Appendix A Proofs and Calcuclations

A.1 ω-continuity of the $\widehat{T}_P$ operator

Let $D = \{d_0, d_1, d_2, \ldots\}$ be an ω-chain, that is, $d_0 \subseteq d_1 \subseteq d_2 \subseteq \ldots$. We need to show that $\widehat{T}_P(\bigcup D) = \bigcup \widehat{T}_P(D)$. The left-hand side:

$$\begin{aligned}
\widehat{T}_P(\bigcup D) &= \bigcup\{\rho(\bigvee Y) \mid Y \in \mathcal{P}^{\text{fin}}(\eta(T_P(\bigcup D)))\} && \text{(def. of } \widehat{T}_P\text{)} \\
&= \bigcup\{\rho(\bigvee Y) \mid Y \in \mathcal{P}^{\text{fin}}(\eta(\bigcup T_P(D)))\} && (T_P \text{ is } \omega\text{-cont.}) \\
&= \{x \mid \exists Y \in \mathcal{P}^{\text{fin}}(\eta(\bigcup T_P(D))).\ x \in \rho(\bigvee Y)\} && \text{(def. of } \bigcup\text{)}
\end{aligned}$$

The right-hand side:

$$\begin{aligned}
\bigcup \widehat{T}_P(D) &= \bigcup\{\widehat{T}_P(d) \mid d \in D\} && \text{(image)} \\
&= \bigcup\{\bigcup\{\rho(\bigvee Y) \mid Y \in \mathcal{P}^{\text{fin}}(\eta(T_P(d)))\} \mid d \in D\} && \text{(def. of } \widehat{T}_P\text{)} \\
&= \{x \mid \exists d \in D.\ \exists Y \in \mathcal{P}^{\text{fin}}(\eta(T_P(d))).\ x \in \rho(\bigvee Y)\} && \text{(def of } \bigcup\text{)}
\end{aligned}$$

Thus, it is enough to show that $\exists d \in D.\ \exists Y \in \mathcal{P}^{\text{fin}}(\eta(T_P(d))).\ x \in \rho(\bigvee Y)$ if and only if $\exists Y \in \mathcal{P}^{\text{fin}}(\eta(\bigcup T_P(D))).\ x \in \rho(\bigvee Y)$. The ($\Rightarrow$) direction is trivial. For the ($\Leftarrow$) direction pick d_N with the lowest $n \in \mathbb{N}$ such that $Y \subseteq \eta(T_P(d_n))$, which exists, since Y is finite.

A.2 Proof of Theorem 4.2

Let $x \leq_L y$. Since L is generated by $\eta^L(H_P)$, there exist $X \subseteq Y \subseteq H_P$ such that $x = \bigvee \eta^L(X)$ and $y = \bigvee \eta^L(Y)$.

$$\begin{aligned}
T_P^L(x) &= T_P^L(\bigvee \eta^L(X)) \\
&= T_P^L([\eta^L](X)) && \text{(def. of } [\eta^L]\text{)} \\
&= [\eta^L](\widehat{T}_P(X)) && \text{(assumption)} \\
&\leq_L [\eta^L](\widehat{T}_P(Y)) && \text{(composition preserves monotonicity)} \\
&= T_P^L([\eta^L](Y)) && \text{(assumption)} \\
&= T_P^L(\bigvee \eta^L(Y)) && \text{(def. of } [\eta^L]\text{)} \\
&= T_P^L(y)
\end{aligned}$$

A.3 Counter Example for Example 4

$$\begin{aligned}
([\eta] \circ \widehat{T}_P)\big(\{\texttt{p(0)}, \texttt{p(1)}\}\big) &= [\eta]\big(\{\texttt{p(0)}, \texttt{p(1)}, \texttt{p(2)}, \texttt{p(3)}\}\big) \\
&= t \text{ such that } t(Q) = \text{if } Q = \texttt{p} \text{ then } 3 \text{ else } \bot \\
&\neq u \text{ such that } u(Q) = \text{if } Q = \texttt{p} \text{ then } 2 \text{ else } \bot \\
&= [\eta]\big(\{\texttt{p(2)}\}\big) \\
&= ([\eta] \circ \widehat{T}_P)\big(\{\texttt{p(1)}\}\big) \\
&= ([\eta] \circ \widehat{T}_P \circ \rho)(t) \text{ such that } t(Q) = \text{if } Q = \texttt{p} \text{ then } 1 \text{ else } \bot \\
&= ([\eta] \circ \widehat{T}_P \circ \rho \circ [\eta])\big(\{\texttt{p(0)}, \texttt{p(1)}\}\big)
\end{aligned}$$

Appendix B Additional Examples

This appendix contains some additional examples that do not fit in the main part of the paper because of the page limit, but which could be useful in understanding the details of our semantics for tabling with answer subsumption.

