Appendix A Unfounded sets by Knorr et al.

In the proof of Proposition 7 of (Knorr et al. 2011), conditions are given which are similar to, but do not coincide with, the conditions in Def 3.1 of this paper. In fact, as shown below, when applied to arbitrary partitions, their definition becomes problematic for our purpose.

Let P_n, N_n be the sequences of P_ω and N_ω, *i.e.*, the sequences in computing the coherent well-founded partition. Let U be the set of all $\mathbf{K}H \notin \Gamma'_{\mathcal{K}}(\mathbf{P}_n)$. Note that $\mathsf{OB}_{\mathcal{O},P_n}$ must be consistent. Then, for each $\mathbf{K}H \in U$, the following conditions are satisfied:

(U1) for each $\mathbf{K}H \leftarrow body$ in $\mathcal{P}$, at least one of the folllowing holds:

- (U1a) some modal $\mathbf{K}$-atom $\mathbf{K}A$ appears in *body* and in $U \cup \mathsf{KA}(\mathcal{K}) \setminus \mathbf{N}_n$;
- (U1b) some modal **not**-atom $\mathbf{not}B$ appear in *body* and in $\mathbf{P}_n$;
- (U1c) $\mathsf{OB}_{\mathcal{O},P_n} \models \neg H$.

(U2) for each S with $S \subseteq \mathbf{P}_n$, on which $\mathbf{K}H$ depends, there is at least one modal $\mathbf{K}$-atom $\mathbf{K}A$ such that $\mathsf{OB}_{\mathcal{O},S \setminus \mathbf{K}A} \not\models H$ and $\mathbf{K}A \in U \cup \mathsf{KA}(\mathcal{K}) \setminus \mathbf{N}_n$.

In a footnote, the authors commented that these conditions resemble the notion of unfounded sets in (Van Gelder et al. 1991).

By this definition, let us consider Example 2 again.

Example 6
Recall $\mathcal{K}_2 = (\mathcal{O}_2, \mathcal{P}_2)$, where $\pi(\mathcal{O}_2) = (a \supset b)$ and $\mathcal{P}_2$ consists of

$$\mathbf{K}a \leftarrow \mathbf{not}\,c. \qquad \mathbf{K}c \leftarrow \mathbf{not}\,a. \qquad \mathbf{K}b \leftarrow \mathbf{K}b.$$

By the alternating fixpoint construction, its coherent well-founded partition is $(\emptyset, \{\mathbf{K}a, \mathbf{K}b, \mathbf{K}c\})$, *i.e.*, it has all **K**-atoms undefined, which is correctly captured by the alternating fixpoint construction as well as by their definition of unfounded set. Thus, their notion of unfounded set serves the purpose of proving the properties of a well-founded semantics.

However, the difference shows up when applied to arbitrary partitions. Let $(T,F) = (\emptyset, \{\mathbf{K}b\})$. Then, based on the above definition, the unfounded set is $\emptyset$. That is, even that $\mathbf{K}b$ is false in the given partition is lost in the result of computing unfounded set. In contrast, by our definition, Definition 3.1, the unfounded set is $\{\mathbf{K}a, \mathbf{K}b\}$.

Appendix B Proofs

Proposition 3.1
Let $\mathcal{K}$ be a normal hybrid MKNF knowledge base, (T,F) a partial partition of $\mathsf{KA}(\mathcal{K})$. If X_1 and X_2 are unfounded sets of $\mathcal{K}$ *w.r.t.* (T,F), then $X_1 \cup X_2$ is an unfounded set of $\mathcal{K}$ *w.r.t.* (T,F).

