Online appendix for the paper

Translating LPOD and CR-Prolog$_2$ into Standard Answer Set Programs

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Appendix A Proof of Proposition 1

Let S be a set of atoms and let σ be a signature. By $S|_\sigma$, we denote the projection of S onto σ. Let S' be a set of atoms. We say S *agrees with* S' *onto* σ if $S|_\sigma = S'|_\sigma$.

In the following proofs, whenever we talk about an LPOD program Π, we refer to (9) as its ordered disjunction part Π_{od}.

Lemma 1
Let Π be an answer set program, S an answer set of Π, and A an atom in S.

- **(a)** S is an answer set of $\Pi \cup \{A \leftarrow body\}$.
- **(b)** S is an answer set of $\Pi \cup \{head \leftarrow body\}$ if $S \not\models body$.
- **(c)** S is an answer set of $\Pi \setminus \{head \leftarrow body\}$ if $S \not\models body$.
- **(d)** S is an answer set of $\Pi \cup \{constraint\}$ if $S \models constraint$.
- **(e)** S is an answer set of $\Pi \setminus \{constraint\}$ if $S \models constraint$.

Here, $body$ is a conjunction of atoms in Π where each atom is possibly preceded by *not*, $head$ is a disjunction of atoms in Π, and $constraint$ is a rule of the form $\leftarrow body$.

Lemma 2
Let Π be an answer set program. Let r be a rule of the form $A \leftarrow B_1, \dots, B_m, not\ C_1, \dots, not\ C_n$ where A, B_i, C_j are atoms. Let S be a set of atoms such that $S \cap \{C_1, \dots, C_n\} = \phi$. Then S is an answer set of $\Pi \cup \{r\}$ iff S is an answer set of $\Pi \cup \{A \leftarrow B_1, \dots, B_m\}$.

Lemma 3
(Proposition 8 in (Ferraris 2011)) Let Π be an ASP program, Q be a set of atoms not occurring in Π. For each $q \in Q$, let $Def(q)$ be a formula that doesn't contain any atoms from Q. Then $X \mapsto X \setminus Q$ is a 1-1 correspondence between the answer sets of $\Pi \cup \{Def(q) \rightarrow q : q \in Q\}$ and the answer sets of Π.

Let Π be an LPOD with signature σ. By the definition of a split program of LPOD, there are $n_1 \times \dots \times n_m$ split programs of Π. Let $\Pi(k_1, \dots, k_m)$ denote a split program of Π, where for $1 \le i \le m$, $k_i \in \{1, \dots, n_i\}$ and rule i in Π is replaced by its k_i-th option:

$$C_i^{k_i} \leftarrow Body_i, not\ C_i^1, \dots, not\ C_i^{k_i-1} \tag{A1}$$

where $Body_i$ is the body of rule i.

Let $AP_\Pi(x_1,\dots,x_m)$, where $x_i \in [0,n_i]$, denote the assumption program obtained from Π by replacing each LPOD rule i with its x_i-th assumption, $O_i(x_i)$:

$$body_i \leftarrow Body_i \tag{A2}$$
$$\bot \leftarrow x_i = 0,\ body_i \tag{A3}$$
$$\bot \leftarrow x_i > 0,\ not\ body_i \tag{A4}$$
$$C_i^j \leftarrow body_i,\ x_i = j \qquad (\text{for } 1 \le j \le n_i) \tag{A5}$$
$$\bot \leftarrow body_i,\ x_i \ne j,\ not\ C_i^1,\dots,not\ C_i^{j-1},\ C_i^j \qquad (\text{for } 1 \le j \le n_i) \tag{A6}$$

where $Body_i$ is the body of rule i, and $body_i$ is an atom not occurring in Π.

Proposition 1 *For any LPOD Π of signature σ and any set S of atoms of σ, S is a candidate answer set of Π iff $S \cup \{body_i \mid S$ satisfies the body of rule i in $\Pi_{od}\}$ is an answer set of some assumption program of Π. More specifically,*

(a) *for any candidate answer set S of Π, let's obtain $x_1,\dots,x_m$ such that, for $1 \le i \le m$,*
- — $x_i = 0$ *if* $S \not\models Body_i$,
- — $x_i = k$ *if* $S \models Body_i$, *and* $C_i^k \in S$, *and* $C_i^j \notin S$ *for* $1 \le j \le k-1$,

then $\phi(S) = S \cup \{body_i \mid S$ satisfies the body of rule i in $\Pi_{od}\}$ is an answer set of $AP_\Pi(x_1,\dots,x_m)$;

(b) *for any answer set S' of any assumption program $AP_\Pi(x_1,\dots,x_m)$, $S'|_\sigma$ is a candidate answer set of Π.*

Proof.

(a) Let S be a candidate answer set of Π. We obtain $x_1,\dots,x_m$ such that, for $1 \le i \le m$,
- — $x_i = 0$ if $S \not\models Body_i$,
- — $x_i = k$ if $S \models Body_i$, and $C_i^k \in S$, and $C_i^j \notin S$ for $1 \le j \le k-1$.

We will prove that $\phi(S)$ is an answer set of $AP_\Pi(x_1,\dots,x_m)$. Since S is a candidate answer set of Π, S must be an answer set of some $\Pi(k_1,\dots,k_m)$. Let's consider any LPOD rule i in Π. We know rule i is replaced by one of its options (A1) in $\Pi(k_1,\dots,k_m)$. Let's obtain Π' from $\Pi(k_1,\dots,k_m)$ by replacing the option of rule i with $O_i(x_i)$. Recall that $Body_i$ represent the body of rule i. Let S' be $S \cup \{body_i \mid S \models Body_i\ \}$. We are going to prove S' is an answer set of Π'.

Since $x_i = j$ is not an atom, rule (A6) is strong equivalent to the following constraint

$$\leftarrow body_i, C_i^j, not\ C_i^1,\dots,not\ C_i^{j-1}, not\ x_i = j$$

thus Lemma 1 (d) applies to this rule. According to the assignments for $x_1,\dots,x_m$, it's obvious that rules (A3), (A4), (A6) are satisfied by $\phi(S)$.

- — If $S \not\models Body_i$, $S' \not\models body_i$. By Lemma 1 (c), S is an answer set of $\Pi(k_1,\dots,k_m)$ minus the option of rule i. Since rules (A3), (A4), (A6) are satisfied by S, and the bodies of rules (A2), (A5) are not satisfied by S, by Lemma 1 (d) and Lemma 1 (b), $S' = S$ is an answer set of Π'.
- — If $S \models Body_i$, then $S' \models body_i$, and $x_i > 0$, and at least one of the atoms in $\{C_i^1,\dots,C_i^{n_i}\}$ must be true, and the first atom among them that is true in S is $C_i^{x_i}$ (S satisfies $C_i^{x_i}$ and S doesn't satisfy C_i^j for $j \in \{1,\dots,x_i-1\}$). Let Π'' be the union of $\Pi(k_1,\dots,k_m)$ and the rule (A2), then by Lemma 3, S' is an answer set of

Π''. Assume for the sake of contradiction that $k_i < x_i$. By rule (A1), at least one of $\{C_i^1, \dots, C_i^{k_i}\}$ must be true in S, which contradicts with the fact that the first atom that is true in S is $C_i^{x_i}$. [5] Then there are 2 cases for k_i:

- if $k_i = x_i$, by Lemma 2, S' is an answer set of $\Pi'' \cup \{C_i^{x_i} \leftarrow body_i\}$ minus rule (A1). Consequently, by Lemma 1 (b), S' is an answer set of Π'' union rule (A5) minus rule (A1). Since rules (A3), (A4), (A6) are satisfied by S', by Lemma 1 (d), S' is an answer set of Π';
- if $k_i > x_i$, "*not* $C_i^{x_i}$" is in the body of rule (A1), then by Lemma 1 (c), S' is an answer set of Π'' minus rule (A1). Since $S \models C_i^{x_i}$, by Lemma 1 (a), S' is an answer set of $\Pi'' \cup \{C_i^{x_i} \leftarrow body_i\}$ minus rule (A1). Consequently, by Lemma 1 (b), S' is an answer set of Π'' union rule (A5) minus rule (A1). Since rules (A3), (A4), (A6) are satisfied by S', by Lemma 1 (d), S' is an answer set of Π'.

Consequently, $\phi(S)$ is an answer set of $AP_\Pi(x_1, \dots, x_m)$, which is obtained from $\Pi(k_1, \dots, k_m)$ by replacing each option of rule i of Π with $O_i(x_i)$ for $1 \le i \le m$.

(b) Let S' be an answer set of program $AP_\Pi(x_1, \dots, x_m)$. Let's consider any LPOD rule i in Π. Let's obtain Π' from $AP_\Pi(x_1, \dots, x_m)$ by replacing $O_i(x_i)$ with the k_i-th option of rule i where $k_i = x_i$ if $x_i > 0$, $k_i = 1$ if $x_i = 0$. We first prove $S = S' \setminus \{body_i\}$ is an answer set of Π'.

Since S' must satisfy rules (A3), (A4), (A6), by Lemma 1 (e), S' is an answer set of $AP_\Pi(x_1, \dots, x_m)$ minus rules (A3), (A4), (A6). By Lemma 1 (c), S' is an answer set of $AP_\Pi(x_1, \dots, x_m) \cup \{C_i^{x_i} \leftarrow body_i\}$ minus rules (A3), (A4), (A5), (A6). Note that by rule (A2), S' satisfies $body_i$ iff S' satisfies $Body_i$. There are 2 cases as follows.

— If $S' \models Body_i$, $S' \models body_i$. Since S' satisfies rules (A3) and (A5), we know $x_i > 0$ and S' satisfies $C_i^{x_i}$. Thus k_i equals to x_i. Assume for the sake of contradiction that the first atom among $\{C_i^1, \dots, C_i^{n_i}\}$ that is true in S' is C_i^j and $j < x_i$. Since S' satisfies rule (A6), S' satisfies $x_i = j$. Contradiction. Thus S' satisfies $C_i^{x_i}$ and doesn't satisfy C_i^j for $j \in \{1, \dots, x_i - 1\}$. By Lemma 2, S' is an answer set of $AP_\Pi(x_1, \dots, x_m)$ union rule (A1) minus rules (A3), (A4), (A5), (A6). By Lemma 3, S is an answer set of Π'.

