : (I,J) \leq_p (I',J') \Leftrightarrow A_P(I,J) \leq_p A_P(I',J')$. $A_P(I,J) \leq_p A_P(I',J')$ is equivalent to $A_P^{\models_3}(I,J) \leq A_P^{\models_3}(I',J') \wedge A_P^{\models_3^\uparrow}(I,J) \geq A_P^{\models_3}(I',J')$ according to how the precision order is defined. Now, $A_P(I,J) \leq_p A_P(I',J')$ means that, for every atom p, it holds that $p \in A_P^{\models_3}(I,J) \implies p \in A_P^{\models_3}(I',J')$ and $p \in A_P^{\models_3^\uparrow}(I',J') \implies p \in A_P^{\models_3}(I,J)$. If $\models_3^\uparrow$ is upper-regular, then for every formula $\psi : (I',J') \models_3^\uparrow \psi$ implies $(I,J) \models_3^\uparrow \psi$. So for every rule $p \leftarrow \psi \in P : (I',J') \models_3^\uparrow \psi$ implies $(I,J) \models_3^\uparrow \psi$. Now, for every atom $p \in A_P^{\models_3^\uparrow}(I',J')$ there exists a rule $p \leftarrow \psi \in P$, such that $(I',J') \models_3^\uparrow \psi$ and therefore $(I,J) \models_3^\uparrow \psi$. Consequently, according to the definition for $A_P^{\models_3^\uparrow}(I,J)$, it follows that $p \in A_P^{\models_3^\uparrow}(I,J)$. Analogously, we see that for every atom $p \in A_P^{\models_3}(I,J)$ there exists a rule $p \leftarrow \psi \in P$, such that $(I,J) \models_3 \psi$. If $\models_3$ is lower-regular, this entails that $(I',J') \models_3 \psi$ and therefore $p \in A_P^{\models_3}(I',J')$ by definition of $A_P^{\models_3}(I',J')$ so $\leq_p$-monotonicity holds.
2. A_P extends T_P, i.e., $\forall I : A_P(I,I) = (T_P(I), T_P(I))$. In other words, $A_P^{\models_3}(I,I) = T_P(I))$ and $A_P^{\models_3^\uparrow}(I,I) = T_P(I))$. $T_P(I))$ is by definition equal to $\{p|p \leftarrow \psi \in P, I \models_2 \psi\}$. Since $\models_3$ and $\models_3^\uparrow$ are lower- and upper-regular respectively, we know that $(I,I) \models_3 \psi$ iff $I \models_2 \psi$ iff $(I,I) \models_3^\uparrow \psi$. Therefore, $A_P^{\models_3}(I,I) = \{p|p \leftarrow \psi \in P, (I,I) \models_3 \psi\} = \{p|p \leftarrow \psi \in P, I \models_2 \psi\} = T_P(I)) = A_P^{\models_3^\uparrow}(I,I) = \{p|p \leftarrow \psi \in P, (I,I) \models_3^\uparrow \psi\}$.
3. In conclusion, $A_P(I,J)$ is an approximating operator if $\models_3$ and $\models_3^\uparrow$ are lower- and upper-regular.

Finally, we have to prove that A_P is isomorphic to the three-valued Φ_P. Let $\mathcal{I} = (I,J)$ be a three-valued interpretation.

- Then $A_P(I,J)_1 = A_P^{\models_3}(I,J) = \{p|(p \leftarrow \psi) \in P, (I,J) \models_3 \psi\} = \{p|(p \leftarrow \psi) \in P, \mathcal{H}^3(I,J)(\psi) = \mathbf{t}\}$. On the other hand, we know that $\Phi_P(I,J)_1$ denotes the set of atoms p such that $lub\{\mathcal{H}^3(I,J)(\psi)|(p \leftarrow \psi) \in P\} = \mathbf{t}$ or thus, such that at least one rule with p in the head has a body that evaluates to $\mathbf{t}$. This corresponds to the set $\{p|(p \leftarrow \psi) \in P, \mathcal{H}^3(I,J)(\psi) = \mathbf{t}\}$. So $A_P(I,J)_1 = \Phi_P(I,J)_1$.

