Supplementary Material for the paper: "Incremental Answer Set Programming with Overgrounding"

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Lemma 1
It is given a program P, a set F of facts, an embedding program $\mathscr{E}$ of $P \cup F$ and an answer set $A \in AS(P \cup F)$. Then, for each $a \in A$ there exists a rule $r_a \in (grnd(P) \cup F)$ s.t. $a \in H(r_a)$ and $r_a \in \mathscr{E}$; thus, $A \subseteq Heads(\mathscr{E})$.

Proof
By Theorem 3.2, each $a \in A$ is associated to an integer value $stage(a)$ and there exists a rule $r_a \in grnd(P) \cup F$, with $a \in H(r_a)$. Note that $r_a \in (grnd(P) \cup F)^A$ since $A \models B(r)$. We now show that $r_a \in \mathscr{E}$ by induction on the *stage* associated to $a \in A$. If $stage(a) = 1$, r_a is such that $B(r_a) = \emptyset$. Hence, since $\mathscr{E}$ is an embedding program for $P \cup F$, and $\mathscr{E} \vdash_b r_a$, it must hold that $r_a \in \mathscr{E}$. Now, (inductive hypothesis) assume that for $stage(a) < j$, $r_a \in \mathscr{E}$. We show that for $stage(a) = j$, $r_a \in \mathscr{E}$. Indeed r_a is such that for each $b \in B^+(r_a)$, $stage(b) < j$, and hence there exists a rule $r_b \in \mathscr{E}$ with $b \in H(r_b)$. Hence $\mathscr{E} \vdash_b r_a$ and thus, since $\mathscr{E}$ is an embedding program for $P \cup F$, $\mathscr{E} \vdash_h r_a$, and $r_a \in \mathscr{E}$. □

Proof of Theorem 4.1
$[AS(grnd(P) \cup F) \subseteq AS(\mathscr{E})]$. Let $A \in AS(grnd(P) \cup F)$. We will show that $(grnd(P) \cup F)^A = \mathscr{E}^A$, thus the statement trivially follows. Indeed, since $\mathscr{E} \subseteq grnd(P) \cup F$, it holds that $\mathscr{E}^A \subseteq (grnd(P) \cup F)^A$. So, if $(grnd(P) \cup F)^A$ and $\mathscr{E}^A$ differ, there must exists a rule $r \in grnd(P) \setminus \mathscr{E}$, and, obviously, such that $r \in (grnd(P) \cup F)^A$. But since $\mathscr{E}$ is an embedding program for $P \cup F$, this means that $\mathscr{E} \nvdash B(r)$. However, $A \models B(r)$, and hence $\forall b \in B^+(r)$ we know that $b \in A$. By Lemma 1, $A \subseteq Heads(\mathscr{E})$, and then $\forall b \in B^+(r)$ there exists a rule $r' \in \mathscr{E}$ such that $b \in H(r')$, thus leading to a contradiction with $\mathscr{E} \nvdash B(r)$.
$[AS(\mathscr{E}) \subseteq AS(grnd(P) \cup F)]$. Let $A \in AS(\mathscr{E})$. We again show that $\mathscr{E}^A = (grnd(P) \cup F)^A$. Similarly to the case above, since $\mathscr{E}^A \subseteq (grnd(P) \cup F)^A$, there must exists a rule $r \in (grnd(P) \cup F) \setminus \mathscr{E}$, s.t. $\mathscr{E} \nvdash B(r)$. Moreover, $A \models B(r)$, and thus $\forall b \in B^+(r)$ we know that $b \in A$. However, A is an answer set for $\mathscr{E}$; this clearly means that $\forall b \in B^+(r)$ there exists $r' \in \mathscr{E}$ such that $b \in H(r')$, which in turn means that $\mathscr{E} \vdash B(r)$, thus leading to a contradiction.

