Appendix A Problems code

In this section we present the Problog and PFL code of the testing problems.

A.1 Workshops Attributes

To all programs of this section we added 50 workshops and an increasing number of attributes.

Problog program

```
series:- person(P),attends(P),sa(P).

0.501::sa(P):-person(P).

attends(P):- person(P),attr(A),at(P,A).

0.3::at(P,A):-person(P),attr(A).
```

PFL program

```
het series1,ch1(P);[1.0, 0.0, 0.0, 1.0];[person(P)].

deputy series,series1;[].

bayes ch1(P),attends(P),sa(P);[1.0,1.0,1.0,0.0,
                               0.0,0.0,0.0,1.0];[person(P)].

bayes sa(P);[0.499,0.501];[person(P)].

het attends1(P),at(P,A);[1.0, 0.0, 0.0, 1.0];[person(P),attr(A)].

deputy attends(P),attends1(P);[person(P)].

bayes at(P,A);[0.7,0.3];[person(P),attr(A)].
```

A.2 Competing Workshops

For the *competing workshops* problem we report only the PFL version. For testing purpose we added to this code 10 workshops and an increasing number of people.

PFL program

```
bayes ch1(P),attends(P),sa(P);[1.0,1.0,1.0,0.0,
                               0.0,0.0,0.0,1.0];[person(P)].

het series1,ch1(P);[1.0, 0.0, 0.0, 1.0];[person(P)].

deputy series,series1;[].

bayes sa(P);[0.499,0.501];[person(P)].

het attends1(P),ch2(P,W);[1.0, 0.0, 0.0, 1.0];[person(P),workshop(W)].

deputy attends(P),attends1(P);[person(P)].

bayes ch2(P,W),hot(W),ah(P,W);[1.0,1.0,1.0,0.0,
                               0.0,0.0,0.0,1.0];[person(P),workshop(W)].

bayes ah(P,W);[0.2,0.8];[person(P),workshop(W)].
```

A.3 Plates

For tha *plates* problem we added 5 individuals for X and an increasing number of individuals for Y.

Problog program

```
f:- e(Y).

e(Y) :- d(Y),n1(Y).
e(Y) :- y(Y),\+ d(Y),n2(Y).

d(Y):- c(X,Y).

c(X,Y):-b(X),n3(X,Y).
c(X,Y):- x(X),\+ b(X),n4(X,Y).

b(X):- a, n5(X).
b(X):- \+ a,n6(X).

a:- n7.

0.1::n1(Y) :-y(Y).
0.2::n2(Y) :-y(Y).
0.3::n3(X,Y) :- x(X),y(Y).
0.4::n4(X,Y) :- x(X),y(Y).
0.5::n5(X) :-x(X).
0.6::n6(X) :-x(X).
0.7::n7.
```