- Then $A_P(I,J)_2 = A_P^{\models_3^{\uparrow}}(I,J) = \{p|(p \leftarrow \psi) \in P, (I,J) \models_3^{\uparrow} \psi\} = \{p|(p \leftarrow \psi) \in P, \mathcal{H}^3(I,J)(\psi) \in \{\mathbf{t},\mathbf{u}\}\}$. $\Phi_P(I,J)_2$ denotes the set of atoms p such that $lub\{\mathcal{H}^3(I,J)(\psi)|(p \leftarrow \psi) \in P\} \in \{\mathbf{t},\mathbf{u}\}$. In words, at least one rule with p in the head should have a body that at least evaluates to $\mathbf{u}$. This corresponds again to the set $\{p|(p \leftarrow \psi) \in P, \mathcal{H}^3(I,J)(\psi) \in \{\mathbf{t},\mathbf{u}\}\}$. So $A_P(I,J)_2 = \Phi_P(I,J)_2$.

□

Proposition 3 (*semi-constructive answer sets*)
If $\models_3$ is lower-monotone, then for $\mathcal{L}$-programs P, I is an answer set of P iff I is the limit of the increasing sequence $\langle I_\alpha \rangle_{\alpha \geq 0}$ where (1) $I_0 = \emptyset$, (2) $I_{\alpha+1} = A_P^{\models_3}(I_\alpha, I)$ if $I_\alpha \subseteq I$, (3) $I_\lambda = \bigcup_{\alpha<\lambda} I_\alpha$ for limit ordinal λ.

Proof
If $\models_3$ is lower-monotone, then for $\mathcal{L}$-programs P we find that $\lambda I : A_P^{\models_3}(I,J) = \{p|\exists(p \leftarrow \psi) \in P : (I,J) \models_3 \psi\}$ is a monotone operator. By the Knaster–Tarski theorem for monotone operators, we know that the described increasing sequence converges to the least fixpoint of the operator. Therefore, all we need to proof is that the least fixpoint of $\lambda J : A_P^{\models_3}(J,I)$ is I iff I is an answer set of P. I is the least fixpoint of $\lambda J : A_P^{\models_3}(J,I)$ iff (i) $A_P^{\models_3}(I,I) = I$ and (ii) there is no $J \subset I$ such that $A_P^{\models_3}(J,I) = J$.

1. (i) and (ii) $\implies$ I is an answer set.
   If $A_P^{\models_3}(J,I) = J$, then for every rule $p \leftarrow \psi \in P$, it holds that if $(J,I) \models_3 \psi$, then $p \in J$ and thus $(J,I) \models_3 p$. Thus if (i), then for every $(p \leftarrow \psi) \in P$, if $(I,I) \models_3 \psi$ and hence $I \models_2 \psi$, then $(I,I) \models_3 p$ and hence $I \models_2 p$. This is condition (1) of the answer set definition (Definition 4).
   Since $\lambda J : A_P^{\models_3}(J,I)$ is monotone and I is the least fixpoint, we find that from (ii), it follows that for every $J \subset I$, there exists a rule $(p \leftarrow \psi) \in P$, such that $(J,I) \models_3 \psi$ and $(J,I) \not\models_3 p$, i.e., there is no $J \subset I$ such that for every rule $(p \leftarrow \psi) \in P$, if $(J,I) \models_3 \psi$ then $(J,I) \models_3 p$. This is condition (2) of Definition 4, so I is an answer set.
2. I is an answer set $\implies$ (i).
   If I is an answer set, then for every $(p \leftarrow \psi) \in P$, if $I \models_2 \psi$, then $I \models_2 p$. By definition of $A_P^{\models_3}$, this entails that $A_P^{\models_3}(I,I) = I' \subseteq I$. Assume $I' \subset I$, then for every rule $p \leftarrow \psi \in P$, if $(I',I) \models_3 \psi$, then by lower-monotonicity of $\models_3$, $(I,I) \models_3 \psi$, hence by definition of $A_P^{\models_3}$, $p \in A_P^{\models_3}(I,I) = I'$, which entails $(I',I) \models_3 p$. Therefore I' violates condition (2) of the answer set definition.
3. I is an answer set $\implies$ (ii).
   Assume that there exists a $J \subset I$, such that $A_P^{\models_3}(J,I) = J$, then it holds that for every rule $(p \leftarrow \psi) \in P$, if $(J,I) \models_3 \psi$, then $(J,I) \models_3 p$, so J violates condition (2) of the answer set definition.

□

Proposition 4
Answer sets and AFT-stable models coincide for $\mathcal{L}$ programs with a lower regular TSR.