Proof
For each $\mathbf{K}a \in X_1$ and the corresponding MKNF rule r, that $body^+(r) \cap X_1 \neq \emptyset$ implies $body^+(r) \cap (X_1 \cup X_2) \neq \emptyset$. Similarly for each $\mathbf{K}a \in X_2$, then $X_1 \cup X_2$ is also an unfounded set of $\mathcal{K}$ *w.r.t.* (T,F). □

Proposition 3.2

Let $\mathscr{K}$ be a normal hybrid MKNF knowledge base, (T,F) a partial partition of $\mathsf{KA}(\mathscr{K})$, and U an unfounded set of $\mathscr{K}$ *w.r.t.* (T,F). For any MKNF model M of $\mathscr{K}$ with $M \models_{\text{MKNF}} \bigwedge_{\mathbf{K}a\in T} \mathbf{K}a \wedge \bigwedge_{\mathbf{K}b\in F} \neg\mathbf{K}b$, $M \models_{\text{MKNF}} \neg\mathbf{K}u$ for each $\mathbf{K}u \in U$.

Proof
Assume that there exists such an MKNF model M with $M \models_{\text{MKNF}} \mathbf{K}u$ for some $\mathbf{K}u \in U$. Let U^* be the greatest unfounded set of $\mathscr{K}$ *w.r.t.* (T,F) and

$$M' = \{I' \mid I' \models \mathsf{OB}_{\mathscr{O},T} \text{ and } I' \models a, \forall a \in \mathsf{KA}(\mathscr{K}) \setminus U^* \text{ with } M \models_{\text{MKNF}} \mathbf{K}a\}.$$

Note that, $\mathsf{OB}_{\mathscr{O},T} \not\models u$ for each $u \in U^*$, and thus $M' \supset M$.

Clearly, $(I',M',M) \models \mathbf{K}\pi(\mathscr{O})$ for each $I' \in M'$, $M' \models_{\text{MKNF}} \neg\mathbf{K}u$ for each $u \in U^*$, and $\{\mathbf{K}a \in \mathsf{KA}(\mathscr{K}) \mid M \models_{\text{MKNF}} \mathbf{K}a\} \setminus U^* = \{\mathbf{K}a \in \mathsf{KA}(\mathscr{K}) \mid M' \models_{\text{MKNF}} \mathbf{K}a\}$. Let us denote the last set by T^*.

For each $r \in \mathscr{P}$, if $body^+(r) \subseteq T^*$ and $\mathbf{K}(body^-(r)) \cap T^* = \emptyset$, then $head(r) \subseteq T^*$ and $head(r) \cap U^* = \emptyset$. So $M' \models_{\text{MKNF}} \pi(r)$. It then follows that $(I',M',M) \models \pi(\mathscr{K})$ for each $I' \in M'$, which contradicts the precondition that M is an MKNF model of $\mathscr{K}$. Therefore, $M \models_{\text{MKNF}} \neg\mathbf{K}u$ for each $\mathbf{K}u \in U$. □

Proposition 3.3
Let $\mathscr{K} = (\mathscr{O},\mathscr{P})$ be a normal hybrid knowledge base and M an MKNF model of $\mathscr{K}$. Define (T,F) by $T = \{\mathbf{K}a \in \mathsf{KA}(\mathscr{K}) \mid M \models_{\text{MKNF}} \mathbf{K}a\}$ and $F = \mathsf{KA}(\mathscr{K}) \setminus T$. Then, F is the greatest unfounded set of $\mathscr{K}$ *w.r.t.* (T,F).

Proof
Let U^* be the greatest unfounded set of $\mathscr{K}$ *w.r.t.* (T,F). We prove $F = U^*$. That $U^* \subseteq F$ follows from Proposition 3.2 under the special case that the given partition (T,F) satisfies $T = \{\mathbf{K}a \in \mathsf{KA}(\mathscr{K}) \mid M \models_{\text{MKNF}} \mathbf{K}a\}$ and $F = \mathsf{KA}(\mathscr{K}) \setminus T$.

