Online appendix for the paper

On Signings and the Well-Founded Semantics

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Re-proofs

This appendix contains proofs of minor variations of theorems in (Kunen 1989), (Dung 1992), and (Turner 1993). This is just a bonus; the results of this paper do not depend on these results. The statements of the theorems now allow a floor, which is not signed, and have other small variations.

The following theorem is a minor variation of a theorem (Theorem 3.3) that was stated in (Kunen 1989) without proof.

Theorem 8 (Kunen 1989)
Let P be a logic program, $\mathcal{P} \subseteq \Pi$ be a downward-closed set of predicates with floor $\mathcal{F}$, let $\mathcal{Q}$ be $\mathcal{P}\backslash\mathcal{F}$, and s be a signing on $\mathcal{Q}$ for P. Let I be a fixed two-valued semantics for $\mathcal{F}$ and let A be a 3-valued model of Clark's completion P^* extending I. Then there is a two-valued model A' of P^* that extends A.
Specifically, A' can be defined as follows:
If $s(p) = +1$ then $\quad p(\vec{a}) \in A'$ iff $not\ p(\vec{a}) \notin A$
If $s(p) = -1$ then $not\ p(\vec{a}) \in A'$ iff $\quad p(\vec{a}) \notin A$

Proof
We need to establish that A' is a model of P^*, Clark's completion of P. Consider $ground(P)$, where the rules of P are grounded by the elements of A, and the corresponding completion $ground(P)^*$, which might involve infinite disjunctions. We will refer to the elements of $ground(P)^*$ of the form $a \leftrightarrow \vee_{i\in I} B_i$ as *defs*.

For each ground atom a, consider all ground rules with a as head, and the corresponding def. If $A(a) = \mathbf{true}$ then the body of some rule evaluates to $\mathbf{true}$ in A. If $A(a) = \mathbf{false}$ then in each rule, some literal evaluates to **false** in A. In both cases, the def for a is also satisfied by A'.

If $A(a) = \mathbf{unknown}$ then no rule body evaluates to $\mathbf{true}$ and some rule body B evaluates to $\mathbf{unknown}$ (that is, no body literal evaluates to **false** and at least one literal evaluates to $\mathbf{unknown}$) in A. Suppose $s(a) = +1$. Then $A'(a) = \mathbf{true}$, from the definition of A'. If an atom $b \in B$ evaluates to $\mathbf{unknown}$ in A then, because $s(b) = +1$, $A'(b) = \mathbf{true}$. If $not\ c \in B$ evaluates to $\mathbf{unknown}$ in A then, because $s(c) = -1$, $A'(c) = \mathbf{false}$. As a result, $A'(B) = \mathbf{true}$. Now suppose $s(a) = -1$. Then $A'(a) = \mathbf{false}$. If an atom $b \in B$ evaluates to $\mathbf{unknown}$ in A then, because $s(b) = -1$, $A'(b) = \mathbf{false}$. If $not\ c \in B$ evaluates to $\mathbf{unknown}$

in A then, because $s(c) = +1$, $A'(c) = \mathbf{true}$. As a result, $A'(B) = \mathbf{false}$. In both cases, the def for a is satisfied by A'. $\square$

When I is not already a 2-valued model of P^*, we can get a second 2-valued model by using $\bar{s}$, instead of s. This is similar to Turner's extension of a well-founded model to a stable model, below, where $\bar{s}$ provides a second stable model, provided the well-founded model is not stable itself.

To present Turner's theorem, we need to define stable models. Let P be a ground logic program and I be a 2-valued interpretation. Then the Gelfond-Lifschitz reduct of P wrt I, denoted by P^I, is obtained by deleting from P those rules with a negative literal that evaluates to **false** in I, and deleting those negative literals that evaluate to $\mathbf{true}$ in I from the remaining rules. A stable model is a 2-valued interpretation S such that the least model of P^S is S (Gelfond and Lifschitz 1988).

The proof of Theorem 2 of (Turner 1993) is very compact and in a notation I am not familiar with. The following proof of a variation of that theorem is also brief, and is more intuitive to me.