PFL program

```
het f1,e(Y);[1.0, 0.0, 0.0, 1.0];[y(Y)].

deputy f,f1;[].

bayes e1(Y),d(Y),n1(Y);[1.0, 1.0, 1.0, 0.0,
                        0.0, 0.0, 0.0, 1.0];[y(Y)].

bayes e2(Y),d(Y),n2(Y);[1.0, 0.0, 1.0, 1.0,
                        0.0, 1.0, 0.0, 0.0];[y(Y)].

bayes e(Y),e1(Y),e2(Y);[1.0, 0.0, 0.0, 0.0,
                        0.0, 1.0, 1.0, 1.0];[y(Y)].

het d1(Y),c(X,Y);[1.0, 0.0, 0.0, 1.0];[x(X),y(Y)].

deputy d(Y),d1(Y);[y(Y)].

bayes c1(X,Y),b(X),n3(X,Y);[1.0, 1.0, 1.0, 0.0,
                            0.0, 0.0, 0.0, 1.0];[x(X),y(Y)].

bayes c2(X,Y),b(X),n4(X,Y);[1.0, 0.0, 1.0, 1.0,
                            0.0, 1.0, 0.0, 0.0];[x(X),y(Y)].

bayes c(X,Y),c1(X,Y),c2(X,Y);[1.0, 0.0, 0.0, 0.0,
                              0.0, 1.0, 1.0, 1.0];[x(X),y(Y)].

bayes b1(X),a,n5(X);[1.0, 1.0, 1.0, 0.0,
                     0.0, 0.0, 0.0, 1.0];[x(X)].

bayes b2(X),a,n6(X);[1.0, 0.0, 1.0, 1.0,
                     0.0, 1.0, 0.0, 0.0];[x(X)].

bayes b(X),b1(X),b2(X);[1.0, 0.0, 0.0, 0.0,
```

```
                    0.0, 1.0, 1.0, 1.0];[x(X)].

bayes a,n7;[1.0, 0.0, 0.0, 1.0];[].

bayes n1(Y);[0.9, 0.1];[y(Y)].
bayes n2(Y);[0.8, 0.2];[y(Y)].
bayes n3(X,Y);[0.7, 0.3];[x(X),y(Y)].
bayes n4(X,Y);[0.6, 0.4];[x(X),y(Y)].
bayes n5(X);[0.5, 0.5];[x(X)].
bayes n6(X);[0.4, 0.6];[x(X)].
bayes n7;[0.3, 0.7];[].
```

Appendix B Definitions

Definition 1 (*counting formula*)
A counting formula is a syntactic construct of the form $\#_{X_i \in C}[F(\mathbf{X})]$, where $X_i \in \mathbf{X}$ is called the counted logvar.

A *grounded* counting formula is a counting formula in which all arguments of the atom $F(\mathbf{X})$, except for the counted logvar, are constants. It defines a counting randvar (CRV) as follows.

Definition 2 (*counting randvar*)
A parametrized counting randvar (PCRV) is a pair $(\#_{X_i}[F(\mathbf{X})], C)$. For each instantiation of $\mathbf{X} \setminus X_i$, it creates a separate counting randvar (CRV). The value of this CRV is a histogram, and it depends deterministically on the values of $F(\mathbf{X})$. Given a valuation for $F(\mathbf{X})$, it counts how many different values of X_i occur for each $r \in range(F)$. The result is a *histogram* of the form $\{(r_1, n_1), \ldots, (r_k, n_k)\}$, with $r_i \in range(F)$ and n_i the corresponding count.

Definition 3 (*multiplicity*)
The multiplicity of a histogram $h = \{(r1, n1), \ldots, (r_k, n_k)\}$ is a multinomial coefficient, defined as

$$\textsc{Mul}(h) = \frac{n!}{\prod_{i=1}^{k} n_i!}.$$

As multiplicities should only be taken into account for (P)CRVs, never for regular PRVs, we define for each PRV A and for each value $v \in range(A)$: $\textsc{Mul}(A, v) = 1$ if A is a regular PRV, and $\textsc{Mul}(A, v) = \textsc{Mul}(v)$ if A is a PCRV. This Mul function is identical to (Milch et al. 2008)'s NUM-ASSIGN.

Definition 4 (*Count function*)
Given a constraint $C_\mathbf{X}$, for any $\mathbf{Y} \subseteq \mathbf{X}$ and $\mathbf{Z} \subseteq \mathbf{X} - \mathbf{Y}$, the function $\textsc{Count}_{\mathbf{Y}|\mathbf{X}} : C_\mathbf{X} \rightarrow \mathbb{N}$ is defined as follows:

$$\textsc{Count}_{\mathbf{Y}|\mathbf{Z}}(t) = |\pi_\mathbf{Y}(\sigma_{\mathbf{Z}=\pi_\mathbf{Z}(t)}(C_\mathbf{X}))|$$

That is, for any tuple t, this function tells us how many values for $\mathbf{Y}$ co-occur with t's value for $\mathbf{Z}$ in the constraint. We define $\textsc{Count}_{\mathbf{Y}|\mathbf{Z}}(t) = 1$ when $\mathbf{Y} = \emptyset$.

