Supplementary Material for the paper: "Incremental maintenance of overgrounded logic programs with tailored simplifications"

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The numbering of definitions, propositions, lemmas and theorems corresponds to the same statement numbering as in the main paper. Additional statements appearing only in this supplementary material are labelled with letters. We assume we are given a program P and a set of facts F.

Proposition A
For a ground logic program G and $A \in AS(G)$, $A \subseteq Heads(G)$.

Proposition B
For a ground logic program G and $A \in AS(G)$, $Facts(G) \subseteq A$.

The following Proposition re-adapts Theorem 6.22 (Leone et al. 1997).

Proposition C
For a given answer set $A \in AS(P \cup F)$, we can assign to each atom $a \in A$ an integer value $stage(a) = i$ so that *stage* encodes a strict well-founded partial order over all atoms in A, in such a way that there exists a rule $r \in grnd(P) \cup F$ structured s.t. $a \in H(r)$, $A \models B(r)$ and for any atom $b \in B^+(r)$, $stage(b) < stage(a)$.

Proposition D
For a given tailored embedding E for $P \cup F$, let us consider the superset $Facts(E)$ of F. We can assign to each atom $a \in Facts(E)$ an integer value $stage'(a) = i$ so that $stage'$ represents a strict well-founded partial order over all atoms in $Facts(E)$, in such a way that $hom(a)$ is structured as follows: $\{a\} = H(hom(a))$, $\forall b \in B^+(hom(a)), b \in Facts(E)$ and $stage'(a) < stage'(b)$.

Lemma E
For a tailored embedding E of $P \cup F$ and an answer set $A \in AS(P \cup F)$, $Facts(E) \subseteq A$.

Proof
The proof is given by induction on the function $stage'$ applied to $Facts(E)$ as given by Proposition D. W.l.o.g. we assign $stage'(a) = 1$ to each atom $a \in Facts(E) \cap Facts(P \cup F)$. These atoms clearly belong to A. We assume then that for each $a \in Facts(E)$ with $stage'(a) < j$ we know that $a \in A$, and show that this implies that for all $a \in Facts(E)$ for which $stage'(a) = j$, $a \in A$ as well. By Proposition D and the inductive hypothesis, we have that $hom(a)$ is such that each $b \in B^+(hom(a))$ belongs to A, and thus $A \models B^+(hom(a))$. Finally, the Lemma is proven by observing that $B^-(hom(a)) = \emptyset$. $\square$

Lemma F
Given a tailored embedding E of $P \cup F$ and an answer set $A \in AS(P \cup F)$. Then, for each $a \in A$ there exists a rule $r_a \in E$ s.t. $hom(r_a) \in (grnd(P) \cup F)$; thus, $A \subseteq Heads(E)$.

Proof
By Proposition C, each $a \in A$ is associated to an integer value $stage(a)$ and there exists a rule $r_a \in grnd(P) \cup F$, with $a \in H(r_a)$. Note that $r_a \in (grnd(P) \cup F)^A$ since $A \models B(r)$. We now show that $r_a \in hom(E)$ by induction on the *stage* associated to $a \in A$. W.l.o.g. we can assign $stage(a) = 1$, whenever r_a is such that $H(r_a) = \{a\}$, $B^+(r) = \emptyset$ and for all b s.t. *not* $b \in B^-(r)$ we have that $b \notin A$. When $stage(a) = 1$, since E is a tailored embedding for $P \cup F$, it is easy to check that $E \vdash_b r_a$, and thus $r_a \in E$.

Now, (inductive hypothesis) assume that for $stage(a) < j$, $r_a \in hom(E)$. We show that for $stage(a) = j$, $r_a \in hom(E)$. Given the above, r_a is such that for each $b \in B^+(r_a)$, $stage(b) < j$, and hence there exists a rule $r_b \in E$ with $b \in H(r_b)$. Hence $E \vdash_b r_a$. Since E is a tailored embedding for $P \cup F$, and thus $E \Vdash r_a$, we have that at least one of cases in Definition 4.2 apply. In particular:

- If the case 1 applies, $E \vdash_b r_a$ implies $r_a \in E$;
- If the case 2 applies, there is clearly a rule $r'_a \in E$ for which $hom(r'_a) = hom(r_a)$;
- If the case 3 applies, it must be that for some *not* $b \in B^-(r_a)$, $b \in Facts(E)$. But on the other hand $A \models B(r)$ and thus $b \notin A$. However, by Lemma E, $b \in A$, which leads to a contradiction.