To show $F \subseteq U^*$, assume $\mathbf{K}a \notin U^*$, from which for any unfounded set U of $\mathscr{K}$ *w.r.t.* (T,F), $\mathbf{K}a \notin U$. By definition (Def. 3.1), for each $R \subseteq \mathscr{P}$ such that $head(R) \cup \mathsf{OB}_{\mathscr{O},T} \models \mathbf{K}a$ and $head(R) \cup \mathsf{OB}_{\mathscr{O},T} \cup \{\neg b\}$ is consistent for any $\mathbf{K}b \in F$, no rule $r \in R$ satisfies any of the three conditions in Def. 3.1, which implies $body^+(r) \subseteq T$ and $\mathbf{K}(body^-(r)) \subseteq F$ and, as M is an MKNF model of $\mathscr{K}$, $head(R) \subseteq T$ and it follows $\mathbf{K}a \in T$. By definition, that $\mathbf{K}a \in T$ implies $\mathbf{K}a \notin F$. □

Theorem 3.1
Let $\mathscr{K}$ be a normal hybrid MKNF knowledge base and (T,F) a partial partition of $\mathsf{KA}(\mathscr{K})$. $U_{\mathscr{K}}(T,F) = \mathsf{KA}(\mathscr{K}) \setminus Atmost_{\mathscr{K}}(T,F)$.

Proof
We first prove that $\mathsf{KA}(\mathscr{K}) \setminus Atmost_{\mathscr{K}}(T,F)$ is an unfounded set of $\mathscr{K}$ *w.r.t.* (T,F), then we prove that for any other unfounded set U, $U \subseteq \mathsf{KA}(\mathscr{K}) \setminus Atmost_{\mathscr{K}}(T,F)$.

(1) Let $X = \mathsf{KA}(\mathscr{K}) \setminus Atmost_{\mathscr{K}}(T,F)$. If X is not an unfounded set of $\mathscr{K}$ *w.r.t.* (T,F), then there exist a $\mathbf{K}$-atom $\mathbf{K}a \in X$ and a set of MKNF rules $R \subseteq \mathscr{P}$ such that $head(R) \cup \mathsf{OB}_{\mathscr{O},T} \models a$ and $head(R) \cup \mathsf{OB}_{\mathscr{O},T} \cup \{\neg b\}$ is consistent for each $\mathbf{K}b \in F$, and for each $r \in R$:

- $body^+(r) \cap F = \emptyset$,
- $\mathbf{K}(body^-(r)) \cap T = \emptyset$, and

- $body^+(r) \cap X = \emptyset$.

Note that for each $r \in R$, $body^+(r) \subseteq Atmost_{\mathscr{K}}(T,F)$. Let $Y = \{\mathbf{K}h \mid h \in head(R)\}$. From the definition of $V_{\mathscr{K}}^{(T,F)}$, $Y \subseteq Atmost_{\mathscr{K}}(T,F)$. It follows $\mathbf{K}a \in Atmost_{\mathscr{K}}(T,F)$, which contradicts the precondition that $\mathbf{K}a \in \mathsf{KA}(\mathscr{K}) \setminus Atmost_{\mathscr{K}}(T,F)$. So X is an unfounded set of $\mathscr{K}$ *w.r.t.* (T,F).

(2) For the sake of contradiction, assume U is an unfounded set of $\mathscr{K}$ *w.r.t.* (T,F) such that $U \nsubseteq \mathsf{KA}(\mathscr{K}) \setminus Atmost_{\mathscr{K}}(T,F)$. Then there exists a $\mathbf{K}$-atom $\mathbf{K}a \in U$ such that $\mathbf{K}a \in Atmost_{\mathscr{K}}(T,F)$.

(a) If there exists an MKNF rule $r \in \mathscr{P}$, $\mathbf{K}a \in head(r)$, $body^+(r) \subseteq Atmost_{\mathscr{K}}(T,F)$, $body^+(r) \cap F = \emptyset$, $\mathbf{K}(body^-(r)) \cap T = \emptyset$, and $\{a, \neg b\} \cup \mathsf{OB}_{\mathscr{O},T}$ is consistent for each $\mathbf{K}b \in F$, then $body^+(r) \cap U \neq \emptyset$.