— If $S' \not\models Body_i$, $S' \not\models body_i$. By lemma 1 (c), S' is an answer set of $AP_\Pi(x_1, \dots, x_m)$ minus rules (A2), (A3), (A4), (A5), (A6). By Lemma 1 (b), $S = S'$ is an answer set of Π'.

So S is an answer set of Π'. Consequently, $S'|_\sigma$ is an answer set of $\Pi(k_1, \dots, k_m)$, where $k_i = x_i$ if $x_i > 0$, $k_i = 1$ if $x_i = 0$. In other words, $S'|_\sigma$ is a candidate answer set of Π.

□

Appendix B Proof of Proposition 2

For any answer set program Π, let $gr(\Pi, x_1, \dots, x_m)$ be a partial grounded program obtained from Π by replacing variables $X_1, \dots, X_m$ in Π with $x_1, \dots, x_m$.

[5] For example, suppose $k_i = 2$, and $x_i = 3$ is the index of the first atom in $\{C_i^1, \dots, C_i^{n_1}\}$ that is true in S. Since S satisfies the k_i-th option of rule i — "$C^2 \leftarrow body, not\ C^1$", and $S \models body$, then either C^1 is true or C^2 is true, which contradicts with the fact that C^3 is the first atom to be true in S.

Let Π be an LPOD of signature σ. In the following proofs, let $\mathsf{lpod2asp}(\Pi)$ be $\Pi_1 \cup \Pi_2 \cup \Pi_3$, where Π_1 consists of the rules in bullets 1 and 2 in section **Generate Candidate Answer Sets**, Π_2 consists of the rules in bullet 3 in the same section, and Π_3 consists of the rules in section **Find Preferred Answer Sets**. Note that $\mathsf{lpod2asp}(\Pi)_{base}$ is $\Pi_1 \cup \Pi_2$.

The proof of **Proposition 2** will use a restricted version of the splitting theorem from (Ferraris et al. 2009), which is reformulated as follows:

Splitting Theorem *Let Π_1, Π_2 be two answer set programs, $\mathbf{p}$, $\mathbf{q}$ be disjoint tuples of distinct atoms. If*

- *each strongly connected component of the dependency graph of $\Pi_1 \cup \Pi_2$ w.r.t. $\mathbf{p} \cup \mathbf{q}$ is a subset of $\mathbf{p}$ or a subset of $\mathbf{q}$,*
- *no atom in $\mathbf{p}$ has a strictly positive occurrence in Π_2, and*
- *no atom in $\mathbf{q}$ has a strictly positive occurrence in Π_1,*

then an interpretation I of $\Pi_1 \cup \Pi_2$ is an answer set of $\Pi_1 \cup \Pi_2$ relative to $\mathbf{p} \cup \mathbf{q}$ if and only if I is an answer set of Π_1 relative to $\mathbf{p}$ and I is an answer set of Π_2 relative to $\mathbf{q}$.

Proposition 2 *The candidate answer sets of an LPOD Π of signature σ are exactly the candidate answer sets on σ of $\mathsf{lpod2asp}(\Pi)_{base}$. In other words, (for any set S, let $\phi(S)$ be $S \cup \{body_i \mid S$ satisfies the body of rule i in Π_{od} $\}$)*

- **(a)** *for any candidate answer set S of Π, there are $x_1, \dots, x_m$ such that $\phi(S)$ is an answer set of $AP_\Pi(x_1, \dots, x_m)$, and there exists an optimal answer set S' of $\Pi_1 \cup \Pi_2$ such that $S' \models ap(x_1, \dots, x_m)$ and $S = shrink(S', x_1, \dots, x_m)$;*
- **(b)** *for any optimal answer set S' of $\Pi_1 \cup \Pi_2$ and any $x_1, \dots, x_m$ such that $S' \models ap(x_1, \dots, x_m)$, $S = shrink(S', x_1, \dots, x_m)$ is a candidate answer set of Π, and $\phi(S)$ is an answer set of $AP_\Pi(x_1, \dots, x_m)$.*

Proof. Let $\Pi_{1,2}$ be $\Pi_1 \cup \Pi_2$. According to the translation, the empty set is always an answer set of $\Pi_{1,2}$ (since the empty set doesn't satisfy the body of any rule in $\Pi_{1,2}$), thus there must exist at least one optimal answer set of $\Pi_{1,2}$. Furthermore, by rule (11), the optimal answer set should contain as many $ap(*)$ as possible. Then $gr(\Pi_{1,2}, x_1, \dots, x_m)$ is $gr(\Pi_1, x_1, \dots, x_m) \cup gr(\Pi_2, x_1, \dots, x_m)$. Let $\Pi_{1,2}^{gr}$ be $\bigcup_{y_i \in \{0,\dots,n_i\}} gr(\Pi_{1,2}, y_1, \dots, y_m)$. Let $\sigma^{\Pi_{1,2}^{gr}}$ be the signature of $\Pi_{1,2}^{gr}$, let $\sigma^{gr(\Pi_{1,2}, x_1, \dots, x_m)}$ be the signature of $gr(\Pi_{1,2}, x_1, \dots, x_m)$, let $\sigma^{gr(\Pi_1, x_1, \dots, x_m)}$ be the signature of $gr(\Pi_1, x_1, \dots, x_m)$, and let $\sigma^{gr(\Pi_2, x_1, \dots, x_m)}$ be $\sigma^{gr(\Pi_{1,2}, x_1, \dots, x_m)} \setminus \sigma^{gr(\Pi_1, x_1, \dots, x_m)}$. We then prove bullets **(a)** and **(b)** as follows.

- **(a)** Let S be a candidate answer set of Π. By Proposition 1, $\phi(S)$ must be an answer set of some $AP_\Pi(x_1, \dots, x_m)$ of Π. Let $\psi(S)$ be

  $$\{a(\mathbf{v}, x_1, \dots, x_m) \mid a(\mathbf{v}) \in S\} \cup \{body_i(x_1, \dots, x_m) \mid S \text{ satisfies the body of rule } i \text{ in } \Pi_{od}\}$$
  $$\cup \{ap(x_1, \dots, x_m), degree(ap(x_1, \dots, x_m), d_1, \dots, d_m)\},$$

  where $d_i = 1$ if $x_i = 0$, $d_i = x_i$ if $x_i > 0$. Our target is to construct an S' from $\psi(S)$ and prove S' is an optimal answer set of $\Pi_{1,2}$ such that $S' \models ap(x_1, \dots, x_m)$, and $S = shrink(S', x_1, \dots, x_m)$.

  First, we prove $\psi(S)$ is an optimal answer set of $gr(\Pi_{1,2}, x_1, \dots, x_m)$.

  1. By the construction of $\psi(S)$, $\psi(S)$ satisfies the reduct of $gr(\Pi_2, x_1, \dots, x_m)$ relative to $\psi(S)$, and is minimal with respect to $\sigma^{gr(\Pi_2, x_1, \dots, x_m)}$. So $\psi(S)$ is an answer set of $gr(\Pi_2, x_1, \dots, x_m)$ with respect to $\sigma^{gr(\Pi_2, x_1, \dots, x_m)}$.

2. Since $\phi(S)$ is a minimal model of the reduct of $AP_\Pi(x_1,\dots,x_m)$ relative to $\phi(S)$, and $\psi(S) \models ap(x_1,\dots,x_m)$, it's easy to check that $\psi(S)$ is a minimal model of the reduct of $gr(\Pi_1,x_1,\dots,x_m)$ relative to $\psi(S)$ with respect to $\sigma^{gr(\Pi_1,x_1,\dots,x_m)}$. So $\psi(S)$ is an answer set of $gr(\Pi_1,x_1,\dots,x_m)$ with respect to $\sigma^{gr(\Pi_1,x_1,\dots,x_m)}$.

By the splitting theorem, $\psi(S)$ is an answer set of $gr(\Pi_{1,2},x_1,\dots,x_m)$. Since $\psi(S)$ satisfies $ap(x_1,\dots,x_m)$, which is the only $ap(*)$ occurring in $gr(\Pi_{1,2},x_1,\dots,x_m)$, $\psi(S)$ must be an optimal answer set of $gr(\Pi_{1,2},x_1,\dots,x_m)$.
Then, we construct an optimal answer set S' of $\Pi_{1,2}$ from any optimal answer set S'' of $\Pi_{1,2}$ such that $S' \models ap(x_1,\dots,x_m)$ and $S = shrink(S',x_1,\dots,x_m)$.
We first show that S'' must satisfy $ap(x_1,\dots,x_m)$. Assume for the sake of contradiction that S'' does not satisfy $ap(x_1,\dots,x_m)$. Since each partial grounded program of $\Pi_{1,2}$ is disjoint from each other, by the splitting theorem, $S''|_{\sigma^{gr(\Pi_{1,2},x_1,\dots,x_m)}}$ is an answer set of $gr(\Pi_{1,2},x_1,\dots,x_m)$ and $S'' \setminus S''|_{\sigma^{gr(\Pi_{1,2},x_1,\dots,x_m)}}$ is an answer set of $\Pi^{gr}_{1,2} \setminus gr(\Pi_{1,2},x_1,\dots,x_m)$. Let S' be the union of $\psi(S)$ and $S'' \setminus S''|_{\sigma^{gr(\Pi_{1,2},x_1,\dots,x_m)}}$, since $\psi(S)$ is an answer set of $gr(\Pi_{1,2},x_1,\dots,x_m)$, by the splitting theorem, S' is an answer set of $\Pi_{1,2}$. Since S' has a lower penalty than S'', S'' is not an optimal answer set of $\Pi_{1,2}$, which contradicts with our initial assumption. So S'' must satisfy $ap(x_1,\dots,x_m)$. Indeed, if there exists an answer set of $AP_\Pi(x_1,\dots,x_m)$,

$$\text{any optimal answer set of } \Pi_{1,2} \text{ must satisfy } ap(x_1,\dots,x_m). \tag{B1}$$

Consequently, S' has the same penalty as S'' in $\Pi_{1,2}$, which means that S' is an optimal answer set of $\Pi_{1,2}$. Besides, S equals to $shrink(\psi(S),x_1,\dots,x_m)$, which equals to $shrink(S',x_1,\dots,x_m)$.