Proof of Proposition 4.1

By contradiction, assume that $\mathscr{E}$ is not an embedding program for $P \cup F$. Then, $\exists r \in (grnd(P) \cup F)$ such that $\mathscr{E} \nvdash r$, that is, $\mathscr{E} \nvdash_h r$ and $\mathscr{E} \vdash_b r$. Since $\mathscr{E} \nvdash_h r$, we have that $r \notin \mathscr{E}$, and hence at least one of the following statements hold: (i) $r \notin \mathscr{E}_1$, (ii) $r \notin \mathscr{E}_2$. Without loss of generality, assume $r \notin \mathscr{E}_1$. By hypothesis, $\mathscr{E}_1$ is an embedding program for $P \cup F$, thus it must hold that $\mathscr{E}_1 \nvdash_b r$. Then, by definition, there exists $b \in B^+(r)$ s.t. $\nexists r' \in \mathscr{E}_1$ with $b \in H(r')$; this implies that such r' cannot exist in $\mathscr{E}$, thus contradicting the fact that $\mathscr{E} \vdash_b r$.

Proof of Theorem 4.2
($\Rightarrow$) Assume that $\mathscr{E}$ is an embedding program for $P \cup F$. By contradiction, assume there is a rule $r \in Inst(P, \mathscr{E}) \cup F$ such that $r \notin \mathscr{E}$. Clearly, $B^+(r) \subseteq Heads(\mathscr{E})$ (by definition of *Inst*). This means that $\mathscr{E} \vdash_b r$, and, since $\mathscr{E} \vdash r$, this implies $\mathscr{E} \vdash_h r$, i.e., $r \in \mathscr{E}$, thus contradicting our assumption.
($\Leftarrow$) Assume that for a set of rules $\mathscr{E}$, $\mathscr{E} \supseteq Inst(P, \mathscr{E}) \cup F$ and, by contradiction, that $\mathscr{E}$ is not an embedding program for $P \cup F$. Then, there must be a rule $r \in grnd(P) \cup F$ such that $\mathscr{E} \nvdash r$. Clearly, $r \notin \mathscr{E}$ (otherwise, we would have $\mathscr{E} \vdash_h r$), and $\mathscr{E} \vdash_b r$. This means that $B^+(r) \subseteq Heads(\mathscr{E})$ and thus $Inst(P, \mathscr{E}) \subseteq \mathscr{E}$ must contain r, contradicting our assumption.

Proof of Theorem 4.3
Let define the monotone operator $\overline{Inst}(P,F) = Inst(P,F) \cup F$, and consider the complete lattice of subsets of $grnd(P) \cup F$ under set containment.

The proof then follows by Theorem 4.2 and by Knaster-Tarski theorem (Tarski 1955) by observing that

$$Inst(P,F)^\infty \cup F = lfp(\overline{Inst}) = inf\{\mathscr{E} \subseteq grnd(P) \cup F \mid \overline{Inst}(P,F) \subseteq \mathscr{E}\} = \bigcap_{\mathscr{E} \in \mathscr{E}\mathscr{S}} \mathscr{E}$$

and that $Inst(P,F)^\infty \cup F$ is the least fixpoint for $\overline{Inst}(P,F)$.

Proof of Theorem 4.4
The proof follows from Theorem 4.1 and Proposition 4.1.

Proof of Theorem 5.1
Let $IU = Inst(P, UF_k)^\infty \cup F_i$, and for each i, $i \leq k$, $IF_i = Inst(P, F_i)^\infty \cup F_i$. Recall that each IF_i is an embedding for $P \cup F_i$.

By monotonicity of *Inst*, we have point (1) above and that $IU \supseteq IF_i$ for each $i \leq k$. Each IF_i is clearly an embedding program for $P \cup F_i$ by Theorem 4.3. We show that point 2 follows by showing that IU is an embedding program for $P \cup F_i$ and from Theorem 4.1.

For a given $i \leq k$, consider a rule $r \in (grnd(P) \cup F_i)$; If $r \in IU$ then $IU \vdash r$. Let us consider the case in which $r \in (grnd(P) \cup F_i) \setminus IU$. Note that $IF_i \vdash r$ and thus $IF_i \nvdash_b r$. Now, either $IU \nvdash_b r$ or $IU \vdash_b r$. In the former case, clearly $IU \vdash r$. In the latter case, we have that $\forall a \in B^+(r)$ $a \in Heads(IU)$, and this means that $r \in IU$ by definition of IU, thus contradicting the assumption that $r \notin IU$. Thus IU is an embedding for $P \cup F_i$ for all $i \leq k$.