Definition 5 (*Count-normalized constraint*)
For any constraint $C_{\mathbf{X}}$, $\mathbf{Y} \subseteq \mathbf{X}$ and $\mathbf{Z} \subseteq \mathbf{X} - \mathbf{Y}$, $\mathbf{Y}$ is count-normalized w.r.t. $\mathbf{Z}$ in $C_{\mathbf{X}}$ if and only if

$$\exists n \in \mathbb{N} : \forall t \in C_{\mathbf{X}} : \textsc{Count}_{\mathbf{Y}|\mathbf{Z}}(t) = n.$$

When such an n exists, we call it the conditional count of $\mathbf{Y}$ given $\mathbf{Z}$ in $C_{\mathbf{X}}$, and denote it $\textsc{Count}_{\mathbf{Y}|\mathbf{Z}}(C_{\mathbf{X}})$.

Definition 6 (*substitution*)
A substitution $\theta = \{X_1 \to t_1, \ldots, X_n \to t_n\} = \{\mathbf{X} \to \mathbf{t}\}$ maps each logvar X_i to a term t_i, which can be a constant or a logvar. When all t_i are constants, θ is called a grounding substitution, and when all are different logvars, a renaming substitution. Applying a substitution θ to an expression α means replacing each occurrence of X_i in α with t_i; the result is denoted $\alpha\theta$.

Definition 7 (*alignment*)
An alignment θ between two parfactors $g = \phi(\mathcal{A})|C$ and $g' = \phi'(\mathcal{A}')|C'$ is a one-to-one substitution $\{\mathbf{X} \to \mathbf{X}'\}$, with $\mathbf{X} \subseteq logvar(\mathcal{A})$ and $\mathbf{X}' \subseteq logvar(\mathcal{A}')$, such that $\rho(\pi_{\mathbf{X}'}(C)) = \pi_{\mathbf{X}'}(C')$ (with ρ the attribute renaming operator).

An alignment tells the multiplication operator that two atoms in two different parfactors represent the same PRV, so it suffices to include it in the resulting parfactor only once.

Appendix C Correctness proof for heterogeneous multiplication

Theorem 1
Given a model $(\mathcal{F}_1, \mathcal{F}_2)$, two heterogeneous parfactors $g_1, g_2 \in \mathcal{F}_2$ and an alignment θ between g_1 and g_2, if the preconditions of the HET-MULTIPLY operator are fulfilled then the postcondition

$$G \setminus \{g_1, g_2\} \cup \{\textsc{het-multiply}(g_1, g_2, \theta)\}$$

holds.

Proof
Immediate from the definition of heterogeneous multiplication. □

Theorem 2
Given a model $(\mathcal{F}_1, \mathcal{F}_2)$, a heterogeneous parfactor $g \in \mathcal{F}_2$ and an atom A_{k+1} to be summed out, if the preconditions of the HET-SUM-OUT operator are fulfilled then the postcondition

$$\mathcal{P}_{G\setminus\{g\}\cup\{\textsc{het-sum-out}(g,(A_1,\ldots,A_k),A_{k+1})\}} = \sum_{RV(A_{k+1})} \mathcal{P}_{\mathcal{G}}$$

holds.

Proof

We prove the formula giving *phi'* in Operator 2 by double induction over $r = \text{COUNT}_{\mathbf{X}^{excl}|\mathbf{X}^{com}}(C)$ and the number n of values at t in the tuple $(a'_1, \dots, a'_k)$. For simplicity we assume that the variable to be summed out, A_{k+1}, is not a counting variable, but the same reasoning can be applied for a counting variable. For $n = 0$, $r = 1$

$$\phi'(f, \dots, f, \mathbf{b}) = \phi(f, \dots, f, f, \mathbf{b}) + \phi(f, \dots, f, t, \mathbf{b})$$