If $\{\mathbf{K}a\} = body^+(r) \cap U$, then there exists another MKNF rule $r' \in \mathscr{P}$ with $\mathbf{K}a \in head(r')$, $body^+(r') \subseteq Atmost_{\mathscr{K}}(T,F)$, $body^+(r') \cap F = \emptyset$, $\mathbf{K}(body^-(r')) \cap T = \emptyset$, and $\{a, \neg b\} \cup \mathsf{OB}_{\mathscr{O},T}$ is consistent for each $\mathbf{K}b \in F$. The process can continue until there exists such an MKNF rule r^* with $\{\mathbf{K}a\} \neq body^+(r^*) \cap U$.

If $\{\mathbf{K}a\} \neq body^+(r) \cap U$, then there exists another $\mathbf{K}$-atom $\mathbf{K}a_1 \in U \cap atmost_{\mathscr{K}}(T,F)$. The argument can repeat indefinitely, which results in a contradiction to the precondition that the set $\mathsf{KA}(\mathscr{K})$ is finite. So there does not exist such an MKNF rule and Case (a) is impossible.

(b) If $\mathsf{OB}_{\mathscr{O},Atmost_{\mathscr{K}}(T,F)} \models a$, then for each set of MKNF rules $R \subseteq \mathscr{P}$ with $\{\mathbf{K}h \mid h \in head(R)\} \subseteq Atmost_{\mathscr{K}}(T,F)$, $\mathsf{OB}_{\mathscr{O},\{\mathbf{K}h \mid h \in head(R)\}} \models a$, and for each $r \in R$, $body^+(r) \subseteq Atmost_{\mathscr{K}}(T,F)$, $body^+(r) \cap F = \emptyset$, $\mathbf{K}(body^-(r)) \cap T = \emptyset$, and $\{a, \neg b\} \cup \mathsf{OB}_{\mathscr{O},T}$ is consistent for each $\mathbf{K}b \in F$, there exists an MKNF rule $r^* \in R$ such that $body^+(r^*) \cap U \neq \emptyset$.

Note that, since such a set R always exists, so does such an MKNF rule r^*. However, from the proof for (a), there does not exist such an MKNF rule r^*. Thus Case (b) is impossible.

So for each unfounded set U of $\mathscr{K}$ *w.r.t.* (T,F), $U \subseteq \mathsf{KA}(\mathscr{K}) \setminus Atmost_{\mathscr{K}}(T,F)$.

From (1) and (2), $U_{\mathscr{K}} = \mathsf{KA}(\mathscr{K}) \setminus Atmost_{\mathscr{K}}(T,F)$. □

Theorem 3.2
Let $\mathscr{K}$ be a normal hybrid MKNF knowledge base. $W_{\mathscr{K}}(\emptyset,\emptyset) = (\mathbf{P}_\omega, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_\omega)$.

Proof
By induction we can prove that $(\mathbf{P}_\omega, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_\omega) \sqsubseteq W_{\mathscr{K}}(\emptyset,\emptyset)$. In the following we show that $W_{\mathscr{K}}(\emptyset,\emptyset) \sqsubseteq (\mathbf{P}_\omega, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_\omega)$.

Let $W_{\mathscr{K}}^{(\emptyset,\emptyset)} \uparrow^k = (T_k, F_k)$. Clearly, $(T_0, F_0) \sqsubseteq (\mathbf{P}_\omega, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_\omega)$. Assuming that $(T_i, F_i) \sqsubseteq (\mathbf{P}_\omega, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_\omega)$, we want to prove that $W_{\mathscr{K}}^{(\emptyset,\emptyset)}(T_i, F_i) \sqsubseteq (\mathbf{P}_\omega, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_\omega)$.

