Appendix for the paper of A Proof Theoretic Study of Soft Concurrent Constraint Programming

ELAINE PIMENTEL
Universidade Federal do Rio Grande do Norte, Natal, Brazil
(*e-mail:* elaine.pimentel@gmail.com)

CARLOS OLARTE
Pontificia Universidad Javeriana Cali, Colombia
Universidade Federal do Rio Grande do Norte, Natal, Brazil
(*e-mail:* carlos.olarte@gmail.com)

VIVEK NIGAM
Universidade Federal da Paraíba, João Pessoa, Brazil
(*e-mail:* vivek.nigam@gmail.com)

submitted 14 February 2014; revised 25 March 2014; accepted 6 May 2014

Appendix A Adequacy Theorem

In this section we will discuss the adequacy theorem. We will start by proving Theorem 2 for the case where the framework used for the specification is *SELL* (i,e, the underlying constraint system is built from an idempotent c-semiring). Later, in Section A.2, we extend this result for the *SELLS* case for non-idempotent c-semirings.

A.1 Adequacy using SELL

Since *SELL* admits a *focused* system (Andreoli 1992), we can use here the same machinery developed in (Nigam et al. 2013).

First of all, notice that, by using simple logical equivalences (such as moving the existential outwards), we can rewrite the constraints to the following shape:

$$c = \exists \overline{x}.([pc_1]_{a_1} \otimes \cdots \otimes [pc_n]_{a_n})$$

where $[pc_1]_{a_1}, \ldots, [pc_n]_{a_n}$ are all of the form $!^{a_i}(!^{a_i}A_1 \otimes \cdots \otimes !^{a_i}A_{mi})$ or of the form $!^{a_i}A$. Observe that the formula above is composed only by positive formulas. Thus, from the focusing discipline, whenever such a formula appears in the left-hand-side, it is decomposed as illustrated by the following derivation:

$$\dfrac{\dfrac{\Delta, [pc_1]_{a_1}, \ldots, [pc_n]_{a_n} \longrightarrow \mathscr{R}}{\Delta, [pc_1]_{a_1} \otimes \cdots \otimes [pc_n]_{a_n} \longrightarrow \mathscr{R}}\ n-1 \times \otimes_L}{\Delta, \exists \overline{x}.([pc_1]_{a_1} \otimes \cdots \otimes [pc_n]_{a_n}) \longrightarrow \mathscr{R}}\ p \times \exists_L$$

Next, the constraints $[pc_1]_{a_1}, \ldots, [pc_n]_{a_n}$ appearing in the premise of this derivation are moved to the contexts $a_1, \ldots, a_n$, respectively. This is all done in a negative phase. That is, focusing on

$\mathscr{P}[\![\textbf{tell}(c)]\!]$ corresponds exactly to the operational semantics of tells: the pre-constraints in c are added to the constraint store, creating fresh names in the process.

On the other hand, if such a constraint c is focused on the right, the derivation will have the shape

$$\dfrac{\dfrac{\dfrac{\Delta_{1\leq a_1} \longrightarrow pc_1}{\Delta_1 -_{[pc_1]_{a_1}}\!\!\rightarrow} \quad \cdots \quad \dfrac{\Delta_{n\leq a_n} \longrightarrow pc_n}{\Delta_n -_{[pc_n]_{a_n}}\!\!\rightarrow}}{\Delta -_{[pc_1]_{a_1}\otimes\cdots\otimes[pc_n]_{a_n}}\!\!\rightarrow}\; n-1\times\otimes}{\Delta -_{\exists\overline{x}.([pc_1]_{a_1}\otimes\cdots\otimes[pc_n]_{a_n})}\!\!\rightarrow}\; p\times\exists_R$$

where $\Delta -_c\!\!\rightarrow$ represents a sequent with left context Δ and focused on the right-hand side formula c. Here $\Delta_{i\leq a_i}$ contains the elements of Δ_i whose contexts are marked with subexponentials greater or equal to a_i. Since a_i and $\mathfrak{u},\mathfrak{p},\mathfrak{d}$ are not related, $\Delta_{i\leq a_i}$ will have only pre-constrains and non-logical axioms. This means that focusing on $\mathscr{P}[\![\textbf{ask}\ c\ \textbf{then}\ P]\!] = !^{\mathfrak{p}}(c \multimap \mathscr{P}[\![P]\!])$ corresponds to proving c only from pre-constraints and non-logical axioms and moving all the other resources to proving $\mathscr{P}[\![P]\!]$.