so the thesis is proved. For $n = 0$, $r > 1$, let us call $\phi'_r(a'_1, \dots, a'_k, \mathbf{b})$ the the value of ϕ' for r. Let us assume that the formula holds for $r - 1$. For $r > 1$, there is an extra valuation $\mathbf{x}_{excl}$ for $\mathbf{X}_{excl}$ given $\mathbf{X}_{com}$ so there is an extra factor $g''(\mathbf{x}_{excl})$. Eliminating A_{k+1} from g'' gives

$$\phi'_1(f, \dots, f, \mathbf{b}) = \phi(f, \dots, f, f, \mathbf{b}) + \phi(f, \dots, f, t, \mathbf{b})$$

This must be multiplied by ϕ'_{r-1} with heterogeneous multiplication as $A_1, \dots, A_k$ are shared obtaining

$$\phi'_r(f, \dots, f, \mathbf{b}) = \sum_{\hat{a}_1 \vee \check{a}_1 = f} \dots \sum_{\hat{a}_k \vee \check{a}_k = f} \phi'_{r-1}(\hat{a}_1, \dots, \hat{a}_k, \mathbf{b}) \times \phi'_1(\check{a}_1, \dots, \check{a}_k, f, \mathbf{b}) =$$

$$= \phi'_{r-1}(f, \dots, f, \mathbf{b}) \times \phi'_1(\check{a}_1, \dots, \check{a}_k, f, \mathbf{b}) =$$
$$= (\phi(f, \dots, f, f, \mathbf{b}) + \phi(f, \dots, f, t, \mathbf{b}))^{r-1} \times (\phi(f, \dots, f, f, \mathbf{b}) + \phi(f, \dots, f, t, \mathbf{b})) =$$
$$= (\phi(f, \dots, f, f, \mathbf{b}) + \phi(f, \dots, f, t, \mathbf{b}))^{r}$$

so the thesis is proved.

For $n > 0$ of values at t in the tuple $\mathbf{a}' = (a'_1, \dots, a'_k)$, we assume that the formula holds for $(n-1, r)$ and $(n, r-1)$. For the defintiion of heterogeneous multiplication

$$\phi'_r(a'_1, \dots, a'_k, \mathbf{b}) \quad = \quad \sum_{\hat{a}_1 \vee \check{a}_1 = a'_1} \dots \sum_{\hat{a}_k \vee \check{a}_k = a'_k} \phi'_{r-1}(\hat{a}_1, \dots, \hat{a}_k, \mathbf{b}) \phi'_1(\check{a}_1, \dots, \check{a}_k, \mathbf{b})$$

By adding and removing

$\sum_{\mathbf{a}<\mathbf{a}'} \sum_{\hat{a}_1 \vee \check{a}_1 = a_1} \dots \sum_{\hat{a}_k \vee \check{a}_k = a_k} \phi'_{r-1}(\hat{a}_1, \dots, \hat{a}_k, \mathbf{b}) \phi'_1(\check{a}_1, \dots, \check{a}_k, \mathbf{b})$