(b) Let S' be an optimal answer set of $\Pi_{1,2}$ and $x_1,\dots,x_m$ a list of integers such that $S' \models ap(x_1,\dots,x_m)$. Our target is to prove $S = shrink(S',x_1,\dots,x_m)$ is a candidate answer set of Π. By Proposition 1, it is sufficient to prove that $\phi(S)$ is an answer set of $AP_\Pi(x_1,\dots,x_m)$.
We first split $\Pi^{gr}_{1,2}$ into $gr(\Pi_1,x_1,\dots,x_m)$ and the remaining part $\Pi^{gr}_{1,2} \setminus gr(\Pi_1,x_1,\dots,x_m)$. Since

1. no atom in $\sigma^{gr(\Pi_1,x_1,\dots,x_m)}$ has a strictly positive occurrence in $\Pi^{gr}_{1,2} \setminus gr(\Pi_1,x_1,\dots,x_m)$,
2. no atom in $\sigma^{\Pi^{gr}_{1,2}} \setminus \sigma^{gr(\Pi_1,x_1,\dots,x_m)}$ has a strictly positive occurrence in $gr(\Pi_1,x_1,\dots,x_m)$, and
3. each strongly connected component of the dependency graph of $\Pi_{1,2}$ w.r.t. $\sigma^{\Pi^{gr}_{1,2}}$ is a subset of $\sigma^{gr(\Pi_1,x_1,\dots,x_m)}$ or $\sigma^{\Pi^{gr}_{1,2}} \setminus \sigma^{gr(\Pi_1,x_1,\dots,x_m)}$,

by the splitting theorem, S' is an answer set of $gr(\Pi_1,x_1,\dots,x_m)$ with respect to $\sigma^{gr(\Pi_1,x_1,\dots,x_m)}$. So $S'|_{\sigma^{gr(\Pi_1,x_1,\dots,x_m)}}$ is a minimal model of the reduct of $gr(\Pi_1,x_1,\dots,x_m)$ relative to $S'|_{\sigma^{gr(\Pi_1,x_1,\dots,x_m)}}$. Since $S'|_{\sigma^{gr(\Pi_1,x_1,\dots,x_m)}} \models ap(x_1,\dots,x_m)$, it's easy to check that $\phi(S)$ is a minimal model of the reduct of $AP_\Pi(x_1,\dots,x_m)$ relative to $\phi(S)$, where the reduct can be obtained from the reduct of $gr(\Pi_1,x_1,\dots,x_m)$ relative to $S'|_{\sigma^{gr(\Pi_1,x_1,\dots,x_m)}}$ by replacing each occurrence of $ap(x_1,\dots,x_m)$ with $\top$, and replacing each occurrence of $a(\mathbf{v},x_1,\dots,x_m)$ by $a(\mathbf{v})$ where $a(\mathbf{v}) \in \sigma$. Thus $\phi(S)$ is an answer set of $AP_\Pi(x_1,\dots,x_m)$. By Proposition 1, S is a candidate answer set of Π.

□

Appendix C Proof of Theorem 1

Let Π be an LPOD of signature σ. Recall that we let $\mathsf{lpod2asp}(\Pi)$ be $\Pi_1 \cup \Pi_2 \cup \Pi_3$, where Π_1 consists of the rules in bullets 1 and 2 in section **Generate Candidate Answer Sets**, Π_2 consists of the rules in bullet 3 in the same section, and Π_3 consists of the rules in section **Find Preferred Answer Sets**.

Lemma 4
Let Π be an LPOD. There is a 1-1 correspondence between the answer sets of $\mathsf{lpod2asp}(\Pi)$ and the answer sets of $\Pi_1 \cup \Pi_2$, and any answer set of $\mathsf{lpod2asp}(\Pi)$ agrees with the corresponding answer set of $\Pi_1 \cup \Pi_2$ on the signature of $\Pi_1 \cup \Pi_2$.

Proof. Let $\Pi_{1,2}$ be $\Pi_1 \cup \Pi_2$. Let's take $\Pi_{1,2}$ as our current program, Π_{cur}, and consider including the translation rules in Π_3 (rules (22) — (36) under each preference criterion) into Π_{cur}. For each criterion, let's include the first rule, e.g., rule (22), into Π_{cur}, it's easy to see that this rule satisfies the condition of Lemma 3. By Lemma 3, there is a 1-1 correspondence between the answer sets of Π_{cur} and the answer sets of $\Pi_{1,2}$. Similarly, if we further include the second rule, e.g., rule (23), into Π_{cur}, there is still a 1-1 correspondence between the answer sets of Π_{cur} and the answer sets of $\Pi_{1,2}$. Similarly, we can include more rules from Π_3 into the current program Π_{cur} in order, and consequently, there is a 1-1 correspondence between the answer sets of $\Pi_{1,2} \cup \Pi_3$ and the answer sets of $\Pi_{1,2}$. Since all the atoms introduced by Π_3 are not in the signature of $\Pi_{1,2}$, any answer set of $\mathsf{lpod2asp}(\Pi)$ agrees with the corresponding answer set of $\Pi_{1,2}$ on the signature of $\Pi_{1,2}$. $\square$

Lemma 5
Let S be a candidate answer set of an LPOD Π. If $\phi(S) = S \cup \{body_i \mid S$ satisfies the body of rule i in $\Pi_{od}\}$ is an answer set of $AP_\Pi(x_1, \dots, x_m)$ for some $x_1, \dots, x_m$, then for $1 \le i \le m$, S satisfies rule i of Π_{od} to degree 1 if $x_i = 0$, to degree x_i if $x_i > 0$. [6]

Proof. Since $\phi(S)$ is an answer set of $AP_\Pi(x_1, \dots, x_m)$, for $1 \le i \le m$, S satisfies rules (A3), (A4), which are equivalent to:

$$x_i = 0 \leftrightarrow \neg body_i$$
$$x_i > 0 \leftrightarrow body_i$$

If $x_i = 0$, $\phi(S) \not\models body_i$. So the body of rule i is not satisfied by S, which means rule i is satisfied at (i.e., satisfied to) degree 1. If $x_i > 0$, $\phi(S) \models body_i$. By rule (A6), the first atom in the head of rule i that is true in $\phi(S)$, and also S, is C^{x_i}, which means that rule i is satisfied by S at degree x_i. $\square$

Lemma 6
Let Π be an LPOD (9). Let $AP_\Pi(x_1, \dots, x_m)$ and $AP_\Pi(y_1, \dots, y_m)$ be two programs that are consistent, where the list $x_1, \dots, x_m$ is different from $y_1, \dots, y_m$. Let S_1 be an answer set of $AP_\Pi(x_1, \dots, x_m)$, S_2 be an answer set of $AP_\Pi(y_1, \dots, y_m)$. Then

- **(a)** there exists an optimal answer set K of $\mathsf{lpod2asp}(\Pi)$ such that $K \models ap(x_1, \dots, x_m)$, $K \models ap(y_1, \dots, y_m)$, $S_1|_\sigma = shrink(K, x_1, \dots, x_m)$, and $S_2|_\sigma = shrink(K, y_1, \dots, y_m)$;
- **(b)** any optimal answer set K of $\mathsf{lpod2asp}(\Pi)$ must satisfy $ap(x_1, \dots, x_m)$ and $ap(y_1, \dots, y_m)$.

[6] This lemma won't hold if $AP_\Pi(x_1, \dots, x_m)$ is replaced by $\Pi(k_1, \dots, k_m)$.

Proof. **(a)** Let $\mathsf{lpod2asp}(\Pi)$ be $\Pi_1 \cup \Pi_2 \cup \Pi_3$ as defined before. By Lemma 4, it is sufficient to prove that there exists an optimal answer set L of $\Pi_1 \cup \Pi_2$ such that $L \vDash ap(x_1, \dots, x_m)$, $L \vDash ap(y_1, \dots, y_m)$, $S_1|_\sigma = shrink(L, x_1, \dots, x_m)$, and $S_2|_\sigma = shrink(L, y_1, \dots, y_m)$.

Let $\Pi_{1,2}$ be $\Pi_1 \cup \Pi_2$. By Proposition 2, there exists an optimal answer set L_2 of $\Pi_{1,2}$ such that $L_2 \vDash ap(y_1, \dots, y_m)$, and $S_2|_\sigma = shrink(L_2, y_1, \dots, y_m)$. Let $\psi(S_1)$ be $\{a(\mathbf{v}, x_1, \dots, x_m) \mid a(\mathbf{v}) \in S_1\} \cup \{body_i(x_1, \dots, x_m) \mid S_1$ satisfies the body of rule i in $\Pi_{od}\} \cup \{ap(x_1, \dots, x_m)$, $degree(ap(x_1, \dots, x_m), d_1, \dots, d_m)\}$, where $d_i = 1$ if $x_i = 0$, $d_i = x_i$ if $x_i > 0$. Let L be the union of $\psi(S_1)$ and $L_2 \setminus L_2|_{\sigma^{gr(\Pi_{1,2}, x_1, \dots, x_m)}}$. It's easy to see that $L \vDash ap(x_1, \dots, x_m)$, $L \vDash ap(y_1, \dots, y_m)$, $S_1|_\sigma = shrink(L, x_1, \dots, x_m)$, and $S_2|_\sigma = shrink(L, y_1, \dots, y_m)$. Besides, L has the same penalty as L_2. So to prove Lemma 6 **(a)**, it is sufficient to prove that L is an answer set of $\Pi_1 \cup \Pi_2$.

First, we prove $\psi(S_1)$ is an answer set of $gr(\Pi_{1,2}, x_1, \dots, x_m)$.