B.1 The Extended Immediate Consequence Operator

This example illustrates why we need to extend the T_P operator to include the results of the lattice operations, that is, why we need the $\widehat{T}_P$ operator. Consider the lattice $\{a, b, c, d\}$, with $a, b \leq c$ and $c \leq d$, which we use in the following program:

```
lub(a,b,c). lub(a,c,c). lub(a,d,d).
lub(b,a,c). lub(b,c,c). lub(b,d,d).
lub(c,d,d).
lub(X,X,X).

:- table p(lattice(lub/3)).

p(a).
p(b).
p(d) :- p(c).
```

The regular immediate consequence gives us $\mathsf{lfp}(T_P) = \{\mathtt{p(a)}, \mathtt{p(b)}\}$, which means that $\rho(\bigvee \eta(\mathsf{lfp}(T_P))) = \{\mathtt{p(c)}\}$. The atom `p(c)` does not follow from the logical inference, but from the lattice's join operator. It is included in the overall answer thanks to the post-processing step, but its logical consequences are not. With the $\widehat{T}_P$ operator we have $\mathsf{lfp}(\widehat{T}_P) = \{\mathtt{p(a)}, \mathtt{p(b)}, \mathtt{p(c)}, \mathtt{p(d)}\}$, and so $\rho(\bigvee \eta(\mathsf{lfp}(\widehat{T}_P))) = \{\mathtt{p(d)}\}$, which is the intended semantics.

B.2 Circular Dependencies

The following example shows what happens when two predicates are interdependent, but only one of them is tabled:

```
:- table even(min).

even(0).
even(X) :- odd(Y), Y is X - 1.
odd(X) :- even(Y), Y is X - 1.
```

Our semantics interprets this program as the set

$$\rho(\bigvee \eta(\mathsf{lfp}(\widehat{T}_P))) = \{\mathtt{even(0)}, \mathtt{odd(1)}, \mathtt{odd(3)}, \mathtt{odd(5)}, \mathtt{odd(7)}, \ldots\}.$$

One other possible candidate would be $\{\mathtt{even(0)}, \mathtt{odd(1)}\}$. It is because we treat subsumption as a post-processing step *per stratum*, which means that inter-dependent predicates are resolved as if no answers were subsumed. Subsumption affects predicates in the strata above. For instance, assume we add the following (non-tabled) predicate to the program:

```
also_odd(X) :-  even(Y), Y is X - 1.
```

It is in a different stratum than `even` and `odd`, so its semantics depends on the semantics of `even` after the subsumption step. This means that the semantics together with the `also_odd` predicate is given as:

$$\{\mathtt{even(0)}, \mathtt{also\_odd(1)}, \mathtt{odd(1)}, \mathtt{odd(3)}, \mathtt{odd(5)}, \mathtt{odd(7)}, \ldots\}$$

Importantly, programs like the one above do not satisfy our correctness condition for the greedy strategy.

B.3 Example of answer subsumption for arbitrary lattices

Consider the shortest path program, now interpreted in the lattice $\langle \mathbb{N} \cup \{\infty\}, \leq_\infty \rangle$, the natural numbers extended with infinity with the canonical order, and abstractions and representations $\lceil\cdot\rceil_e^\infty$, $\lfloor\cdot\rfloor_e^\infty$, $\lceil\cdot\rceil_p^\infty$ and $\lfloor\cdot\rfloor_p^\infty$, where:

$$\lceil \texttt{nt} \rceil_e^\infty = 1 \qquad\qquad \lfloor x \rfloor_e^\infty = \texttt{nt}$$

$$\lceil \texttt{d} \rceil_p^\infty = \begin{cases} \infty & \text{if } \texttt{d} = \texttt{infty} \\ d & \text{otherwise} \end{cases} \qquad\qquad \lfloor d \rfloor_p^\infty = \begin{cases} \texttt{d} & \text{if } d \in \mathbb{N} \\ \texttt{infty} & \text{otherwise} \end{cases}$$

This lattice computes the *longest* path in the graph, demonstrating that a change in the interpreting lattice can change the result of the program entirely. Now suppose the edge `e(c,a,nt)` is added to the program, creating a cycle. The least fixed-point semantics $\mathsf{lfp}(\widehat{T}_P)$ becomes infinite, and $\rho(\bigvee_{x \in \mathsf{lfp}(\widehat{T}_P)} \eta^\infty(x))$ contains only paths of infinite length, represented by atoms such as `p(a,c,infty)`. Although such least fixed-points are not constructively computable in practice, from a theoretical point of view they demonstrate the essence of this approach quite well.