$T_{\mathscr{K}}^{(\emptyset,\emptyset)}(T_i, F_i) = T^*_{\mathscr{K},\mathsf{KA}(\mathscr{K}) \setminus F_i}(T_i) \subseteq \mathbf{P}_\omega$, $Atmost_{\mathscr{K}}(T_i, \emptyset) = \Gamma'_{\mathscr{K}}(T_i) \supseteq \Gamma'_{\mathscr{K}}(\mathbf{P}_\omega)$. By induction, we can assume that $Atmost_{\mathscr{K}}(T_k, F_j) \supseteq \Gamma'_{\mathscr{K}}(\mathbf{P}_\omega)$ for each $0 \leq k \leq i$ and $0 \leq j < i$. We want to prove that $Atmost_{\mathscr{K}}(T_i, F_i) \supseteq \Gamma'_{\mathscr{K}}(\mathbf{P}_\omega)$.

Let $F_i = \mathsf{KA}(\mathscr{K}) \setminus Atmost_{\mathscr{K}}(T_{i-1}, F_{i-1})$. If $\Gamma'_{\mathscr{K}}(\mathbf{P}_\omega) \nsubseteq Atmost_{\mathscr{K}}(T_i, F_i)$, then there are two possible cases.

Case 1: There exists $r \in \mathscr{P}$ such that $body^+(r) \subseteq \Gamma'_{\mathscr{K}}(\mathbf{P}_\omega)$ and $body^+(r) \cap F_i \neq \emptyset$. Then $\Gamma'_{\mathscr{K}}(\mathbf{P}_\omega) \cap (\mathsf{KA}(\mathscr{K}) \setminus Atmost_{\mathscr{K}}(T_{i-1}, F_{i-1})) \neq \emptyset$, thus $\Gamma'_{\mathscr{K}}(\mathbf{P}_\omega) \nsubseteq Atmost_{\mathscr{K}}(T_{i-1}, F_{i-1})$, which conflicts to the assumption for the induction. So this case is impossible.

Case 2: There exists $\mathbf{K}a \in \mathsf{KA}(\mathscr{K})$ such that $\mathbf{K}a \in \Gamma'_{\mathscr{K}}(\mathbf{P}_\omega)$, $\mathbf{K}a \notin Atmost_{\mathscr{K}}(T_i, F_i)$, $\{a, \neg b\} \cup \mathsf{OB}_{\mathscr{O},T_i}$ for some $\mathbf{K}b \in F_i$ is inconsistent, and $\{a\} \cup \mathsf{OB}_{\mathscr{O},T_i}$ is consistent. Then $\mathsf{OB}_{\mathscr{O},T_i} \models a \supset b$, thus $\mathbf{K}b \in \Gamma'_{\mathscr{K}}(\mathbf{P}_\omega)$. $\mathbf{K}b \in F_i$ implies $\mathbf{K}b \notin Atmost_{\mathscr{K}}(T_{i-1}, F_{i-1})$. Then $\Gamma'_{\mathscr{K}}(\mathbf{P}_\omega) \not\subseteq Atmost_{\mathscr{K}}(T_{i-1}, F_{i-1})$, which conflicts to the assumption for the induction. So this case is also impossible.

Then it is impossible that $\Gamma'_{\mathscr{K}}(\mathbf{P}_\omega) \not\subseteq Atmost_{\mathscr{K}}(T_i, F_i)$. So $\Gamma'_{\mathscr{K}}(\mathbf{P}_\omega) \subseteq Atmost_{\mathscr{K}}(T_i, F_i)$ and $U_{\mathscr{K}}(T_i, F_i) \subseteq \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_\omega$. So $W^{(\emptyset,\emptyset)}_{\mathscr{K}}(T_i, F_i) \sqsubseteq (\mathbf{P}_\omega, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_\omega)$ and $W_{\mathscr{K}}(\emptyset, \emptyset) \sqsubseteq (\mathbf{P}_\omega, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}_\omega)$. □

Theorem 3.3

Let $\mathscr{K}$ be a normal hybrid MKNF knowledge base and (T, F) a partial partition of $\mathsf{KA}(\mathscr{K})$. $(\mathbf{P}^{(T,F)}_i, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}^{(T,F)}_i) \sqsubseteq W_{\mathscr{K}}(T, F) \sqsubseteq E_{\mathscr{K}}(T, F)$, for each $i > 0$.