We conclude that either the case 1 or the case 2, i.e., $a \in Heads(E)$. $\square$

Proposition 4.1
An embedding E for $P \cup F$ is a tailored embedding for $P \cup F$.

Proof
The proof is given in the main text. $\square$

Theorem 4.1
[Equivalence]. Given a tailored embedding program E for $P \cup F$, then $AS(grnd(P) \cup F) = AS(E)$.

Proof
We show that a given set of atoms A is in $AS(grnd(P) \cup F)$ iff A is in $AS(E)$. We split the proof in two parts.

[$\mathbf{AS(grnd(P) \cup F) \subseteq AS(E)}$]. Let $A \in AS(grnd(P) \cup F)$. We show that A is a minimal model of E^A. First we show that A is model for E^A. Indeed, let us assume that there is a simplified rule $r \in E^A$ such that $A \not\models r$. This can happen only if $A \models B(r)$ but $A \not\models H(r)$. However, $A \models hom(r)$, which implies that either:

- $A \not\models B(hom(r))$. This implies that $\exists l \in B(hom(r))$ such that $A \not\models l$. We have an immediate contradiction if $l \in B(r)$. Contradiction arises also if $l \notin B(r)$: indeed, since E is a tailored embedding, l does not appear in $B(r)$ only if the case 2 of Definition 4.2 has been applied, which means that a simplification of type 3 has been applied. By Lemma E, we have a contradiction, since $Facts(E) \subseteq A$ implies that l must appear in A.

- $A \models B(hom(r))$ and thus $A \models H(hom(r))$. Note that $A \models H(hom(r))$ implies that $A \models H(r)$ since $H(r) = H(hom(r))$.

We then show that there is no smaller model for E^A. Let us assume that there exist a set A', $A' \subset A$, which is a model for E^A and thus A is not a minimal model of E^A. Note that A is a minimal model of $(grnd(P) \cup F)^A$ and thus there must exist $r \in (grnd(P) \cup F)^A$ for which $A' \not\models r$.

Such a rule can be either such that:

(a) There is no $s \in E$ s.t. $r = hom(s)$;
(b) There is $s \in E$ s.t. $r = hom(s)$ and $s \notin E^A$;
(c) There exists $s \in E^A$ s.t. $r = hom(s)$.

We show that r cannot fall in the cases *(a)* and *(b)*, while the case *(c)* implies that A' cannot be a model for E^A.

Case (a). Since $r \in (grnd(P) \cup F)^A$ it is the case that $A \models H(r)$ and $A \models B(r)$. However, by Lemma F, we know that $A \subseteq Heads(E)$. Also, we know that $E \Vdash r$, but there is no $s \in E$ for which $r = hom(s)$. This means that r should be tailored either by the case 1 or 3 of Definition 4.2.

If the case 1 applies, then it must be that $E \nvdash_b r$ or $E \vdash_h r$. On the one hand, Lemma F forces us to conclude that $E \vdash_b r$; thus it should be the case that $E \vdash_h r$, which contradicts the assumption that r has no $s \in E$ for which $r = hom(s)$. If the case 3 applies, there exists *not* $a \in B^-(r)$ s.t. $a \in Facts(E)$. But by Lemma E, $Facts(E) \subseteq A$, which contradicts $A \models B(r)$.

Case (b). In this case, there is $s \in E$ s.t. $r = hom(s)$ and $s \notin E^A$; Again, note that $A \models H(r)$ and $A \models B(r)$, which in turn implies that $A \models B(s)$ and $A \models H(s)$. Thus this case cannot apply, since it turns out that $s \in E^A$.

Case (c). Since the two cases above cannot apply, r must fall in this latter case. Since $A' \not\models r$, it must be the case that $A' \not\models H(r)$ and $A' \models B(r)$. Note that $B(s) \subseteq B(r)$ and $H(s) = H(r)$. Thus, $A' \not\models H(s)$ and $A' \models B(s)$, which implies $A' \not\models s$. We conclude that A' cannot be a model for E^A.

[$\mathbf{AS(E)} \subseteq \mathbf{AS(grnd(P)} \cup \mathbf{F)}$]. Let $A \in AS(E)$. We first show that $A \models (grnd(P) \cup F)$. We split all the rules of $(grnd(P) \cup F)^A$ in two disjoint sets: $hom(E^A)$ and $(grnd(P) \cup F) \setminus hom(E^A)$.