Proof
If $\models_3$ is lower-regular, then the AFT-stable models are defined. Recall from Section 2 that, according to the constructive test, x is a stable model iff x is the limit of the *lfp* construction of $\lambda z : A(z,x)_1$ with $A(z,x)$ an approximator on L^c. From Proposition 2, we know that $A_P(I,J) = (A_P^{\models_3}(I,J), A_P^{\models_3^\uparrow}(I,J))$ is an approximator for T_P if $\models_3$ and $\models_3^\uparrow$ are lower- and upper-regular. This holds as long as we choose an upper-regular $\models_3^\uparrow$. Then the first component $A_P(I,J)_1$ corresponds to $A_P^{\models_3}(I,J)$. Thus J is a stable model iff J is the limit of the *lfp* construction of $\lambda I : A_P^{\models_3}(I,J)$ iff J is an answer set (by Proposition 3). □

Proposition 5
For $I \in [\bot, J]$, $I \models_2 P^J$ iff for every rule $p \leftarrow \psi \in P$, if $(I,J) \models_{GL} \psi$ then $(I,J) \models_{GL} p$. J is a GL-answer set of P iff J is an AFT-stable model of P under strong Kleene truth assignment.

Proof
1. For every rule r^J in P^J there exists a counterpart r in P, however, the opposite is not true. Since the head of a rule is never changed in the reduct, proving equivalence of the relations corresponds to showing for every rule $p \leftarrow \psi$ that $(I,J) \models_{GL} \psi$ if and only if $(p \leftarrow \psi)^J$ exists and $I \models_2 \psi^J$. Any rule $p \leftarrow \psi$ in P is of the form $p \leftarrow l_1 \wedge ... \wedge l_n \wedge \neg t_1 \wedge ... \wedge \neg t_m$ with $l_1, ..., l_n, t_1, ..., t_m$ atoms. If there exists an atom t_i such that $t_i \in J$, then the rule is deleted during the first step of the construction of the reduct. Consequently, for every I we expect $(I,J) \not\models_{GL} \psi$. Indeed, from $t_i \in J$ the satisfaction relation deduces that $(I,J) \not\models_{GL} \neg t_i$ and therefore $(I,J) \not\models_{GL} \psi$. If for every $j \in [1,m]$ it holds that $t_j \notin J$, then $p \leftarrow \psi$ is transformed to $(p \leftarrow \psi)^I$ during the second step of the reduct-construction. This new rule is now given by $p \leftarrow l_1 \wedge ... \wedge l_n$. From the definition of the satisfaction relation $\models_2$ we know that $I \models_2 \psi^J$ if and only if $l_1, ..., l_n \in I$ and because $(I,J) \models_{GL} \neg t_i$ for all i, we have $(I,J) \models_{GL} \psi$. Hence $I \models_2 \psi^J$ iff $(I,J) \models_{GL} \psi$.
2. We now know that for $I \in [\bot, J]$, $I \models_2 P^J$ iff $(I,J) \models_{GL} P$ with P^J the GL-reduct of P for J. Since $\models_{GL}$ is lower-regular and P is non-disjunctive, we know that J is an AFT stable model iff J is an answer set of P by Proposition 4. J is an answer set of P iff $J \models_2 P$ and there is no $I \subset J$ such that $(I,J) \models_{GL} P$. Hence, J is an answer set of P iff $J \models_2 P$ and there is no $I \subset J$ such that $I \models_2 P^J$ iff J is a GL-answer set.

□

Proposition 6
Let $\models_a, \models_b$ be TSR's that coincide with $\models_{GL}$ on aggregate free bodies. If $\models_a \leq_p \models_b$ and J is an answer set associated with $\models_a$ (an a-answer set), then J is a b-answer set.

Proof
Assume that J is an a-answer set. This means that (i) for every $(p \leftarrow \psi) \in P$, if $J \models_2 \psi$, then $J \models_2 p$ and (ii) for every $I \subset J$ it holds that there exists a rule $(p \leftarrow \psi) \in P$ such that $(I,J) \models_a \psi$ and $(I,J) \not\models_a p$. For J to be a b-answer set, observe that (i) holds as it is a a-answer set; it remains to show that (ii) holds. Consider the same rule as above. By

$\models_a \leq_p \models_b$, we find that $(I, J) \models_b \psi$. Moreover, in the considered programs, p can only be a propositional atom. If $(I, J) \not\models_a p$, then $p \notin I$ and thus $(I, J) \not\models_b p$. Consequently, J is a b-answer set of P. □

Proposition 7
The TSRs $\models_{triv}$, $\models_{ult}$ and $\models_{bnd}$ are lower-regular. Since $\models_{triv} \leq_p \models_{bnd} \leq_p \models_{ult}$, an answer set of $\models_{triv}$ is one of $\models_{bnd}$, and one of $\models_{bnd}$ is one of $\models_{ult}$.