Proof

Let $W_{\mathscr{K}}(T, F) = (T^*, F^*)$. We start with $(\mathbf{P}^{(T,F)}_0, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}^{(T,F)}_0) = (T, F)$. It can be verified that $(\mathbf{P}^{(T,F)}_1, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}^{(T,F)}_1) \sqsubseteq (T^*, F^*)$. Assuming that $(\mathbf{P}^{(T,F)}_i, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}^{(T,F)}_i) \sqsubseteq (T^*, F^*)$ $(i > 0)$, we want to prove that $(\Gamma_{\mathscr{K}}(\mathbf{N}^{(T,F)}_i), \mathsf{KA}(\mathscr{K}) \setminus \Gamma'_{\mathscr{K}}(\mathbf{P}^{(T,F)}_i)) \sqsubseteq (T^*, F^*)$, which can be similarly proved by the proof for Theorem 3.2. So $(\mathbf{P}^{(T,F)}_i, \mathsf{KA}(\mathscr{K}) \setminus \mathbf{N}^{(T,F)}_i) \sqsubseteq W_{\mathscr{K}}(T, F)$, for each $i > 0$. □

Theorem 5.1

Let $\mathscr{K}$ be a normal hybrid MKNF knowledge base and (T, F) the well-founded partition of $\mathscr{K}$. An MKNF interpretation M is an MKNF model of $\mathscr{K}$ iff M is an MKNF model of $\mathscr{K}^{(T,F)}$.

Proof

We use $\mathscr{M}(\mathscr{K})$ to be the set of all MKNF models of $\mathscr{K}$. Assuming that for a partial partition (T', F') of $\mathsf{KA}(\mathscr{K})$, $\mathscr{M}(\mathscr{K}) = \mathscr{M}(\mathscr{K}^{(T',F')})$ and for each $M \in \mathscr{M}(\mathscr{K})$, $M \models_{\mathrm{MKNF}} \bigwedge_{Ka \in T'} \mathbf{K}a \wedge \bigwedge_{Kb \in F'} \neg \mathbf{K}b$. We want to prove that $\mathscr{M}(\mathscr{K}) = \mathscr{M}(\mathscr{K}^{W^{(\emptyset,\emptyset)}_{\mathscr{K}}(T',F')})$ and for each $M \in \mathscr{M}(\mathscr{K})$, $M \models_{\mathrm{MKNF}} \bigwedge_{Ka \in T^{(\emptyset,\emptyset)}_{\mathscr{K}}(T',F')} \mathbf{K}a \wedge \bigwedge_{\mathbf{K}b \in U^{(\emptyset,\emptyset)}_{\mathscr{K}}(T',F')} \neg \mathbf{K}b$.

From Proposition 3.2, it is easy to verify that, for each $M \in \mathscr{M}(\mathscr{K})$, $M \models_{\mathrm{MKNF}} \bigwedge_{Ka \in T^{(\emptyset,\emptyset)}_{\mathscr{K}}(T',F')} \mathbf{K}a \wedge \bigwedge_{\mathbf{K}b \in U^{(\emptyset,\emptyset)}_{\mathscr{K}}(T',F')} \neg \mathbf{K}b$. Then $\mathscr{M}(\mathscr{K}) = \mathscr{M}(\mathscr{K}^{W^{(\emptyset,\emptyset)}_{\mathscr{K}}(T',F')})$.

Then the theorem can be proved from the fact that the well-founded partition is equivalent to $W^{(\emptyset,\emptyset)}_{\mathscr{K}} \uparrow^\infty$. □