For a rule $r \in hom(E^A)$, let s be such that $r = hom(s)$. We have that $A \models B(s)$ and $A \models H(s)$. Since $H(r) = H(s)$, this latter implies that $A \models H(r)$. Let us examine each literal $l \in B^*(s)$, which has been eliminated by the case 2 of Definition 4.2. We have that $l \in Facts(E)$, and thus $A \models l$ by Proposition B. We can thus conclude that $A \models B(r)$ and, consequently $A \models r$.

Let us now consider a rule $r \in (grnd(P) \cup F) \setminus hom(E^A)$. We show that $A \models r$. Let us assume, by contradiction that $A \not\models r$, i.e., $A \models B(r)$ but $A \not\models H(r)$. We distinguish two subcases: either $r \in hom(E)$, or $r \notin hom(E)$.

If $r \in hom(E)$, we let s be such that $r = hom(s)$. Since $r \notin hom(E^A)$, we have that $s \notin E^A$, i.e., $A \not\models B(s)$ which implies $A \not\models B(r)$, which contradicts the assumption that $A \not\models r$. If $r \notin hom(E)$, we however know that $E \Vdash r$. This can be either because of the case 1 or the case 3 of Definition 4.2.

If r falls in the case 1, we have that $hom(r) = r$ and either $E \nvdash_b r$ or $E \vdash_h r$. Since $r \notin hom(E)$, it must then be that $E \nvdash_b r$, i.e., there exists at least one $a \in B^+(r)$ s.t. it does not exist a rule $r' \in E$ for which $E \vdash_h r'$. Then, $a \notin A$ by proposition A and thus $A \not\models B(r)$.

If r falls in the case 3, we have that there exist a literal *not* $a \in B^-(r)$ for which $a \in Facts(E)$. Clearly, by proposition B, $a \in A$, and thus $A \not\models B(r)$.

Thus $A \models (grnd(P) \cup F)^A$. We know show that A is a minimal model for $(grnd(P) \cup F)^A$. Let us consider a set $A' \subset A$ and assume that $A' \models (grnd(P) \cup F)^A$. However, we know that A is a minimal model of E^A and thus $A' \not\models E^A$. We can show that this implies that $A' \not\models (grnd(P) \cup F)^A$. Indeed if $A' \not\models E^A$, then there exists a rule $r \in E^A$ for which $A' \not\models r$. This, as we will show implies that $A' \not\models hom(r)$ (note that it can be easily shown that $hom(r)$ belongs to $(grnd(P) \cup F)^A$).

Indeed, we know that $A' \models B(r)$ and $A' \not\models H(r)$. Also it is the case that $A' \models B(r), B^*(r)$. In fact if we assume, by contradiction, that $A' \not\models B(r), B^*(r)$ there should exist a literal $l \in B^*(r)$ for which $A' \not\models l$. l cannot be negative since $A \models l$ and $A' \subset A$. If l is positive, the case 2 of Definition 4.2 tells us that $l \in Facts(E)$, i.e., $Facts(E) \not\subset A'$, which in turn implies that A' cannot be a model for $(grnd(P) \cup F)^A$. This concludes the proof. $\square$

Proposition 4.2
[Intersection]. Given two tailored embeddings E_1 and E_2 for $P \cup F$, $E_1 \sqcap E_2$ is a tailored embedding for $P \cup F$.

Proof
Let $E = E_1 \sqcap E_2$, and let us consider a rule $r \in (grnd(P) \cup F)$. We show that $E \Vdash r$. Preliminarily, we observe two facts which hold by definition of simplified intersection and by the fact that both E_1 and E_2 are tailored embeddings. We are given a literal a and one of E_1 or E_2 (w.l.o.g., we choose E_1):

(a) $a \in Facts(E_1)$ implies that $a \in Facts(E)$.
(b) $a \notin Heads(E_1)$ implies that $a \notin Heads(E)$;

By contradiction, let us assume that $E \nVdash r$, and we split the proof in two parts, depending on whether $r \in hom(E)$ or whether $r \notin hom(E)$.