Theorem 9 (Turner 1993)
Let P be a logic program, $\mathcal{P} \subseteq \Pi$ be a downward-closed set of predicates with floor $\mathcal{F}$, let $\mathcal{Q}$ be $\mathcal{P}\backslash\mathcal{F}$, and s be a signing on $\mathcal{Q}$ for P. Let I be a fixed stable model for $\mathcal{F}$ and let W be the well-founded model of P extending I. Then there is a stable model S of P that extends W.
Specifically, S can be defined as follows:
If $s(p) = +1$ then $\quad p(\vec{a}) \in S$ iff $not\ p(\vec{a}) \notin W$
If $s(p) = -1$ then $not\ p(\vec{a}) \in S$ iff $\quad p(\vec{a}) \notin W$

Proof
We need to prove that S is a stable model. W is a 3-valued model of P^* (Van Gelder et al. 1991) so, by Theorem 8, S is a 2-valued model of P^* and, hence, also a model of P.

We use a characterization of stable models established in Theorem 2.5 of (Dung 1992). A model S of P is stable iff for all A, if A is unfounded wrt S then $A \cap S = \emptyset$.

We first prove by induction on the Kleene sequence that no atom $a \in W$ is in an unfounded set wrt S. If $a \in \mathcal{W} \uparrow (\beta + 1)$ then there is a rule a :- B in $ground(P)$ such that B evaluates to $\mathbf{true}$ in $\mathcal{W} \uparrow \beta$. By the induction hypothesis, every atom $b \in B$ is not in an unfounded set wrt S. Hence, a is not in an unfounded set wrt S, since B is neither **false** in S nor does B contain an atom from an unfounded set wrt S. If $a \in \mathcal{W} \uparrow \alpha$, for limit ordinal α, then $a \in \mathcal{W} \uparrow \beta$ for some $\beta < \alpha$ and, by the induction hypothesis, a is not in an unfounded set wrt S. Hence, if A is unfounded wrt S, then $A \subseteq S\backslash W$.

Let A be unfounded wrt S. Then, for any $a \in A$, $a \in S\backslash W$ and, by the definition of S, $s(a) = +1$. Furthermore, for any rule a :- B for a, if $b \in B$ then $s(b) = +1$ and if $not\ b \in B$ then $s(b) = -1$. Thus, for any $a \in A$ and rule a :- B, if B evaluates to **false** in S then, from the definition of S, B evaluates to **false** in W. (If $b \in B$ is **false** in S and $s(b) = +1$, then $not\ b \in W$; if $not\ b \in B$ is **false** in S $s(b) = -1$, then $b \in W$.) It follows that A is unfounded wrt W. Consequently, $not\ a \in W$, which contradicts $a \in S$. Hence, $A = \emptyset$.

Thus, by the characterization above, S is stable. $\square$

As observed above, a second stable model can be obtained by using $\bar{s}$, if W is not stable. Along similar lines, but not using signings, (Gire 1992; Gire 1994) showed that, supposing P is order-consistent, if W is not stable then P has at least two stable models. (P is order-consistent (Fages 1994) if $\leq_{\pm}$ is well-founded, where $p \leq_{\pm} q$ iff $p \leq_{+1} q$ and $p \leq_{-1} q$.)

Using the construction in the previous theorem, based on Dung's characterization of stable models (Dung 1992), we can give a shorter and more direct proof of a variant of Theorem 5.11 of (Dung 1992) than in that paper. The proof still uses the same idea as Dung. The statement of this theorem uses a signing, rather than strictness in (Dung 1992), but these two concepts are very closely related (see Section 2.1 of the paper).

The *sceptical stable semantics* extending an interpretation I is the set of all literals true in every stable model extending I.

Theorem 10 (Dung 1992)
Let P be a logic program, $\mathcal{P} \subseteq \Pi$ be a downward-closed set of predicates with floor $\mathcal{F}$, and let $\mathcal{Q}$ be $\Pi \backslash \mathcal{F}$. Let I be a fixed stable model for $\mathcal{F}$. Let W be the well-founded model of P extending I, and T be the sceptical stable semantics of P extending I.

If $\mathcal{Q}$ has a signing for P then $T = W$.

Proof
Every stable model extends the well-founded model, so $W \subseteq T$. Suppose, to obtain a contradiction, $\mathcal{Q}$ has a signing s for P and there is a literal L in $T \backslash W$. Let the predicate of L be p. Since the semantics of $\mathcal{F}$ is common to T and W, $p \notin \mathcal{F}$.

By Theorem 9, there is a stable model S extending I of P corresponding to s, and another S' corresponding to $\bar{s}$. Since $L \notin W$, by the construction in that theorem either $L \notin S$ or $L \notin S'$. But this contradicts the original supposition that L appears in every stable model. Thus, there is no such L and, hence, $T = W$. $\square$

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