Continuing this exercise, we can go case by case and prove that, indeed, one focus step corresponds to one operational step, hence proving Theorem 2 with the highest level of adequacy (on derivations).

A.2 Adequacy using SELLS

The ideas above cannot be used in order to show that the adequacy theorem also holds for *SELLS*. The reason is that it is not trivial how to define a focused system to *SELLS*. Thus we will show that, in the proof of constraints, no encoded processes, procedure calls or procedure definitions are used. This is due to the fact that $\mathfrak{u},\mathfrak{p},\mathfrak{d}$ are unrelated, and $\mathfrak{p},\mathfrak{d}$ are linear.

Lemma 1

Assume the subexponential signature Σ used to build Soft-CCP. Let $\Delta\cup\{p,c\}$ be a set of formulas, where: Δ contains the encoding of non-logical axioms and constraints; c is a constraint and p is the encoding of a process or of a procedure call. Let b be the subexponential $\mathfrak{p}$ or $\mathfrak{d}$. Then the sequents $\Delta, !^b p \longrightarrow c$ and $\Delta, p \longrightarrow c$ are not provable in $SELLS_\Sigma$.

Proof

The proof is by contradiction. Assume that the sequent $\Delta, !^b p \longrightarrow c$ (resp. $\Delta, p \longrightarrow c$) is provable and consider a proof π of it with smallest height. The last rule applied in π cannot be an initial rule, because $!^b p$ (resp. p) is linear. One possible action is to derelict the formula $!^b p$ obtaining the sequent $\Delta, p \longrightarrow c$, which reduces the two cases to one. Another possibility would be applying some non logical axiom $!^{\top\mathscr{A}}(\forall\overline{x}(d \multimap e))$ in Δ. But since d,e are constraints, this will lead to a premise with the formula $!^b p$ (resp. p) in the context. Moreover, introducing the formula c is either not possible: when c is of the form $!^a pc$, b is unrelated to a (resp. the linear formula p is in the context); or when possible, that is, when c's main connective is an $\exists$ or a $\otimes$, then $!^b p$ (resp. p) is in the context of one of the premises. Finally, we can introduce the formula p if it is the encoding of a process, such as an ask. But again one of the resulting premises will again contain a formula of the form $!^b p'$ in the context, where p' is the encoding of a process. Thus there is no such minimal proof. □

Lemma 2
Assume the subexponential signature Σ used to build Soft-CCP. Let $\Delta \cup \{f, c\}$ be a set of formulas, where Δ contains the encoding of logical axioms and constraints; c is a constraint, and $!^{\mathfrak{u}} f$ is the encoding of a process definition $p(\overline{x}) \stackrel{\Delta}{=} P$. Then the sequent $\Delta, !^{\mathfrak{u}} f \longrightarrow c$ is provable in $SELLS_{\Sigma}$ if and only if $\Delta \longrightarrow c$ is provable.

Proof
The $(\Leftarrow)$ direction is straightforward as one only needs to weaken $!^{\mathfrak{u}} f$.

The $(\Rightarrow)$ direction is as follows. The only way to prove the sequent $\Delta, !^{\mathfrak{u}} f \longrightarrow c$ is by weakening $!^{\mathfrak{u}} f$. As in the proof of Lemma 1, either we cannot introduce c or when it is introduced the formula $!^{\mathfrak{u}} f$ still appears in the context of the premise. Moreover, we cannot derelict $!^{\mathfrak{u}} f$, because the resulting sequent would contain a linear formula and using the same reasoning in Lemma 1 we can show that this resulting sequent is not provable. Contracting $!^{\mathfrak{u}} f$ also does not help in the proof, as the new occurrence of $!^{\mathfrak{u}} f$ would also need to be weakened. $\square$

Hence even without using focusing in order to control the flow of the proof, we have a neat way of controlling its shape, using the subexponential structure and linearity.

Appendix B Cut-elimination for *SELLS*

We prove now Theorem 3. We shall omit the subindex "$\mathscr{A}$" in $\times_{\mathscr{A}}$ and $+_{\mathscr{A}}$ since in this context it is clear that $\times$ and $+$ refer to the operands of the c-semiring.

We start by proving the following result, which is a substitution lemma for $\preceq$.

Lemma 3
Let Σ be a subexponential signature constructed on a c-semiring. Then if $b \preceq a \times c$ and $a \preceq d$, then $b \preceq d \times c$.