Proof
From definition 7.1 in the paper by Pelov *et al.* (2007), it follows that $\mathcal{H}^{triv}, \mathcal{H}^{ult}$ and $\mathcal{H}^{bnd}$ are regular. Then the first part of this proposition follows from Proposition 1, the second part from Proposition 6. □

Proposition 8
For aggregate programs containing only positive conditions in aggregate atoms, the TSR $\models_{GZ}$ is identical to the TSR $\models_{triv}$ and lower-regular for consistent pairs, i.e., with (I, J) a consistent pair, $(I, J) \models_{GZ} a^{Aggr}$ iff $(I, J) \models_{triv} a^{Aggr}$.

Proof
We only need to proof this for aggregate atoms as $\models_{GZ}$ is truth functional.

1. $(I, J) \models_{GZ} a^{Aggr}$ entails $(I, J) \models_{triv} a^{Aggr}$. By definition of $\models_{GZ}$, $(I, J) \models_{GZ} a^{Aggr}$ iff $J \models_2 a^{Aggr}$ and $(I, J) \models_{GZ} \bigwedge\{cond_j \in Cond(a^{Aggr})^1 | J \models_2 cond_j\}$. The conditions are positive, so they are evaluated in I, hence, given that $(I, J) \models_{GZ} a^{Aggr}$, $(I, J) \models_{GZ} \bigwedge\{cond_j \in Cond(a^{Aggr}) | J \models_2 cond_j\}$ and $I \models_2 \bigwedge\{cond_j \in Cond(a^{Aggr}) | J \models_2 cond_j\}$ and, for conditions in this set, $cond_i^I = cond_i^J$. For the other conditions (such that $J \not\models_2 cond_i$), they are false in J hence also false in $I \subseteq J$. So, for all conditions, $cond_i^I = cond_i^J$ and $(I, J) \models_{triv} a^{Aggr}$.
2. $(I, J) \models_{triv} a^{Aggr}$ entails $(I, J) \models_{GZ} a^{Aggr}$.
   By definition of $\models_{triv}$, if $(I, J) \models_{triv} a^{Aggr}$, then $J \models_2 a^{Aggr}$ and ${cond_i}^J = {cond_i}^I$ for every condition $cond_i \in Cond(a^{Aggr})$. Therefore, for every condition $cond_i \in \{cond_j \in Cond(a^{Aggr}) | J \models_2 cond_j\}$ it holds that $I \models_2 cond_i$. By definition of $\models_{GZ}$ for conjunction of literals, then $(I, J) \models_{GZ} \bigwedge\{cond_j \in Cond(a^{Aggr}) | J \models_2 cond_j\}$. Hence, $(I, J) \models_{GZ} a^{Aggr}$.
3. Since the satisfaction relation $\models_{GZ}$ is equivalent to $\models_{triv}$ for negation-free aggregate atoms and $\models_{triv}$ is lower-regular, $\models_{GZ}$ is also lower-regular.

□

Proposition 9
(i) $\models_{MR}$ extends $\models_2$, i.e., $(I, I) \models_{MR} \psi$ iff $I \models_2 \psi$. (ii) $\models_{MR}$ is lower-monotone, i.e., if $I \subseteq I'$ and $(I, J) \models_{MR} \psi$, then $(I', J) \models_{MR} \psi$.

Proof

[1] For $a^{Aggr} = Agg(\{a_1 : cond_1, \cdot, a_n : cond_n\}) * w$, $Cond(a^{Aggr}) = \{cond_i | i \in [1, n]\}$

1. The satisfaction relation is truth-functional hence we only need to prove this for aggregate atoms. We know that $(I, I) \models_{MR} a^{Aggr}$ iff $I \models_2 a^{Aggr}$ and there exists an interpretation $Z \subseteq I$ such that $Z \models_2 a^{Aggr}$. Take $Z = I$ and it is clear that $I \models_2 a^{Aggr}$ implies that there exists a Z that satisfies the requirements. This observation leads to $(I, I) \models_{MR} a^{Aggr}$ iff $I \models_2 a^{Aggr}$.
2. The part of the definition for $\models_{MR}$ that was taken from $\models_{GL}$ satisfies this property since $(I, J) \leq_p (I', J)$ and $\models_{GL}$ is lower-regular. For the rule regarding aggregates, it is clear that if $(I, J) \models_{MR} a^{Aggr}$, then $J \models_2 a^{Aggr}$ and there exists a $Z \subseteq I \subseteq I'$ such that $Z \models_2 a^{Aggr}$. Clearly, this means that $(I', J) \models_{MR} a^{Aggr}$, so $\models_{MR}$ is lower-monotone.