1. By the construction of $\psi(S_1)$, $\psi(S_1)$ satisfies the reduct of $gr(\Pi_2, x_1, \dots, x_m)$ relative to $\psi(S_1)$, and is minimal with respect to $\sigma^{gr(\Pi_2, x_1, \dots, x_m)}$. So $\psi(S_1)$ is an answer set of $gr(\Pi_2, x_1, \dots, x_m)$ relative to $\sigma^{gr(\Pi_2, x_1, \dots, x_m)}$.
2. Since S_1 is a minimal model of the reduct of $AP_\Pi(x_1, \dots, x_m)$ relative to S_1, and $\psi(S_1) \vDash ap(x_1, \dots, x_m)$, it's easy to check that $\psi(S_1)$ is a minimal model of the reduct of $gr(\Pi_1, x_1, \dots, x_m)$ relative to $\psi(S_1)$ with respect to $\sigma^{gr(\Pi_1, x_1, \dots, x_m)}$. So $\psi(S_1)$ is an answer set of $gr(\Pi_1, x_1, \dots, x_m)$ relative to $\sigma^{gr(\Pi_1, x_1, \dots, x_m)}$.

By the splitting theorem, $\psi(S_1)$ is an answer set of $gr(\Pi_{1,2}, x_1, \dots, x_m)$.

Second, let $\Pi_{1,2}^{gr}$ be $\bigcup_{y_i \in \{0, \dots, n_i\}} gr(\Pi_{1,2}, y_1, \dots, y_m)$. Since each partial grounded program of $\Pi_{1,2}$ is disjoint from each other, by the splitting theorem, $L_2|_{\sigma^{gr(\Pi_{1,2}, x_1, \dots, x_m)}}$ is an answer set of $gr(\Pi_{1,2}, x_1, \dots, x_m)$ and $L_2 \setminus L_2|_{\sigma^{gr(\Pi_{1,2}, x_1, \dots, x_m)}}$ is an answer set of $\Pi_{1,2}^{gr} \setminus gr(\Pi_{1,2}, x_1, \dots, x_m)$.

Finally, by the splitting theorem, L is an answer set of $\Pi_1 \cup \Pi_2$.

(b) Let $\mathsf{lpod2asp}(\Pi)$ be $\Pi_1 \cup \Pi_2 \cup \Pi_3$ as defined before. By Lemma 4, it is sufficient to prove that any optimal answer set L of $\Pi_1 \cup \Pi_2$ must satisfy $ap(x_1, \dots, x_m)$ and $ap(y_1, \dots, y_m)$. Since S_1 is an answer set of $AP_\Pi(x_1, \dots, x_m)$, and S_2 is an answer set of $AP_\Pi(y_1, \dots, y_m)$, by (B1), any optimal answer set L of $\Pi_1 \cup \Pi_2$ must satisfy $ap(x_1, \dots, x_m)$ and $ap(y_1, \dots, y_m)$. $\square$

Lemma 7

The candidate answer sets of an LPOD Π of signature σ are exactly the candidate answer sets on σ of $\mathsf{lpod2asp}(\Pi)$. In other words, (for any set S of atoms, let $\phi(S)$ be $S \cup \{body_i \mid S$ satisfies the body of rule i in Π_{od} $\})$

- **(a)** for any candidate answer set S of Π, there are $x_1, \dots, x_m$ such that $\phi(S)$ is an answer set of $AP_\Pi(x_1, \dots, x_m)$, and there exists an optimal answer set K of $\mathsf{lpod2asp}(\Pi)$ such that $K \vDash ap(x_1, \dots, x_m)$ and $S = shrink(K, x_1, \dots, x_m)$;
- **(b)** for any optimal answer set K of $\mathsf{lpod2asp}(\Pi)$ and any $x_1, \dots, x_m$ such that $K \vDash ap(x_1, \dots, x_m)$, $S = shrink(K, x_1, \dots, x_m)$ is a candidate answer set of Π, and $\phi(S)$ is an answer set of $AP_\Pi(x_1, \dots, x_m)$.

Proof.

- **(a)** Let S be a candidate answer set of Π. Let $\mathsf{lpod2asp}(\Pi)$ be $\Pi_1 \cup \Pi_2 \cup \Pi_3$ as defined before. By Proposition 2, there are $x_1, \dots, x_m$ such that $\phi(S)$ is an answer set of $AP_\Pi(x_1, \dots, x_m)$, and there exists an optimal answer set S' of $\Pi_1 \cup \Pi_2$ such that $S' \vDash ap(x_1, \dots, x_m)$ and $S = shrink(S', x_1, \dots, x_m)$. By Lemma 4, there exists an answer set K of $\mathsf{lpod2asp}(\Pi)$

such that K agrees with S' on the signature of $\Pi_1 \cup \Pi_2$. Thus $K \vDash ap(x_1, \dots, x_m)$ and $S = shrink(K, x_1, \dots, x_m)$. Since the signature of $\Pi_1 \cup \Pi_2$ includes all $ap(*)$ atoms and S' is an optimal answer set of $\Pi_1 \cup \Pi_2$, K is an optimal answer set of $\mathsf{lpod2asp}(\Pi)$.

(b) Let K be an optimal answer set of $\mathsf{lpod2asp}(\Pi)$ such that $K \vDash ap(x_1, \dots, x_m)$ for some $x_1, \dots, x_m$. By Lemma 4, there exists an answer set S' of $\Pi_1 \cup \Pi_2$ such that S' and K agrees on the signature of $\Pi_1 \cup \Pi_2$, which means $shrink(S', x_1, \dots, x_m) = shrink(K, x_1, \dots, x_m)$, and $S' \vDash ap(x_1, \dots, x_m)$. Besides, since K and S' satisfy the same set of $ap(*)$ atoms, and K is an optimal answer set of $\mathsf{lpod2asp}(\Pi)$, S' is an optimal answer set of $\Pi_1 \cup \Pi_2$. By Proposition 2, $S = shrink(S', x_1, \dots, x_m)$ is a candidate answer set of Π, and $\phi(S)$ is an answer set of $AP_\Pi(x_1, \dots, x_m)$.

□

Lemma 8
Under each of the four preference criteria, the preferred answer sets of an LPOD Π of signature σ are exactly the preferred answer sets on σ of $\mathsf{lpod2asp}(\Pi)$. In other words,

(a) for any preferred answer set S of Π, there exists an optimal answer set K of $\mathsf{lpod2asp}(\Pi)$ and there are $x_1, \dots, x_m$ such that $K \vDash pAS(x_1, \dots, x_m)$ and $S = shrink(K, x_1, \dots, x_m)$;

(b) for any optimal answer set K of $\mathsf{lpod2asp}(\Pi)$ and any $x_1, \dots, x_m$ such that $K \vDash pAS(x_1, \dots, x_m)$, $S = shrink(K, x_1, \dots, x_m)$ is a preferred answer set of Π.

Proof. (**a**) Let Π be an LPOD (9) of signature σ. Let S be a preferred answer set of Π; and let S_2 be any candidate answer set of Π with different satisfaction degrees compared to S. For any set of atoms S', let $\phi(S') = S' \cup \{body_i \mid S'$ satisfies the body of rule i in $\Pi_{od}\}$. By Proposition 1, we know $\phi(S)$ is an answer set of $AP_\Pi(x_1, \dots, x_m)$ for some $x_1, \dots, x_m$, and $\phi(S_2) = S_2 \cup \{body_i \mid S_2$ satisfies the body of rule i for some $1 \le i \le m$ $\}$ is an answer set of $AP_\Pi(y_1, \dots, y_m)$ for some $y_1, \dots, y_m$, where by Lemma 5, the list $x_1, \dots, x_m$ is not the same as $y_1, \dots, y_m$.

By Lemma 6 (**a**), there exists an optimal answer set K of $\mathsf{lpod2asp}(\Pi)$ such that $K \vDash ap(x_1, \dots, x_m)$, $K \vDash ap(y_1, \dots, y_m)$, $S = shrink(K, x_1, \dots, x_m)$, and $S_2 = shrink(K, y_1, \dots, y_m)$.

Then it is sufficient to prove $K \vDash pAS(x_1, \dots, x_m)$, which by rules (26), (31), (34), (37), suffices to proving $K \not\vDash prf(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m))$ no matter what S_2 we are choosing. Assume for the sake of contradiction that $K \vDash prf(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m))$, we will derive a contradiction for each preference criterion. Note that

- $K \vDash degree(ap(x_1, \dots, x_m), d_1, \dots, d_m)$

iff (by rules (18), (19), (20), and given $K \vDash ap(x_1, \dots, x_m)$)

- for $1 \le i \le m$, $d_i = 1$ if $x_i = 0$, $d_i = x_i$ if $x_i > 0$

iff (by Lemma 5, and given S is a candidate answer set of Π, and given $\phi(S)$ is an answer set of $AP_\Pi(x_1, \dots, x_m)$)

- the satisfaction degrees of S are $d_1, \dots, d_m$.

Similarly,

- $K \vDash degree(ap(y_1, \dots, y_m), e_1, \dots, e_m)$

iff

- the satisfaction degrees of S_2 are $e_1, \dots, e_m$.

1. **Cardinality-preferred:**
   - $K \models prf(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m))$

   iff (by rule (25))
   - there exists a number d such that $0 \leq d \leq maxdegree - 1$ and
     — $K \models prf2degree(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m), d + 1)$
     — $K \models equ2degree(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m), Y)$ for $1 \leq Y \leq d$

   iff (by rules (23), (24))
   - there exists a number d such that $0 \leq d \leq maxdegree - 1$ and
     — there exist n_1 and n_2 such that $K \models card(ap(y_1, \dots, y_m), d + 1, n_1)$, $K \models card(ap(x_1, \dots, x_m), d + 1, n_2)$, and $n_1 > n_2$
     — for each $1 \leq Y \leq d$, there exists a number n such that $K \models card(ap(y_1, \dots, y_m), Y, n)$ and $K \models card(ap(x_1, \dots, x_m), Y, n)$

   iff (by rule (22))
   - there exists a number d such that $0 \leq d \leq maxdegree - 1$ and
     — there exist n_1 and n_2 such that S_2 satisfies n_1 rules at degree d, S satisfies n_2 rules at degree $d + 1$, and $n_1 > n_2$
     — for each $1 \leq Y \leq d$, there exists a number n such that both S_2 and S satisfy n rules at degree Y

   iff (by the semantics of LPOD)
   - S_2 is cardinality-preferred to S

   which violates the fact that S is a preferred answer set.
2. **Inclusion-preferred:**
   - $K \models prf(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m))$

   iff (by rule (30))
   - there exists a number d such that $0 \leq d \leq maxdegree - 1$ and
     — $K \models prf2degree(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m), d + 1)$
     — $K \models equ2degree(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m), Y)$ for $1 \leq Y \leq d$

   iff (by rules (27), (28), (29))
   - there exists a number d such that $0 \leq d \leq maxdegree - 1$ and
     — $K \not\models equ2degree(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m), d + 1)$ and for $1 \leq i \leq m$, whenever S satisfies rule i at degree $d + 1$, S_2 must also satisfy rule i at degree $d + 1$; [7]
     — for each $1 \leq Y \leq d$, S satisfies rule i at degree Y iff S_2 satisfies rule i at degree Y for $1 \leq i \leq m$ [8]

[7] The atom $\{D_{11} \neq X; D_{21} = X\}1$ is true in K iff the number of atoms in this set that is satisfied by K is smaller or equal to 1, which means that this atom is true iff $K \models \neg(\{D_{11} \neq X \wedge D_{21} = X)$ iff $K \models (D_{21} = X \rightarrow \{D_{11} = X)$. In the case $X = d + 1$, this atom is true iff "whenever S_2 satisfies rule 1 at degree $d + 1$, S must satisfies rule 1 at degree $d + 1$".