we get

$\phi'_r(a'_1, \dots, a'_k, \mathbf{b}) =$

$$\begin{aligned}
= & \sum_{\hat{a}_1 \vee \check{a}_1 = a'_1} \dots \sum_{\hat{a}_k \vee \check{a}_k = a'_k} \phi'_{r-1}(\hat{a}_1, \dots, \hat{a}_k, \mathbf{b}) \phi'_1(\check{a}_1, \dots, \check{a}_k, \mathbf{b}) + \\
& + \sum_{\mathbf{a}<\mathbf{a}'} \sum_{\hat{a}_1 \vee \check{a}_1 = a_1} \dots \sum_{\hat{a}_k \vee \check{a}_k = a_k} \phi'_{r-1}(\hat{a}_1, \dots, \hat{a}_k, \mathbf{b}) \phi'_1(\check{a}_1, \dots, \check{a}_k, \mathbf{b}) - \\
& - \sum_{\mathbf{a}<\mathbf{a}'} \sum_{\hat{a}_1 \vee \check{a}_1 = a_1} \dots \sum_{\hat{a}_k \vee \check{a}_k = a_k} \phi'_{r-1}(\hat{a}_1, \dots, \hat{a}_k, \mathbf{b}) \phi'_1(\check{a}_1, \dots, \check{a}_k, \mathbf{b}) = \\
= & \sum_{\hat{a}_1 \le a'_1} \sum_{\check{a}_1 \le a'_1} \dots \sum_{\hat{a}_k \le a'_k} \sum_{\check{a}_k \le a'_k} \phi'_{r-1}(\hat{a}_1, \dots, \hat{a}_k, \mathbf{b}) \phi'_1(\check{a}_1, \dots, \check{a}_k, \mathbf{b}) - \\
& - \sum_{\mathbf{a}<\mathbf{a}'} \sum_{\hat{a}_1 \vee \check{a}_1 = a_1} \dots \sum_{\hat{a}_k \vee \check{a}_k = a_k} \phi'_{r-1}(\hat{a}_1, \dots, \hat{a}_k, \mathbf{b}) \phi'_1(\check{a}_1, \dots, \check{a}_k, \mathbf{b}) =
\end{aligned}$$

For the definition of heterogeneous multiplication we obtain $\phi'_r(a'_1, \dots, a'_k, \mathbf{b}) =$

$$= \left(\sum_{\hat{\mathbf{a}} \le \mathbf{a}'} \phi'_{r-1}(\hat{a}_1, \dots, \hat{a}_k, \mathbf{b}) \right) \times \left(\sum_{\check{\mathbf{a}} \le \mathbf{a}'} \phi'_1(\check{a}_1, \dots, \check{a}_k, \mathbf{b}) \right) - \sum_{\mathbf{a}<\mathbf{a}'} \phi'_r(a_1, \dots, a_k, \mathbf{b})$$

By applying the inductive hypothesis for $r-1$ we get $\phi'_r(a'_1,\ldots,a'_k,\mathbf{b}) =$

$$\begin{aligned}
= &\left(\sum_{\hat{\mathbf{a}}\leq\mathbf{a}'}\left(\sum_{\mathbf{a}\leq\hat{\mathbf{a}}}\phi(a_1,\ldots,a_k,f,\mathbf{b})+\phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^{r-1}-\sum_{\mathbf{a}<\hat{\mathbf{a}}}\phi'_{r-1}(a_1,\ldots,a_k,\mathbf{b})\right)\times\\
&\times\left(\sum_{\check{\mathbf{a}}\leq\mathbf{a}'}\phi'_1(\check{a}_1,\ldots,\check{a}_k,\mathbf{b})\right)-\sum_{\mathbf{a}<\mathbf{a}'}\phi'_r(a_1,\ldots,a_k,\mathbf{b}) =\\
= &\left(\sum_{\hat{\mathbf{a}}\leq\mathbf{a}'}\left(\sum_{\mathbf{a}\leq\hat{\mathbf{a}}}\phi(a_1,\ldots,a_k,f,\mathbf{b})+\phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^{r-1}\right)\times\left(\sum_{\check{\mathbf{a}}\leq\mathbf{a}'}\phi'_1(\check{a}_1,\ldots,\check{a}_k,\mathbf{b})\right)-\\
&-\left(\sum_{\hat{\mathbf{a}}\leq\mathbf{a}'}\sum_{\mathbf{a}<\hat{\mathbf{a}}}\phi'_{r-1}(a_1,\ldots,a_k,\mathbf{b})\right)\times\left(\sum_{\check{\mathbf{a}}\leq\mathbf{a}'}\phi'_1(\check{a}_1,\ldots,\check{a}_k,\mathbf{b})\right)\\
&-\sum_{\mathbf{a}<\mathbf{a}'}\phi'_r(a_1,\ldots,a_k,\mathbf{b}) =\\
= &\left(\sum_{\mathbf{a}\leq\mathbf{a}'}\phi(a_1,\ldots,a_k,f,\mathbf{b})+\phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^{r-1}\times\left(\sum_{\mathbf{a}\leq\mathbf{a}'}\phi'_1(\check{a}_1,\ldots,\check{a}_k,\mathbf{b})\right)+\\
&+\left(\sum_{\hat{\mathbf{a}}<\mathbf{a}'}\left(\sum_{\mathbf{a}\leq\hat{\mathbf{a}}}\phi(a_1,\ldots,a_k,f,\mathbf{b})+\phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^{r-1}\right)\times\left(\sum_{\check{\mathbf{a}}\leq\mathbf{a}'}\phi'_1(\check{a}_1,\ldots,\check{a}_k,\mathbf{b})\right)-\\
&-\left(\sum_{\hat{\mathbf{a}}\leq\mathbf{a}'}\sum_{\mathbf{a}<\hat{\mathbf{a}}}\phi'_{r-1}(a_1,\ldots,a_k,\mathbf{b})\right)\times\left(\sum_{\check{\mathbf{a}}\leq\mathbf{a}'}\phi'_1(\check{a}_1,\ldots,\check{a}_k,\mathbf{b})\right)-\\
&-\sum_{\mathbf{a}<\mathbf{a}'}\phi'_r(a_1,\ldots,a_k,\mathbf{b}) =
\end{aligned}$$