□

Proposition 10
For convex aggregate atoms, $\models_{MR}$ behaves lower-regular and equivalent with $\models_{ult}$.

Proof
1. We have already shown that it extends the satisfaction relation $\models_2$ for arbitrary aggregate atoms, including convex ones. All that is left to prove is its $\leq_p$-monotone behaviour. If $(I, J) \leq_p (I', J')$ and $(I, J) \models_{MR} a^{Aggr}$ with a^{Aggr} a convex aggregate atom, then $J \models_2 a^{Aggr}$ and there exists an interpretation $Z \subseteq I$ such that $Z \models_2 a^{Aggr}$. According to the definition of convex aggregate atoms, this implies that for every interpretation Z' such that $Z \subseteq Z' \subseteq J$, $Z' \models_2 a^{Aggr}$. This includes the interpretations I, I' and J'. From this, it is trivial to see that $(I', J') \models_{MR} a^{Aggr}$.
2. If $(I, J) \models_{MR} a^{Aggr}$, then $J \models_2 a^{Aggr}$ and there exists an interpretation $Z \subseteq I$ such that $Z \models_2 a^{Aggr}$. Since a^{Aggr} is convex, this means that for every X such that $Z \subseteq (I \subseteq) X \subseteq J$, $X \models_2 a^{Aggr}$. Thus, $(I, J) \models_{ult} a^{Aggr}$.
3. If $(I, J) \models_{ult} a^{Aggr}$, then $X \models_2 a^{Aggr}$ for every $I \subseteq X \subseteq J$, thus $J \models_2 a^{Aggr}$ and $I \models_2 a^{Aggr}$ with $I \subseteq I$. Therefore, $(I, J) \models_{MR} a^{Aggr}$.

□

Proposition 11
$\models_{FPL}$ extends $\models_2$, i.e., $(I, I) \models_{FPL} \psi$ iff $I \models_2 \psi$. For conjunctions of aggregate free literals, $\models_{FPL}$ coincides with $\models_{GL}$.

Proof
1. $\models_{FPL}$ extends $\models_2$:
   By definition $(I, I) \models_{FPL} \psi$ iff $I \models_2 \psi$ and $I \models_2 \psi$. Hence $(I, I) \models_{FPL} \psi$ iff $I \models_2 \psi$.
2. $\models_{FPL}$ coincides with $\models_{GL}$ for conjunctions of aggregate free literals:
   For literals l_i it is easy to see that: $(I, J) \models_{FPL} l_i$ iff $(I, J) \models_{GL} l_i$ since if $I \models_2 p$, then $J \models_2 p$ and if $J \models_2 \neg p$, then $I \models_2 \neg p$. For conjunctions of literals $\bigwedge_{i=1}^n l_i$, we have $(I, J) \models_{FPL} \bigwedge_{i=1}^n l_i$ iff $I \models_2 \bigwedge_{i=1}^n l_i$ and $J \models_2 \bigwedge_{i=1}^n l_i$. By the definition of $\models_2$ for conjunctions, this is equivalent to $I \models_2 l_i$ and $J \models_2 l_i$ for every l_i in the conjunction. This in turn is equivalent to $(I, J) \models_{FPL} l_i$. By means of the aforementioned result for literals, this is equivalent with $(I, J) \models_{GL} l_i$. By definition of $\models_{GL}$ for conjunctions of literals, we then find $(I, J) \models_{FPL} \bigwedge_{i=1}^n l_i$ iff $(I, J) \models_{GL} \bigwedge_{i=1}^n l_i$.

□

Proposition 12
For convex aggregate atoms, the ternary satisfaction relation $\models_{FPL}$ behaves lower-regular and equivalent to $\models_{ult}$.