Proof
Let's assume that $b \preceq a \times c$ and $a \preceq d$. We prove $b \preceq c \times d$. Recall that $x \preceq y$ if $x + y = y$ (by definition). Then $b \preceq a \times c$ iff $b + a \times c = a \times c$ and $a \preceq d$ iff $a + d = d$. By c-semiring properties, $\times$ distributes on $+$. Then, multiplying c on $a + d = d$ we get $c \times (a + d) = a \times c + c \times d = c \times d$. Hence, $a \times c \preceq c \times d$. By using the fact that $b \preceq a \times c$, we conclude $b \preceq c \times d$. $\square$

Proof of Theorem 3 We first show that Cut permutes over the promotion rule as shown below:

$$\dfrac{\dfrac{!^{a_1}F_1, \ldots, !^{a_n}F_n \longrightarrow G}{!^{a_1}F_1, \ldots, !^{a_n}F_n \longrightarrow !^{a}G}\ !^{a}{}_{R_S} \qquad \dfrac{!^{d_1}G_1, \ldots, !^{d_m}G_m, !^{a}G \longrightarrow F}{!^{d_1}G_1, \ldots, !^{d_m}G_m, !^{a}G \longrightarrow !^{b}F}\ !^{b}{}_{R_S}}{!^{a_1}F_1, \ldots, !^{a_n}F_n, !^{d_1}G_1, \ldots, !^{d_m}G_m \longrightarrow !^{b}F}\ Cut \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{!^{a_1}F_1, \ldots, !^{a_n}F_n \longrightarrow G}{!^{a_1}F_1, \ldots, !^{a_n}F_n \longrightarrow !^{a}G}\ !^{a}{}_{R_S} \qquad !^{d_1}G_1, \ldots, !^{d_m}G_m, !^{a}G \longrightarrow F}{!^{a_1}F_1, \ldots, !^{a_n}F_n, !^{d_1}G_1, \ldots, !^{d_m}G_m \longrightarrow F}\ Cut}{!^{a_1}F_1, \ldots, !^{a_n}F_n, !^{d_1}G_1, \ldots, !^{d_m}G_m \longrightarrow !^{b}F}\ !^{b}{}_{R_S}$$

The derivation above is possible since, from the left premise of the first derivation, $a \preceq a_1 \times \cdots \times a_n$ and, from the right premise of the same derivation, $b \preceq a \times d_1 \times \cdots \times d_m$. Thus from the Lemma 3, we have that $b \preceq a_1 \times \cdots \times a_n \times d_1 \times \cdots \times d_m$, *i.e.*, the last $!^b$ can be introduced.

For the rest of the cases, the proof is similar to *SELL*. The more interesting cases are:

- Promotion + dereliction

$$\dfrac{\dfrac{\Gamma \longrightarrow G}{\Gamma \longrightarrow !^a G}\ !^a{}_{R_S} \quad \dfrac{\Delta, G \longrightarrow F}{\Delta, !^a G \longrightarrow F}\ !^a{}_L}{\Gamma,\Delta \longrightarrow F}\ Cut \qquad \rightsquigarrow \qquad \dfrac{\Gamma \longrightarrow G \quad \Delta, G\longrightarrow F}{\Gamma,\Delta\longrightarrow F}\ Cut$$

- Promotion + weakening

$$\dfrac{\dfrac{\Gamma \longrightarrow G}{\Gamma \longrightarrow !^a G}\ !^a{}_{R_S} \quad \dfrac{\Delta \longrightarrow F}{\Delta, !^a G \longrightarrow F}\ !^a{}_L}{\Gamma,\Delta \longrightarrow F}\ Cut \qquad \rightsquigarrow \qquad \dfrac{\Delta \longrightarrow F}{\Gamma,\Delta\longrightarrow F}\ W$$

We can weaken Γ since applying the $!^a{}_{R_S}$ rule in the left premise forces Γ to have the shape $!^{a_1}F_1,\ldots,!^{a_n}F_n$, with $a \preceq a_1 \times \ldots \times a_n$. On the other hand, from the right-premise, $a \in U$, *i.e.*, formulas of the form $!^a F$ are allowed to contract and weaken. Since U is upwardly closed with respect to $\preceq$, we also have $a_1,\ldots,a_n \in U$. Thus $!^{a_1}F_1,\ldots,!^{a_n}F_n$ can also be weakened.