($r \in hom(E)$). This implies that there are rules $s \in E_1$, $q \in E_2$ and $t \in E$ such that $r = hom(s) = hom(q) = hom(t)$. Note that, for each (positive) literal $l \in B^*(t)$, the case 2 of Definition 4.2 can be applied i.e., $l \in Facts(E_1)$ or $l \in Facts(E_2)$ which implies $l \in Facts(E)$ (Fact (a) above);

($r \notin hom(E)$). In this case we have that either $r \notin hom(E_1)$ or $r \notin hom(E_2)$. W.l.o.g. we assume $r \notin hom(E_1)$. By Definition 4.2, this can be the case if either

1. $E_1 \nvdash_h r$ because there exists $a \in B^+(r)$ and $a \notin Heads(E_1)$. Note that Fact (b) implies that $a \notin Heads(E)$, hence $E \Vdash r$.
2. $E_1 \nvdash_b r$; this implies that $E \nvdash_b r$ hence $E \Vdash r$;
3. $E_1 \nvdash_h r$ because there exists *not* $a \in B^-(r)$, and $a \in Facts(E_1)$. Note that Fact (a) implies that $a \in Facts(E)$, hence $E \Vdash r$.

$\square$

Theorem 4.2
Let **TE** be the set of tailored embeddings of $P \cup F$; let $\mathscr{E} = Inst^\infty(P, F) \cup F$. Then,

$$Simpl^\infty(\mathscr{E}) = \prod_{T \in \mathbf{TE}} T.$$

Proof
Let $\mathscr{T} = \prod_{T \in \mathbf{TE}} T$. By Proposition 2 we notice that $\mathscr{E} = \prod_{E \in \mathbf{ES}} E$. The single argument operator *Simpl* is both deflationary and monotone when restricted over the complete lattice $(L, \sqsubseteq)$, where $L = \{T \in \mathbf{TE} | T \sqsubseteq \mathscr{E}\}$: thus, the iterative sequence $E^0 = sup_{\sqsubseteq}(L) = \mathscr{E}$, $E^{i+1} = Simpl(E^i)$ converges to the least fixpoint $inf_{\sqsubseteq}(\{T \in L | Simpl(T) \sqsubseteq T\}) = \mathscr{T} = Simpl(\mathscr{T})$. $\square$

Theorem 5.1
Let $G_1 = \text{INCRINST}(P, \emptyset, F_1)$. For each i s.t. $1 < i \leq n$, let $G_i = \text{INCRINST}(P, G_{i-1}, F_i)$. Then for each i s.t. $1 \leq i \leq n$, $AS(G_i \cup F_i) = AS(P \cup F_i)$.

Proof
The proof is given by induction on the shot indices. Let $AS_i = AS(P \cup F_i)$. In the base case $(i = 1)$, $AS(G_1 \cup F_1) = AS_1$ since the DESIMPL step has no effect and the ΔINST step coincides with the typical grounding procedure of (Faber et al. 2012). In the inductive case ($i > 1$), we assume that $G_i \cup F_i$ is a tailored embedding for $P \cup F_i$, and we show that $G_{i+1} \cup F_{i+1}$ is a tailored embedding for $P \cup F_{i+1}$. Let $G_{i+1} = \text{INCRINST}(P, G_i, F_{i+1})$. At the final iteration of the INCRINST algorithm, we have that $G_{i+1} = DG \cup Simpl^{\infty}(NR, DG \cup NR \cup F_{i+1})$, where DG is a desimplified version of G_i and NR is an additional set of rules both obtained by repeated application of DESIMPL and INCRINST steps.

Observe that $DG \cup F_i$ is such that $G_i \cup F_i \sqsubseteq DG_i \cup F_i$ and is a tailored embedding for $P \cup F_i$; then, let $AG_{i+1} = Inst^{\infty}(P, DG \cup F_{i+1})$. $DG \cup AG_{i+1} \cup F_{i+1}$ is a tailored embedding for $P \cup F_{i+1}$; it then follows that $DG \cup Simpl^{\infty}(AG_{i+1}) \cup F_{i+1}$ is a tailored embedding for $P \cup F_{i+1}$. Let $CG_{i+1} = \{s \in AG_{i+1} \mid \nexists r \in DG \text{ s.t. } hom(r) = hom(s)\}$. $DG_i \cup CG_{i+1} \cup F_{i+1}$ is a tailored embedding for $P \cup F_{i+1}$. Then we show that $CG_{i+1} \sqsubseteq NR$. It follows that $DG \cup NR \cup F_{i+1} = G_{i+1} \cup F_{i+1}$ is a tailored embedding for $P \cup F_{i+1}$, and that thus $DG \cup Simpl^{\infty}(NR, DG \cup NR \cup F_{i+1})$ is a tailored embedding for $P \cup F_{i+1}$, which concludes the proof. $\square$

References

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