Proof
1. Firstly, we proof: If $(I,J) \models_{ult} a^{Aggr}$, then $(I,J) \models_{FPL} a^{Aggr}$.
   If $(I,J) \models_{ult} a^{Aggr}$, then $X \models_2 a^{Aggr}$ for every $I \subseteq X \subseteq J$. Consequently, $I \models_2 a^{Aggr}$ and $J \models_2 a^{Aggr}$, hence $(I,J) \models_{FPL} a^{Aggr}$.
2. Secondly, we proof: If $(I,J) \models_{FPL} a^{Aggr}$, then $(I,J) \models_{MR} a^{Aggr}$.
   If $(I,J) \models_{FPL} a^{Aggr}$, then $I \models_2 a^{Aggr}$ and $J \models_2 a^{Aggr}$. Thus $J \models_2 a^{Aggr}$ and $I \models_2 a^{Aggr}$ with $I \subseteq I$. Therefore, $(I,J) \models_{MR} a^{Aggr}$.
3. This means that $\models_{FPL}$ is placed between $\models_{ult}$ and $\models_{MR}$ in the precision order. However, for programs with only convex aggregates, $\models_{ult}$ and $\models_{MR}$ coincide. Therefore, $\models_{FPL}$ will also coincide with these semantics.
4. Consequently, $\models_{FPL}$ is a lower-regular relation when we only consider convex aggregates.

□

Proposition 13
$\models_F$ extends $\models_2$, i.e., $(I,I) \models_F \psi$ if $I \models_2 \psi$.

Proof
The truth-assignment is truth-functional, therefore, we only need to prove the property for aggregate atoms. We know that $(I,I) \models_F a^{Aggr}$ iff $I \models_2 a^{Aggr}$ and $I \models_2 Agg(\{a_i : cond_i \in \{a_1 : cond_1, \ldots, a_n : cond_n\} | I \models_2 cond_i\}) * w$. It follows that if $I \models_2 a^{Aggr}$, then $I \models_2 Agg(\{a_i : cond_i \in \{a_1 : cond_1, \ldots, a_n : cond_n\} | I \models_2 cond_i\}) * w$. This observation leads to $(I,I) \models_F a^{Aggr}$ iff $I \models_2 a^{Aggr}$. □

Proposition 14
For convex aggregate atoms, $\models_F$ behaves lower-monotone, i.e., if $I \subseteq I'$ and $(I,J) \models_F \psi$, then $(I',J) \models_F \psi$.

Proof
$\models_F$ extends $\models_{GL}$ and the latter is lower-monotone. So we only need to consider a ψ that is a convex aggregate atom. If a^{Aggr} is a convex aggregate atom, then $a_J^{Aggr} = Agg(\{a_i : cond_i \in \{a_1 : cond_1, \ldots, a_n : cond_n\} | I \models_2 cond_i\}) * w$ is also convex since the transformation only deletes conditions from the original convex aggregate. This means that if $X \subseteq Y \subseteq Z$, $X \models_2 a_J^{Aggr}$, and $Z \models_2 a_J^{Aggr}$, then $Y \models_2 a^{Aggr}$. It is easy to see that if $J \models_2 a^{Aggr}$, then $J \models_2 a_J^{Aggr}$. Therefore, if $(I,J) \models_F a^{Aggr}$ and $(I,J) \leq_p (I',J)$, then $I' \models_2 a_J^{Aggr}$ and thus $(I',J) \models_F a^{Aggr}$. □

Proposition 15
For anti-monotone aggregate atoms, $\models_F$ behaves lower-regular and equivalent with $\models_{ult}$.

Proof

Let $a_J^{Aggr} = Agg(\{a_i : cond_i \in \{a_1 : cond_1, \ldots, a_n : cond_n\} | J \models_2 cond_i\}) * w$.

1. If $(I, J) \models_{ult} a^{Aggr}$, then $X \models_2 a^{Aggr}$ for every $I \subseteq X \subseteq J$. Thus, $J \models a^{Aggr}$ and therefore $J \models a_J^{Aggr}$. Since a^{Aggr} is anti-monotone and the only transformation between a^{Aggr} and a_J^{Aggr} deletes conditions, a_J^{Aggr} must be anti-monotone. Hence $I \models a_J^{Aggr}$ and $(I, J) \models_F a^{Aggr}$.
2. If $(I, J) \models_F a^{Aggr}$, then $J \models_2 a^{Aggr}$. From the anti-monotonicity of a^{Aggr} it follows that for all $X \subseteq J$, $X \models_2 a^{Aggr}$. Hence, $(I, J) \models_{ult} a^{Aggr}$.
3. An anti-monotone aggregate atom is convex and $\models_F$ and $\models_{ult}$ coincide for convex aggregate atoms. Therefore, for anti-monotone aggregate atoms, $\models_F$ behaves lower-regular.

□

References

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