[8] The atom $C_1 = \{D_{11} = X; D_{21} = X\}$ is true in K iff C_1 is the number of atoms in this set that is satisfied by K. Then $C_1 = 0 \vee C_1 = 2$ iff $D_{11} = X \leftrightarrow D_{21} = X$, which can be read as "$S$ satisfies rule 1 at degree X iff S_2 satisfies rule 1 at degree X".

iff

- there exists a number d such that $0 \leq d \leq maxdegree - 1$ and
  - — the rules satisfied by S is a proper subset of the rules satisfied by S_2 at degree $d+1$
  - — the rules satisfied by S is exactly the rules satisfied by S_2 at degrees $\{1, \dots, d\}$

iff (by the semantics of LPOD)

- S_2 is inclusion-preferred to S

which violates the fact that S is a preferred answer set.

3. **Pareto-preferred:**

- $K \models prf(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m))$

iff (by rule (33))

- there exists 2 lists $e_1, \dots, e_m$ and $d_1, \dots, d_m$ such that
  - — $K \models degree(ap(y_1, \dots, y_m), e_1, \dots, e_m)$
  - — $K \models degree(ap(x_1, \dots, x_m), d_1, \dots, d_m)$
  - — $K \not\models equ(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m))$, and
  - — $e_1 \leq d_1, \dots, e_m \leq d_m$

iff (by rule (32))

- there exists 2 lists $e_1, \dots, e_m$ and $d_1, \dots, d_m$ such that
  - — $K \models degree(ap(y_1, \dots, y_m), e_1, \dots, e_m)$
  - — $K \models degree(ap(x_1, \dots, x_m), d_1, \dots, d_m)$
  - — $e_1 \leq d_1, \dots, e_m \leq d_m$, and there exists an i such that $e_i < d_i$

iff (by the semantics of LPOD)

- S_2 is Pareto-preferred to S

which violates the fact that S is a preferred answer set.

4. **Penalty-Sum-preferred:**

- $K \models prf(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m))$

iff (by rule (36))

- there exist n_1 and n_2 such that
  - — $K \models sum(ap(y_1, \dots, y_m), n_1)$
  - — $K \models sum(ap(x_1, \dots, x_m), n_2)$, and
  - — $n_1 < n_2$

iff (by rule (35))

- there exist n_1 and n_2 such that
  - — the sum of the satisfaction degrees of all rules for S_2 is n_1
  - — the sum of the satisfaction degrees of all rules for S is n_2, and
  - — $n_1 < n_2$

iff (by the semantics of LPOD)

- S_2 is penalty-sum-preferred to S

which violates the fact that S is a preferred answer set.

(**b**) Let Π be an LPOD (9) of signature σ; let K be an optimal answer set of $\mathsf{lpod2asp}(\Pi)$; and let K satisfy $pAS(x_1, \dots, x_m)$. By rules (26), (31), (34), (37), $K \vDash ap(x_1, \dots, x_m)$. By Lemma 7, $S = shrink(K, x_1, \dots, x_m)$ is an answer set of $AP_\Pi(x_1, \dots, x_m)$. We will prove that S is a preferred answer set of Π.

Assume for the sake of contradiction that there exists a candidate answer set S_2 of Π and S_2 is preferred to S. By Proposition 1, S_2 is also an answer set of $AP_\Pi(y_1, \dots, y_m)$ for some $y_1, \dots, y_m$, where by Lemma 5, the list $y_1, \dots, y_m$ is not the same as $x_1, \dots, x_m$. By Lemma 6 **(b)**, K must satisfy $ap(y_1, \dots, y_m)$. Since $K \vDash pAS(x_1, \dots, x_m)$, by rules (26), (31), (34), (37), to prove a contradiction, it is sufficient to prove $K \vDash prf(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m))$.

By Lemma 7, $shrink(K, y_1, \dots, y_m)$ is a candidate answer set of Π. By Lemma 7 and Lemma 5, $shrink(K, y_1, \dots, y_m)$ has the same satisfaction degrees as S_2. So $shrink(S', y_1, \dots, y_m)$ is preferred to S. As we proved in bullet **(a)**, under any of the four criterion, $shrink(S', y_1, \dots, y_m)$ is preferred to S iff $K \vDash prf(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m))$. Since $shrink(S', y_1, \dots, y_m)$ is preferred to S, $K \vDash prf(ap(y_1, \dots, y_m), ap(x_1, \dots, x_m))$. □

Theorem 1 *Under any of the four preference criteria, the candidate (preferred, respectively) answer sets of an LPOD Π of signature σ are exactly the candidate (preferred, respectively) answer sets on σ of* $\mathsf{lpod2asp}(\Pi)$.

Proof. The proof follows from Lemma 7 and Lemma 8. □

Appendix D Proof of Proposition 3

Let's review the definition of $AP_\Pi(x_1, \dots, x_m)$. Let Π be a CR-Prolog$_2$ program of signature σ, where its rules are rearranged such that the cr-rules are of indices $1, \dots, k$, the ordered cr-rules are of indices $k+1, \dots, l$, and the ordered rules are of indices $l+1, \dots, m$. These 3 sets of rules are called Π_{cr}, Π_{ocr}, Π_{or} respectively, and the remaining part in Π is called Π_r. For each rule i in $\Pi_{ocr} \cup \Pi_{or}$, let n_i denote the number of atoms in $head(i)$. Let D_i be the set $\{0, 1\}$ for $1 \le i \le k$; $\{0, \dots, n_i\}$ for $k+1 \le i \le l$; $\{1, \dots, n_i\}$ for $l+1 \le i \le m$. $AP_\Pi(x_1, \dots, x_m)$ denotes an assumption program obtained from Π as follows, where $x_i \in D_i$.

- $AP_\Pi(x_1, \dots, x_m)$ contains Π_r
- for each cr-rule $i : \mathit{Head}_i \stackrel{+}{\leftarrow} \mathit{Body}_i$ in Π_{cr}, $AP_\Pi(x_1, \dots, x_m)$ contains

$$\mathit{Head}_i \leftarrow \mathit{Body}_i, x_i = 1 \tag{D1}$$

- for each ordered rule or ordered cr-rule $i : C_i^1 \times \dots \times C_i^{n_i} \stackrel{(+)}{\leftarrow} \mathit{Body}_i$ in $\Pi_{or} \cup \Pi_{ocr}$, for $1 \le j \le n_i$, $AP_\Pi(x_1, \dots, x_m)$ contains

$$C_i^j \leftarrow \mathit{Body}_i, x_i = j \tag{D2}$$

- $AP_\Pi(x_1, \dots, x_m)$ also contains the following rules:

$$\begin{array}{l}
\mathit{isPreferred}(R1, R2) \leftarrow \mathit{prefer}(R1, R2).\\
\mathit{isPreferred}(R1, R3) \leftarrow \mathit{prefer}(R1, R2), \mathit{isPreferred}(R2, R3).\\
\leftarrow \mathit{isPreferred}(R, R).\\
\leftarrow x_{r_1} > 0, x_{r_2} > 0, \mathit{isPreferred}(r_1, r_2). \qquad (1 \le r_1, r_2 \le l)
\end{array}$$

Proposition 3 *For any CR-Prolog$_2$ program Π of signature σ, a set X of atoms is the projection* *of a generalized answer set of Π onto σ iff X is the projection of an answer set of an assumption program of Π onto σ. In other words,*

(a) *for any generalized answer set S of Π, there exists an assumption program $AP_\Pi(x_1, \dots, x_m)$ of Π and one of its answer set S' such that $S|_\sigma = S'|_\sigma$;*
(b) *for any answer set S' of any assumption program $AP_\Pi(x_1, \dots, x_m)$ of Π, there exists a generalized answer set S of Π such that $S'|_\sigma = S|_\sigma$.*

Proof. Let Π be a CR-Prolog$_2$ program. According to the semantics of CR-Prolog$_2$, S is a generalized answer set of Π iff S is an answer set of H'_Π, where H'_Π is obtained from Π as follows. [9]

- H'_Π contains Π_r
- for each cr-rule i : $\mathit{Head}_i \overset{+}{\leftarrow} \mathit{Body}_i$ in Π_{cr}, H'_Π contains

$$\mathit{Head}_i \leftarrow \mathit{Body}_i, \mathit{appl}(i) \tag{D3}$$

- for each ordered cr-rule $i : C_i^1 \times \cdots \times C_i^{n_i} \overset{+}{\leftarrow} \mathit{Body}_i$ in Π_{ocr}, for $1 \le j \le n_i$, H'_Π contains

$$C^j \leftarrow \mathit{Body}_i, \mathit{appl}(i), \mathit{appl}(\mathit{choice}(i, j)) \tag{D4}$$
$$\mathit{fired}(i) \leftarrow \mathit{appl}(\mathit{choice}(i, j)) \tag{D5}$$
$$\mathit{prefer}(\mathit{choice}(i, j), \mathit{choice}(i, j + 1)) \quad (1 \le j \le n_i - 1) \tag{D6}$$
$$\leftarrow \mathit{Body}_i, \mathit{appl}(i), \mathit{not\ fired}(i) \tag{D7}$$