- Promotion + contraction

$$\dfrac{\dfrac{\Gamma \longrightarrow G}{\Gamma \longrightarrow !^a G}\ !^a{}_{R_S} \quad \dfrac{\Delta, !^a G, !^a G \longrightarrow F}{\Delta, !^a G \longrightarrow F}\ !^a{}_L}{\Gamma,\Delta \longrightarrow F}\ Cut \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\Gamma \longrightarrow G}{\Gamma \longrightarrow !^a G}\ !^a{}_{R_S} \quad \dfrac{\dfrac{\Gamma \longrightarrow G}{\Gamma \longrightarrow !^a G}\ !^a{}_{R_S} \quad \Delta, !^a G, !^a G \longrightarrow F}{\Delta, \Gamma, !^a G \longrightarrow F}\ Cut}{\Gamma,\Gamma,\Delta \longrightarrow F}\ Cut}{\Gamma,\Delta \longrightarrow F}\ C$$

- When Cut permutes over structural rules.

$$\dfrac{\dfrac{!^a H, !^a H, \Gamma \longrightarrow G}{!^a H, \Gamma \longrightarrow G}\ C \quad \Delta, G \longrightarrow F}{!^a H, \Gamma, \Delta \longrightarrow F}\ Cut \qquad \rightsquigarrow \qquad \dfrac{\dfrac{!^a H, !^a H, \Gamma \longrightarrow G \quad \Delta, G \longrightarrow F}{!^a H, !^a H, \Gamma, \Delta \longrightarrow F}\ Cut}{!^a H, \Gamma, \Delta \longrightarrow F}\ C$$

$$\dfrac{\dfrac{\Gamma \longrightarrow G}{!^a H, \Gamma \longrightarrow G}\ W \quad \Delta, G \longrightarrow F}{!^a H, \Gamma, \Delta \longrightarrow F}\ Cut \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\Gamma \longrightarrow G \quad \Gamma, G \longrightarrow F}{\Gamma, \Delta \longrightarrow F}\ Cut}{!^a H, \Gamma, \Delta \longrightarrow F}\ W$$

- Some other principal cases

$$\dfrac{\dfrac{\Gamma_1 \longrightarrow A \quad \Gamma_2 \longrightarrow B}{\Gamma_1, \Gamma_2 \longrightarrow A \otimes B}\ \otimes_R \quad \dfrac{\Delta, A, B \longrightarrow F}{\Delta, A \otimes B \longrightarrow F}\ \otimes_L}{\Gamma_1, \Gamma_2, \Delta \longrightarrow F}\ Cut \qquad \rightsquigarrow \qquad \dfrac{\Gamma_1 \longrightarrow A \quad \dfrac{\Gamma_2 \longrightarrow B \quad \Delta, A, B \longrightarrow F}{\Gamma_2, \Delta, A \longrightarrow F}\ Cut}{\Gamma_1, \Gamma_2, \Delta \longrightarrow F}\ Cut$$

$$\dfrac{\dfrac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \,\&\, B}\ \otimes_R \quad \dfrac{\Delta, A \longrightarrow F}{\Delta, A \,\&\, B \longrightarrow F}\ \&_L}{\Gamma, \Delta \longrightarrow F}\ Cut \qquad \rightsquigarrow \qquad \dfrac{\Gamma \longrightarrow A \quad \Delta, A \longrightarrow F}{\Gamma, \Delta \longrightarrow F}\ Cut$$

$$\dfrac{\dfrac{\overset{\Xi_1}{\Gamma \longrightarrow G[t/x]}}{\Gamma \longrightarrow \exists x.G}\ \exists_R \quad \dfrac{\overset{\Xi_2}{\Delta, G[e/x] \longrightarrow F}}{\Delta, \exists x.G \longrightarrow F}\ \exists_L}{\Gamma, \Delta \longrightarrow F}\ Cut \qquad \rightsquigarrow \qquad \dfrac{\overset{\Xi_1}{\Gamma \longrightarrow G[t/x]} \quad \overset{\Xi_2[t/e]}{\Delta, G[t/x] \longrightarrow F}}{\Gamma, \Delta \longrightarrow F}\ Cut$$

The proof of the right premise of the right figure, $\Xi_2[t/e]$ is a *SELLS* proof using the usual eigenvariable argument. This can be proved by induction on the height of proofs.

Appendix C Constraint systems as cylindric algebras

We shall now recall the abstract and general definition of constraint systems as cylindric algebras as in (de Boer et al. 1995).