By applying the formula for $r = 1$
$\phi'_r(a'_1,\ldots,a'_k,\mathbf{b}) =$

$$\begin{aligned}
= &\left(\sum_{\mathbf{a}\leq\mathbf{a}'}\phi(a_1,\ldots,a_k,f,\mathbf{b})+\phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^{r-1}\times\left(\sum_{\mathbf{a}\leq\mathbf{a}'}\phi(a_1,\ldots,a_k,f,\mathbf{b})+\phi(a_1,\ldots,a_k,t,\mathbf{b})\right)+\\
&+\left(\sum_{\hat{\mathbf{a}}<\mathbf{a}'}\left(\sum_{\mathbf{a}\leq\hat{\mathbf{a}}}\phi(a_1,\ldots,a_k,f,\mathbf{b})+\phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^{r-1}\right)\times\left(\sum_{\check{\mathbf{a}}\leq\mathbf{a}'}\phi'_1(\check{a}_1,\ldots,\check{a}_k,\mathbf{b})\right)-\\
&-\left(\sum_{\hat{\mathbf{a}}<\mathbf{a}'}\sum_{\mathbf{a}<\hat{\mathbf{a}}}\phi'_{r-1}(a_1,\ldots,a_k,\mathbf{b})\right)\times\left(\sum_{\check{\mathbf{a}}\leq\mathbf{a}'}\phi'_1(\check{a}_1,\ldots,\check{a}_k,\mathbf{b})\right)-\\
&-\sum_{\mathbf{a}<\mathbf{a}'}\phi'_r(a_1,\ldots,a_k,\mathbf{b}) =\\
= &\left(\sum_{\mathbf{a}\leq\mathbf{a}'}\phi(a_1,\ldots,a_k,f,\mathbf{b})+\phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^{r}-\sum_{\mathbf{a}<\mathbf{a}'}\phi'_r(a_1,\ldots,a_k,\mathbf{b})+\\
&+\left(\sum_{\hat{\mathbf{a}}<\mathbf{a}'}\left(\sum_{\mathbf{a}\leq\hat{\mathbf{a}}}\phi(a_1,\ldots,a_k,f,\mathbf{b})+\phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^{r-1}\right)\times\left(\sum_{\check{\mathbf{a}}\leq\mathbf{a}'}\phi'_1(\check{a}_1,\ldots,\check{a}_k,\mathbf{b})\right)-\\
&-\left(\sum_{\hat{\mathbf{a}}<\mathbf{a}'}\sum_{\mathbf{a}<\hat{\mathbf{a}}}\phi'_{r-1}(a_1,\ldots,a_k,\mathbf{b})\right)\times\left(\sum_{\check{\mathbf{a}}\leq\mathbf{a}'}\phi'_1(\check{a}_1,\ldots,\check{a}_k,\mathbf{b})\right)
\end{aligned}$$