- for each ordered rule $i : C_i^1 \times \cdots \times C_i^{n_i} \leftarrow \mathit{Body}_i$ in Π_{or}, for $1 \le j \le n_i$, H'_Π contains

$$C^j \leftarrow \mathit{Body}_i, \mathit{appl}(\mathit{choice}(i, j)) \tag{D8}$$
$$\mathit{fired}(i) \leftarrow \mathit{appl}(\mathit{choice}(i, j)) \tag{D9}$$
$$\mathit{prefer}(\mathit{choice}(i, j), \mathit{choice}(i, j + 1)) \quad (1 \le j \le n_i - 1) \tag{D10}$$
$$\leftarrow \mathit{Body}_i, \mathit{not\ fired}(i) \tag{D11}$$

- H'_Π also contains:

$$\mathit{isPreferred}(R1, R2) \leftarrow \mathit{prefer}(R1, R2). \tag{D12}$$
$$\mathit{isPreferred}(R1, R3) \leftarrow \mathit{prefer}(R1, R2), \mathit{isPreferred}(R2, R3). \tag{D13}$$
$$\leftarrow \mathit{isPreferred}(R, R). \tag{D14}$$
$$\leftarrow \mathit{appl}(R1), \mathit{appl}(R2), \mathit{isPreferred}(R1, R2). \tag{D15}$$

- and for each $A \in atoms(H_\Pi, \{\mathit{appl}\})$, H'_Π also contains

$$\{A\}. \tag{D16}$$

Note that rule (D12) can be considered as two rules: (D12r), in which each variable is grounded by an index of a cr-rule; and (D12a), in which each variable is grounded by a term $choice(*)$. Similarly, each of the rules (D13), (D14), (D15) can be considered as two rules.

The (propositional) signature of H'_Π is $\sigma \cup atoms(H'_\Pi, \{appl, fired, prefer, isPreferred\})$, while the (propositional) signature of $AP_\Pi(x_1, \dots, x_m)$ is $\sigma \cup atoms(AP_\Pi(x_1, \dots, x_m), \{isPreferred\})$, which is a subset of the signature of H'_Π.

[9] Note that H'_Π is similar to H_Π (which is defined in Section 3.1 of the paper) except that H'_Π contains a choice rule $\{A\}$ for each $A \in atoms(H_\Pi, \{appl\})$.

(a) Let S be a generalized answer set of Π. Then S is an answer set of H'_Π. We obtain $x_1, \dots, x_m$ such that

— for $1 \le i \le k$: $x_i = 0$ if $S \not\models \textit{appl}(i)$,
$x_i = 1$ if $S \models \textit{appl}(i)$;

— for $k+1 \le i \le l$: $x_i = 0$ if $S \not\models \textit{appl}(i)$,
$x_i = j$ if $S \models \textit{appl}(i)$ and $S \models \textit{appl}(\textit{choice}(i,j))$, [10]
$x_i = 1$ if $S \models \textit{appl}(i)$ and $S \not\models \textit{appl}(\textit{choice}(i,j))$ for all j (in the case when $S \not\models \textit{Body}_i$);

— for $l+1 \le i \le m$: $x_i = j$ if $S \models \textit{appl}(\textit{choice}(i,j))$,
$x_i = 1$ if $S \not\models \textit{appl}(\textit{choice}(i,j))$ for all j.

Then it is sufficient to prove that the projection of S onto

$$\sigma \cup atoms(AP_\Pi(x_1, \dots, x_m), \{\textit{isPreferred}\})$$

is an answer set of $AP_\Pi(x_1, \dots, x_m)$. This is equivalent to proving S is a minimal model of the reduct of $AP_\Pi(x_1, \dots, x_m)$ relative to $\sigma \cup atoms(AP_\Pi(x_1, \dots, x_m), \{\textit{isPreferred}\})$. The assumption program $AP_\Pi(x_1, \dots, x_m)$ is similar to H'_Π except that

1. $AP_\Pi(x_1, \dots, x_m)$ does not contain the constraints: (D7), (D11), (D14a,) (D15a)
2. $AP_\Pi(x_1, \dots, x_m)$ does not contain the definitions for $fired(*)$, $prefer(\textit{choice}(*), \textit{choice}(*))$, and $isPreferred(\textit{choice}(*), \textit{choice}(*))$: (D5), (D6), (D9), (D10), (D12a), (D13a)
3. $AP_\Pi(x_1, \dots, x_m)$ uses the value assignments for x_i to represent $appl(*)$ in H'_Π

Let $(H'_\Pi)_{i,\dots,j}$ denote the set of rules in H'_Π translated by rules $(i), \dots, (j)$.
First, let's obtain Π_1 from H'_Π by removing the constraints (D7), (D11), (D14a,) (D15a). In other words, Π_1 is $H'_\Pi \setminus (H'_\Pi)_{D7,D11,D14a,D15a}$. By Lemma 1 (e), S is an answer set of Π_1.
Second, let's obtain Π_2 from Π_1 by removing the definitions for $fired(*)$, $prefer(\textit{choice}(*), \textit{choice}(*))$, and $isPreferred(\textit{choice}(*), \textit{choice}(*))$. In other words, Π_2 is $\Pi_1 \setminus (H'_\Pi)_{D5,D6,D9,D10,D12a,D13a}$. Let σ_1 be the propositional signature of Π_1 and let σ_2 be the propositional signature of Π_2. We will use the splitting theorem to split Π_1 into Π_2 and $(H'_\Pi)_{D5,D6,D9,D10,D12a,D13a}$. Since

1. no atom in σ_2 has a strictly positive occurrence in $(H'_\Pi)_{D5,D6,D9,D10,D12a,D13a}$,
2. no atom in $\sigma_1 \setminus \sigma_2$ has a strictly positive occurrence in Π_2, and
3. each strongly connected component of the dependency graph of Π_1 w.r.t. σ_1 is a subset of σ_2 or $\sigma_1 \setminus \sigma_2$,

by the splitting theorem, S is an answer set of Π_2 relative to σ_2, where σ_2 equals to $\sigma \cup atoms(\Pi_2, \{\textit{appl}\}) \cup atoms(\Pi_2, \{\textit{isPreferred}\})$.
Third, by the assignments of $x_i, \dots, x_m$, we know

— for $1 \le i \le k$: $S \models \textit{appl}(i)$ iff $x_i = 1$,
— for $k+1 \le i \le l$: $S \models \textit{Body}_i \wedge \textit{appl}(i) \wedge \textit{appl}(\textit{choice}(i,j))$ iff $S \models \textit{Body}_i$ and $x_i = j$
— for $l+1 \le i \le m$: $S \models \textit{Body}_i \wedge \textit{appl}(\textit{choice}(i,j))$ iff $S \models \textit{Body}_i$ and $x_i = j$.

Note that we can obtain $AP_\Pi(x_1, \dots, x_m)$ from Π_2 by

— for $1 \le i \le k$, replacing $\textit{appl}(i)$ with $x_i = 1$ in rule (D3);

[10] Since S is an answer set of H'_Π, by rules (D6), (D12), (D13), and (D15), S cannot satisfy $\textit{appl}(\textit{choice}(i,j))$ for two different j.

— for $k+1 \leq i \leq l$, replacing $\mathit{appl}(i) \wedge \mathit{appl}(\mathit{choice}(i,j))$ with $x_i = j$ in rule (D4);
— for $l+1 \leq i \leq m$, replacing $\mathit{appl}(\mathit{choice}(i,j))$ with $x_i = j$ in rule (D8)
— for $1 \leq i \leq l$, replacing $\mathit{appl}(i)$ with $x_i > 0$ in (grounded) rule (D15).

Since S is a minimal model of the reduct of Π_2 relative to $\sigma \cup atoms(H_\Pi, \mathit{appl}) \cup atoms(\Pi_2, \{\mathit{isPreferred}\})$, S is a minimal model of the reduct of $AP_\Pi(x_1, \dots, x_m)$ relative to $\sigma \cup atoms(\Pi_2, \{\mathit{isPreferred}\})$. Since

$$atoms(\Pi_2, \{\mathit{isPreferred}\}) = atoms(AP_\Pi(x_1, \dots, x_m), \{\mathit{isPreferred}\}),$$

S is a minimal model of the reduct of $AP_\Pi(x_1, \dots, x_m)$ relative to

$$\sigma \cup atoms(AP_\Pi(x_1, \dots, x_m), \{\mathit{isPreferred}\}).$$

(b) Let $AP_\Pi(x_1, \dots, x_m)$ be an assumption program of Π, and S_{sp} be an answer set of $AP_\Pi(x_1, \dots, x_m)$.

$$\begin{aligned} \text{Let } S = S_{sp} \;\; & \cup\{\mathit{appl}(i) \mid 1 \leq i \leq k, x_i = 1\} \\ & \cup\{\mathit{appl}(i), \mathit{appl}(\mathit{choice}(i,j)), fired(i) \mid k+1 \leq i \leq l, x_i = j, j > 0\} \\ & \cup\{\mathit{appl}(\mathit{choice}(i,j)), fired(i) \mid l+1 \leq i \leq m, x_i = j\} \\ & \cup\{prefer(\mathit{choice}(i,j), \mathit{choice}(i,j+1)) \mid k+1 \leq i \leq m, 1 \leq j \leq n_i\} \\ & \cup\{isPreferred(\mathit{choice}(i,j_1), \mathit{choice}(i,j_2)) \mid k+1 \leq i \leq m, 1 \leq j_1 < j_2 \leq n_i\} \end{aligned}$$

It is sufficient to prove S is an answer set of H'_Π.
Let Π_1 be $H'_\Pi \setminus (H'_\Pi)_{D7,D11,D14a,D15a}$. Let Π_2 be $\Pi_1 \setminus (H'_\Pi)_{D5,D6,D9,D10,D12a,D13a}$.
First, we prove

$$\begin{aligned} S_{sp} \;\; & \cup\{\mathit{appl}(i) \mid 1 \leq i \leq k, x_i = 1\} \\ & \cup\{\mathit{appl}(i), \mathit{appl}(\mathit{choice}(i,j)) \mid k+1 \leq i \leq l, x_i = j, j > 0\} \\ & \cup\{\mathit{appl}(\mathit{choice}(i,j)) \mid l+1 \leq i \leq m, x_i = j\}, \end{aligned}$$

denoted by S_2, is an answer set of Π_2. Let's compare the reduct of $AP_\Pi(x_1, \dots, x_m)$ relative to S_{sp} and the reduct of Π_2 relative to S_2. The reduct of Π_2 relative to S_2 can be obtained from the reduct of $AP_\Pi(x_1, \dots, x_m)$ relative to S_{sp} by adding the facts

1. $\mathit{appl}(i)$ for $1 \leq i \leq k$ and $x_i = 1$,
2. $\mathit{appl}(i)$ and $\mathit{appl}(\mathit{choice}(i,j))$ for $k+1 \leq i \leq l$, and $x_i = j, j > 0$,
3. $\mathit{appl}(\mathit{choice}(i,j))$ for $l+1 \leq i \leq m$, and $x_i = j$;

and replacing

1. $x_i = 1$ by $\mathit{appl}(i)$ for $1 \leq i \leq k$,
2. $x_i = j$, where $j > 0$, by $\mathit{appl}(i) \wedge \mathit{appl}(\mathit{choice}(i,j))$ for $k+1 \leq i \leq l$,
3. $x_i = j$ by $\mathit{appl}(\mathit{choice}(i,j))$ for $l+1 \leq i \leq m$.