Definition 1 (*Constraint System*)
A cylindric constraint system is a structure $\mathbf{C} = \langle \mathscr{C}, \leq, \sqcup, \mathbf{1}, \mathbf{0}, Var, \exists, D\rangle$ such that:
- $\langle \mathscr{C}, \leq, \sqcup, \mathbf{1}, \mathbf{0}\rangle$ is a lattice with $\sqcup$ the *lub* operation (representing the logical *and*), and **1**, **0** the least and the greatest elements in $\mathscr{C}$ respectively (representing `true` and `false`). Elements in $\mathscr{C}$ are called *constraints* with typical elements $c, c', d, d' \ldots$. If $c \leq d$ and $d \leq c$ we write $c \cong d$. If $c \leq d$ and $c \not\cong d$, we write $c < d$.
-*Var* is a denumerable set of variables.
-For each $x \in Var$ the function $\exists x : \mathscr{C} \to \mathscr{C}$ is a cylindrification operator satisfying: (E1) $\exists x(c) \leq c$; (E2) If $c \leq d$ then $\exists x(c) \leq \exists x(d)$; (E3) $\exists x(c \sqcup \exists x(d)) \cong \exists x(c) \sqcup \exists x(d)$; (E4) $\exists x \exists y(c) \cong \exists y \exists x(c)$.
- For each $x, y \in Var$, the constraint $d_{xy} \in D$ is a *diagonal element* and it satisfies: (D1) $d_{xx} \cong \mathbf{1}$; (D2) If z is different from x, y then $d_{xy} \cong \exists z(d_{xz} \sqcup d_{zy})$; (D3) If x is different from y then $c \leq d_{xy} \sqcup \exists x(c \sqcup d_{xy})$.
- We say that d entails c, notation $d \models c$, iff $c \leq d$.

The cylindrification operators model a sort of existential quantification, helpful for hiding information. Properties (E1) to (E4) are standard.

The diagonal element d_{xy} can be thought of as the equality $x = y$. Properties (D1) to (D3) are standard and they allow the definition of substitutions of the form $[y/x]$ required, for instance, to represent the substitution of formal and actual parameters in procedure calls. By using these properties, it is easy to prove that $c[y/x] \cong \exists x.(c \sqcup d_{xy})$, where $c[y/x]$ represents abstractly the constraint obtained from c by replacing the variables x by y. As it is customary, we shall assume that the constraint system under consideration contains an equality theory. Hence, we shall use indistinguishably the notation d_{xy} and $x = y$ to denote diagonal elements.

Theorem 1 (*Constraint System*)
Let $\mathbb{C} = \langle \mathscr{A}, \mathscr{C}, \models\rangle$ be as in Definition 4. Then, the structure $\langle \mathscr{C}, \leq, \otimes, \mathbf{1}, \mathbf{0}, Var, \exists, D\rangle$ is a cylindric constraint system where $D = \{!^{\top_{\mathscr{A}}}(x = y) \mid x, y \in Var\}$ and $c \leq d$ iff $d \models c$.

Proof
Recall that $c \leq d$ iff the sequent $!^{\top_A}\delta_1, \ldots, !^{\top_A}\delta_n, d \longrightarrow c$ is provable in SELL where δ_i is an axioms in Δ (see Definition 4). Abusing of the notation, we shall write sequents as the one above as $!^{\top_A}\Delta, d \longrightarrow c$.

Properties (E1) to (E4) of $\exists$ (interpreted as $\exists$) are easy.

Note that the constraint system contains an equality theory and then, Δ define the meaning of "=". Observe also that diagonal elements are marked with the largest subexponential $\top_{\mathscr{A}}$ (which is unbounded). Then, it is easy to see that the following sequents are provable: $!^{\top_{\mathscr{A}}}\Delta \longrightarrow !^{\top_{\mathscr{A}}}(x = x) \equiv \mathbf{1}$; $!^{\top_A}\Delta \longrightarrow !^{\top_{\mathscr{A}}}(x = y) \equiv \exists z.(!^{\top_{\mathscr{A}}}(x = z) \otimes !^{\top_{\mathscr{A}}}(z = y))$ whenever z is different from x and y; and $!^{\top_{\mathscr{A}}}\Delta, !^{\top_{\mathscr{A}}}(x = y), \exists x.(c \otimes !^{\top_{\mathscr{A}}}(x = y)) \longrightarrow c$ if x is different from y. Then, properties $(D1)$ to $(D3)$ hold.

Finally, we note that according to Definition 4, every constraint c is a *classical* formula. Then it follows that for any c, d, the sequents $!^{\top_{\mathscr{A}}}\Delta, c \longrightarrow \mathbf{1}$, $!^{\top_{\mathscr{A}}}\Delta, \mathbf{0} \longrightarrow c$ and $!^{\top_{\mathscr{A}}}\Delta, c, d \longrightarrow c$ are also provable. This shows that indeed $\langle \mathscr{C}, \leq, \otimes, \mathbf{1}, \mathbf{0}\rangle$ is a lattice where $\otimes$ is the lub and **1** (resp. **0**) the least (resp. greatest) element. □