By collecting the factor $\sum_{\mathbf{\check{a}}\leq\mathbf{a}'} \phi'_1(\check{a}_1,\ldots,\check{a}_k,\mathbf{b})\ \phi'_r(a'_1,\ldots,a'_k,\mathbf{b}) =$

$$\begin{aligned}
= \quad & \left(\sum_{\mathbf{a}\leq\mathbf{a}'} \phi(a_1,\ldots,a_k,f,\mathbf{b}) + \phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^r - \sum_{\mathbf{a}<\mathbf{a}'} \phi'_r(a_1,\ldots,a_k,\mathbf{b}) + \\
& + \left(\sum_{\mathbf{\hat{a}}<\mathbf{a}'}\left(\sum_{\mathbf{a}\leq\mathbf{\hat{a}}} \phi(a_1,\ldots,a_k,f,\mathbf{b}) + \phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^{r-1} - \sum_{\mathbf{\hat{a}}<\mathbf{a}'}\sum_{\mathbf{a}\leq\mathbf{\hat{a}}} \phi'_{r-1}(a_1,\ldots,a_k,\mathbf{b})\right) \times \\
& \times \left(\sum_{\mathbf{\check{a}}\leq\mathbf{a}'} \phi'_1(\check{a}_1,\ldots,\check{a}_k,\mathbf{b})\right) = \\
= \quad & \left(\sum_{\mathbf{a}\leq\mathbf{a}'} \phi(a_1,\ldots,a_k,f,\mathbf{b}) + \phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^r - \sum_{\mathbf{a}<\mathbf{a}'} \phi'_r(a_1,\ldots,a_k,\mathbf{b}) + \\
& + \left(\sum_{\mathbf{\hat{a}}<\mathbf{a}'}\left(\sum_{\mathbf{a}\leq\mathbf{\hat{a}}} \phi(a_1,\ldots,a_k,f,\mathbf{b}) + \phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^{r-1} - \sum_{\mathbf{\hat{a}}<\mathbf{a}'}\sum_{\mathbf{a}<\mathbf{\hat{a}}} \phi'_{r-1}(a_1,\ldots,a_k,\mathbf{b}) - \right. \\
& \left. - \sum_{\mathbf{a}<\mathbf{a}'} \phi'_{r-1}(a_1,\ldots,a_k,\mathbf{b})\right) \times (\sum_{\mathbf{a}\leq\mathbf{a}'} \phi'_1(a_1,\ldots,a_k,\mathbf{b})) = \\
= \quad & \left(\sum_{\mathbf{a}\leq\mathbf{a}'} \phi(a_1,\ldots,a_k,f,\mathbf{b}) + \phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^r - \sum_{\mathbf{a}<\mathbf{a}'} \phi'_r(a_1,\ldots,a_k,\mathbf{b}) + \\
& + \left(\sum_{\mathbf{a}<\mathbf{a}'} \phi'_{r-1}(a_1,\ldots,a_k,\mathbf{b}) - \sum_{\mathbf{a}<\mathbf{a}'} \phi'_{r-1}(a_1,\ldots,a_k,\mathbf{b})\right) \times \sum_{\mathbf{a}\leq\mathbf{a}'} \phi'_1(a_1,\ldots,a_k,\mathbf{b}) = \\
= \quad & \left(\sum_{\mathbf{a}\leq\mathbf{a}'} \phi(a_1,\ldots,a_k,f,\mathbf{b}) + \phi(a_1,\ldots,a_k,t,\mathbf{b})\right)^r - \sum_{\mathbf{a}<\mathbf{a}'} \phi'_r(a_1,\ldots,a_k,\mathbf{b})
\end{aligned}$$

□

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