Since S_{sp} is a minimal model of the reduct of $AP_\Pi(x_1, \dots, x_m)$ relative to S_{sp}, and since

1. for $1 \leq i \leq k$, $S_2 \models \mathit{appl}(i)$ iff $x_i = 1$,
2. for $k+1 \leq i \leq l$, $S_2 \models \mathit{appl}(i) \wedge \mathit{appl}(\mathit{choice}(i,j))$ iff $x_i = j \wedge j > 0$,
3. for $l+1 \leq i \leq m$, $\mathit{appl}(\mathit{choice}(i,j))$ iff $x_i = j$;

S_2 is a minimal model of the reduct of Π_2 relative to S_2.

Second, we prove S is an answer set of Π_1. Note that S equals

$$\begin{array}{ll} S_2 & \cup\{fired(i) \mid k+1 \leq i \leq l, x_i = j, j > 0\} \\ & \cup\{fired(i) \mid l+1 \leq i \leq m, x_i = j\} \\ & \cup\{prefer(choice(i,j), choice(i,j+1)) \mid k+1 \leq i \leq m, 1 \leq j \leq n_i\} \\ & \cup\{isPreferred(choice(i,j_1), choice(i,j_2)) \mid k+1 \leq i \leq m, 1 \leq j_1 < j_2 \leq n_i\}. \end{array}$$

Let σ_1 be the propositional signature of Π_1 and let σ_2 be the propositional signature of Π_2. We will use the splitting theorem to construct Π_1 from Π_2 and $(H'_\Pi)_{D5,D6,D9,D10,D12a,D13a}$. Note that

1. no atom in σ_2 has a strictly positive occurrence in $(H'_\Pi)_{D5,D6,D9,D10,D12a,D13a}$,
2. no atom in $\sigma_1 \setminus \sigma_2$ has a strictly positive occurrence in Π_2, and
3. each strongly connected component of the dependency graph of Π_1 w.r.t. σ_1 is a subset of σ_2 or $\sigma_1 \setminus \sigma_2$,

Since S is an answer set of Π_2 relative to σ_2, and it's easy to check that S is an answer set of $(H'_\Pi)_{D5,D6,D9,D10,D12a,D13a}$ relative to $\sigma_1 \setminus \sigma_2$, S is an answer set of Π_1.
Third, since S satisfies rules (D7), (D11), (D14a), (D15a), by Lemma 1 (d), S is an answer set of H'_Π.

□

Appendix E Proof of Theorem 2

We first review some definitions. Let Π be a CR-Prolog$_2$ program. Let S be an optimal answer set of crp2asp(Π). Let $x_1, \dots, x_m$ be a list of integers such that $x_i \in D_i$. If $S \vDash ap(x_1, \dots, x_m)$, we define the set $shrink(S, x_1, \dots, x_m)$ as a *generalized answer set on* σ of crp2asp(Π); if $S \vDash candidate(x_1, \dots, x_m)$, we define the set $shrink(S, x_1, \dots, x_m)$ as a *candidate answer set on* σ of crp2asp(Π); if $S \vDash pAS(x_1, \dots, x_m)$, we define the set $shrink(S, x_1, \dots, x_m)$ as a *preferred answer set on* σ of crp2asp(Π).

Theorem 2 *For any CR-Prolog$_2$ program Π of signature σ,*

- **(a)** *The projections of the generalized answer sets of Π onto σ are exactly the generalized answer sets on σ of* crp2asp(Π).
- **(b)** *The projections of the candidate answer sets of Π onto σ are exactly the candidate answer sets on σ of* crp2asp(Π).
- **(c)** *The preferred answer sets of Π are exactly the preferred answer sets on σ of* crp2asp(Π).

Proof. **(a):** Let Π be a CR-Prolog$_2$ program of signature σ. By Proposition 3, it is sufficient to prove that the projections (onto σ) of the answer sets of all assumption programs $AP_\Pi(x_1, \dots, x_m)$ of Π are exactly the generalized answer sets on σ of crp2asp(Π) such that

- for any answer set S of any $AP_\Pi(x_1, \dots, x_m)$, there exists an optimal answer set S' of crp2asp(Π) such that $S' \vDash ap(x_1, \dots, x_m)$ and $S_\sigma = shrink(S', x_1, \dots, x_m)$;
- for any generalized answer set on σ, $shrink(S', x_1, \dots, x_m)$, of crp2asp($\Pi$) (where S' is an optimal answer set of crp2asp(Π) and $S' \vDash ap(x_1, \dots, x_m)$), there exists an answer set S of $AP_\Pi(x_1, \dots, x_m)$ such that $S_\sigma = shrink(S', x_1, \dots, x_m)$.

Let $\mathsf{crp2asp}(\Pi) = \Pi_{base} \cup \Pi_{pref}$, where Π_{pref} is the set of rules translated from rules (48), (53), (54), (55), (56). We use Lemma 3 to prove that there is a 1-1 correspondence between the answer sets of $\mathsf{crp2asp}(\Pi)$ and the answer sets of Π_{base}, while an answer set of $\mathsf{crp2asp}(\Pi)$ agrees with the corresponding answer set of Π_{base} on the signature of Π_{base}. Let's take Π_{base} as our current program, Π_{cur}, and consider including the translation rules in Π_{pref} into Π_{cur}. If we include rules (48) and (53), by Lemma 3, there is a 1-1 correspondence between the answer sets of Π_{cur} and the answer sets of Π_{base}. Similarly, we can include rules (54), (55), (56) in order into Π_{cur}, and find that there is a 1-1 correspondence between the answer sets of $\Pi_{base} \cup \Pi_{pref}$ and the answer sets of Π_{base}, while an answer set of $\Pi_{base} \cup \Pi_{pref}$ agrees with the corresponding answer set of Π_{base} on the signature of Π_{base}. Since the predicates introduced by Π_{pref} are not in σ, it is sufficient to prove that the projections of the answer sets of all assumption programs $AP_\Pi(x_1, \dots, x_m)$ of Π onto σ are exactly the generalized answer sets on σ of Π_{base}.

According to the translation, the empty set is always an answer set of Π_{base}, thus there must exist at least one optimal answer set of Π_{base}. Furthermore, by rule (44), the optimal answer set should contain as many $ap(*)$ as possible. Let $gr(\Pi_{base}, x_1, \dots, x_m)$ be a partial grounded program obtained from Π_{base} by replacing variables $X_1, \dots, X_m$ with $x_1, \dots, x_m$. Since each partial grounded program is disjoint from each other, by the splitting theorem, it is sufficient to prove a 1-1 correspondence ϕ between the answer sets of $AP_\Pi(x_1, \dots, x_m)$ and the optimal answer sets of $gr(\Pi_{base}, x_1, \dots, x_m)$ such that

(a.1) For any answer set S of $AP_\Pi(x_1, \dots, x_m)$, $\phi(S) = \{a(\mathbf{v}, x_1, \dots, x_m) \mid a(\mathbf{v}) \in S\} \cup \{ap(x_1, \dots, x_m)\}$ is an optimal answer set of $gr(\Pi_{base}, x_1, \dots, x_m)$.

(a.2) For any optimal answer set S' of $gr(\Pi_{base}, x_1, \dots, x_m)$, if $S' \not\models ap(x_1, \dots, x_m)$, then $AP_\Pi(x_1, \dots, x_m)$ has no answer set; if $S' \models ap(x_1, \dots, x_m)$, then

$$S = \{a(\mathbf{v}) \mid a(\mathbf{v}, x_1, \dots, x_m) \in S'\} \setminus \{sp\}$$

is an answer set of $AP_\Pi(x_1, \dots, x_m)$.

To prove bullet **(a.1)**, let S be an answer set of $AP_\Pi(x_1, \dots, x_m)$, and let $\phi(S)$ be $\{a(\mathbf{v}, x_1, \dots, x_m) \mid a(\mathbf{v}) \in S\} \cup \{ap(x_1, \dots, x_m)\}$. Since $\phi(S)$ satisfies $ap(x_1, \dots, x_m)$, which is the only $ap(*)$ in $gr(\Pi_{base}, x_1, \dots, x_m)$, if we prove $\phi(S)$ is an answer set of $gr(\Pi_{base}, x_1, \dots, x_m)$, $\phi(S)$ must be an optimal answer set of $gr(\Pi_{base}, x_1, \dots, x_m)$. Note that, if we ignore the suffix $x_1, \dots, x_m$ in the reduct of $gr(\Pi_{base}, x_1, \dots, x_m)$ relative to $\phi(S)$, it is almost the same as the reduct of $AP_\Pi(x_1, \dots, x_m)$ relative to S except that the former has one more atom sp. Since S is a minimal model of the reduct of $AP_\Pi(x_1, \dots, x_m)$ relative to S, and $\phi(S) \models ap(x_1, \dots, x_m)$, $\phi(S)$ is a minimal model of the reduct of $gr(\Pi_{base}, x_1, \dots, x_m)$ relative to $\phi(S)$. Thus $\phi(S)$ is an answer set of $gr(\Pi_{base}, x_1, \dots, x_m)$.

To prove bullet **(a.2)**, let S' be an optimal answer set of $gr(\Pi_{base}, x_1, \dots, x_m)$. There are 2 cases as follows.

1. $ap(x_1, \dots, x_m) \notin S'$. We will prove $AP_\Pi(x_1, \dots, x_m)$ has no answer set. Assume for the sake of contradiction that there exists an answer set S of $AP_\Pi(x_1, \dots, x_m)$, by the bullet **(a.1)** that we just proved, $\phi(S)$ is an optimal answer set of $gr(\Pi_{base}, x_1, \dots, x_m)$. Since $\phi(S) \models ap(x_1, \dots, x_m)$, by rule (44), it has lower penalty than S', thus S' is not an optimal answer set, which is not the case. So $AP_\Pi(x_1, \dots, x_m)$ has no answer set.
2. $ap(x_1, \dots, x_m) \in S'$. Since S' is a minimal model of the reduct of $gr(\Pi_{base}, x_1, \dots, x_m)$, if we remove all occurrence of $ap(x_1, \dots, x_m)$ and $x_1, \dots, x_m$ in both S' and the reduct

of $gr(\Pi_{base}, x_1, \ldots, x_m)$ relative to S', the set of atoms $S = \{a(\mathbf{v}) \mid a(\mathbf{v}, x_1, \ldots, x_m) \in S'\} \setminus \{sp\}$ should be a minimal model of the new program, which is the reduct of $AP_\Pi(x_1, \ldots, x_m)$. Thus S is an answer set of $AP_\Pi(x_1, \ldots, x_m)$.

(b): To prove Theorem 2 **(b)**, it is sufficient to prove

(b.1) for any candidate answer set S of Π, there exist an optimal answer set S' of $\mathsf{crp2asp}(\Pi)$ and a list $x_1, \ldots, x_m$ such that $S' \models candidate(x_1, \ldots, x_m)$, and $S_\sigma = shrink(S', x_1, \ldots, x_m)$;

(b.2) for any optimal answer set S' of $\mathsf{crp2asp}(\Pi)$, if $S' \models candidate(x_1, \ldots, x_m)$, there exists a candidate answer set S of Π such that $S_\sigma = shrink(S', x_1, \ldots, x_m)$.

Let Π be a CR-Prolog$_2$ program with signature σ; Π' be its translation $\mathsf{crp2asp}(\Pi)$.

To prove bullet **(b.1)**, let S be a candidate answer set of Π, then by the semantics of CR-Prolog$_2$, S must be a generalized answer set of Π. We obtain $x_1, \ldots, x_m$ such that,

- for $1 \leq i \leq k$: $x_i = 0$ if $S \not\models$ *appl*(i),
  $x_i = 1$ if $S \models$ *appl*(i);
- for $k+1 \leq i \leq l$: $x_i = 0$ if $S \not\models$ *appl*(i),
  $x_i = j$ if $S \models$ *appl*(i) and $S \models$ *appl*$(choice(i,j))$,
  $x_i = 1$ if $S \models$ *appl*(i) and $S \not\models$ *appl*$(choice(i,j))$ for any j;
- for $l+1 \leq i \leq m$: $x_i = j$ if $S \models$ *appl*$(choice(i,j))$,
  $x_i = 1$ if $S \not\models$ *appl*$(choice(i,j))$ for any j.

Note that the signature of $AP_\Pi(x_1, \ldots, x_m)$ is $\sigma' = \sigma \cup atoms(AP_\Pi(x_1, \ldots, x_m), \{isPreferred\})$. As we proved in the proof of Proposition 3, S is an answer set of $AP_\Pi(x_1, \ldots, x_m)$ with respect to σ'. Then $S_{\sigma'}$ is an answer set of $AP_\Pi(x_1, \ldots, x_m)$. By the first bullet in the proof for Theorem 2 (a), $\phi(S_{\sigma'}) = \{a(\mathbf{v}, x_1, \ldots, x_m) \mid a(\mathbf{v}) \in S_{\sigma'}\} \cup \{ap(x_1, \ldots, x_m)\}$ is an optimal answer set of $gr(\Pi_{base}, x_1, \ldots, x_m)$. Then there exists an optimal answer set S' of Π' such that $S' \models ap(x_1, \ldots, x_m)$ and $S_\sigma = shrink(S', x_1, \ldots, x_m)$.

Then, it suffices to proving $S' \models candidate(x_1, \ldots, x_m)$. Assume for the sake of contradiction that $S' \not\models candidate(x_1, \ldots, x_m)$.

- $S' \not\models candidate(x_1, \ldots, x_m)$

iff (by rule (54))

- there exists an AP such that $S' \models dominate(AP, ap(x_1, \ldots, x_m))$

iff (by rule (48) and (53))

- there exist $i \in \{k+1, \ldots, m\}$ and a list $x'_1, \ldots, x'_m$ such that $S' \models ap(x'_1, \ldots, x'_m)$, $0 < x'_i$, and $x'_i < x_i$, or
- there exist $r_1, r_2 \in \{1, \ldots, l\}$ and a list $x'_1, \ldots, x'_m$ such that $S' \models ap(x'_1, \ldots, x'_m)$, $S' \models isPreferred(r_1, r_2, x'_1, \ldots, x'_m)$, $S' \models isPreferred(r_1, r_2, x_1, \ldots, x_m)$, $x'_{r_1} > 0$, and $x_{r_2} > 0$

iff (by the first 2 bullets in the proof for Theorem 2 (a) and by the assignments of x_i)

- there exists $i \in \{k+1, \ldots, m\}$, a generalized answer set A, and $x_i, x'_i \in \{1, \ldots, n_i\}$ such that $A \models$ *appl*$(choice(i, x'_i))$, $S \models$ *appl*$(choice(i, x_i))$, and $x'_i < x_i$
- there exist $r_1, r_2 \in \{1, \ldots, l\}$, and a generalized answer set A such that $A \models isPreferred(r_1, r_2)$, $S \models isPreferred(r_1, r_2)$, $A \models$ *appl*(r_1), and $S \models$ *appl*(r_2)

iff (by the definition of dominate)

- there exists a generalized answer set A that dominates S

which contradicts with the fact that S is a candidate answer set. Thus $S' \models candidate(x_1, \dots, x_m)$ and $S_\sigma = shrink(S', x_1, \dots, x_m)$.

To prove bullet **(b.2)**, let S' be an optimal answer set of Π' and $S' \models candidate(x_1, \dots, x_m)$ for some list $x_1 \dots, x_m$. By rule (54), $S' \models ap(x_1, \dots, x_m)$. Then by bullet (a), there exists a generalized answer set S of Π such that $S_\sigma = shrink(S', x_1, \dots, x_m)$. Then it is sufficient to prove S is a candidate answer set of Π.

Assume for the sake of contradiction that S is not a candidate answer set of Π, then there must exists a generalized answer set A that dominates S. By the "iff" statements above, we can derive $S' \not\models candidate(x_1, \dots, x_m)$, which leads to a contradiction.

(c): Let Π be a CR-Prolog$_2$ program with signature σ; Π' be its translation $\mathsf{crp2asp}(\Pi)$. To prove Theorem 2 (c), it is sufficient to prove

- **(c.1)** for any preferred answer set S of Π, there exists an optimal answer set S' of Π' such that $S' \models pAS(x_1, \dots, x_m)$ for some $x_1, \dots, x_m$, and $S_\sigma = shrink(S', x_1, \dots, x_m)$
- **(c.2)** for any optimal answer set S' of Π', if $S' \models pAS(x_1, \dots, x_m)$ for some $x_1, \dots, x_m$, there exists a preferred answer set S of Π such that $S_\sigma = shrink(S', x_1, \dots, x_m)$.

To prove bullet **(c.1)**, let S be a preferred answer set of Π, then S must be a candidate answer set of Π. By Theorem 2 (b), there exists an optimal answer set S' of Π' and a list $x_1, \dots, x_m$ such that $S' \models candidate(x_1, \dots, x_m)$ and $S_\sigma = shrink(S', x_1, \dots, x_m)$. Then it is sufficient to prove $S' \models pAS(x_1, \dots, x_m)$.

Assume for the sake of contradiction that $S' \not\models pAS(x_1, \dots, x_m)$.

- $S' \not\models pAS(x_1, \dots, x_m)$

iff (since $S' \models candidate(x_1, \dots, x_m)$, and by rule (56))

- there exists a AP such that $S' \models lessCrRulesApplied(AP, ap(x_1, \dots, x_m))$

iff (by rule (55))

- there exist a list $x'_1, \dots, x'_m$ such that $S' \models candidate(x'_1, \dots, x'_m)$, $x'_i \leq x_i$ for $1 \leq i \leq m$, and there exists a j such that $x'_j < x_j$

iff (since $S' \not\models dominate(ap(x'_1, \dots, x'_m), ap(x_1, \dots, x_m))$, by rule (48))

- there exist a list $x'_1, \dots, x'_m$ such that $S' \models candidate(x'_1, \dots, x'_m)$, $x'_i \leq x_i$ for $1 \leq i \leq m$, there exists a j such that $x'_j < x_j$, and for any $x'_i < x_i$, $x'_i = 0$

iff (by the assignments of x_i)

- there exist a candidate answer set A such that the atoms of the form $appl(*)$ in A is a proper subset of those in S

which contradicts with the fact that S is a preferred answer set.

To prove bullet **(c.2)**, let S' be an optimal answer set of Π' and $S' \models pAS(x_1, \dots, x_m)$ for some list $x_1, \dots, x_m$. By rules (56) and (54), $S' \models candidate(x_1, \dots, x_m)$ and $S' \models ap(x_1, \dots, x_m)$. Then by Theorem 2 (b), there exists a candidate answer set S of Π such that $S_\sigma = shrink(S', x_1, \dots, x_m)$. Then it is sufficient to prove S is a preferred answer set of Π.

Assume for the sake of contradiction that S is not a preferred answer set of Π, then there must exists a candidate answer set A such that the atoms of the form $appl(*)$ in A is a proper subset of those in S. By the "iff" statements above, we can derive $S' \not\models pAS(x_1, \ldots, x_m)$, which leads to a contradiction. $\square$