Appendix for the paper of Causal Graph Justifications of Logic Programs∗

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Appendix A. Auxiliary figures

Associativity	*Commutativity*	*Absorption*
$t + (u{+}w) \;=\; (t{+}u) + w$	$t + u \;=\; u + t$	$t \;=\; t + (t * u)$
$t * (u{*}w) \;=\; (t * u) * w$	$t * u \;=\; u * t$	$t \;=\; t * (t{+}u)$

Distributive	*Identity*	*Idempotence*	*Annihilator*
$t + (u{*}w) \;=\; (t{+}u) * (t{+}w)$	$t \;=\; t + 0$	$t \;=\; t + t$	$1 \;=\; 1 + t$
$t * (u{+}w) \;=\; (t * u) + (t * w)$	$t \;=\; t * 1$	$t \;=\; t * t$	$0 \;=\; 0 * t$

Fig. 1. Sum and product satisfy the properties of a completely distributive lattice.

Appendix B. An example of causal action theory

In this section we consider a more elaborated example from Pearl (2000).

Example 1
Consider the circuit in Figure 2 with two switches, a and b, and a lamp l. Note that a is the main switch, while b only affects the lamp when a is up. Additionally, when the light is on, we want to track which wire section, v or w, is conducting current to the lamp. □

As commented by Pearl (2000), the interesting feature of this circuit is that, seen from outside as a black box, it behaves exactly as a pair of independent, parallel switches, so it is impossible to detect the causal dependence between a and b by a mere observation of performed actions and their effects on the lamp. Figure 2 also includes a possible representation for this scenario; let us call it program P_1. It uses a pair of fluents $up(X)$ and $down(X)$ for the position of switch X, as

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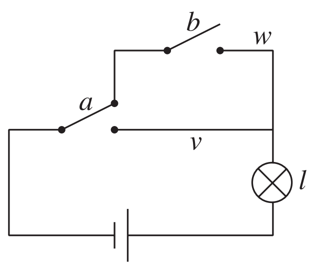
well as *on* and *off* to represent the state of the lamp. Fluents $up(X)$ and $down(X)$ (respectively, *on* and *off*) can be seen as the strong negation of each other, although we do not use an operator for that purpose[1]. Action $m(X,D)$ stands for "move switch X in direction $D \in \{u,d\}$" (*up* and *down*, respectively). Actions between state t and $t+1$ are located in the resulting state. Finally, we have also labelled inertia laws (by i) to help keeping track of fluent justifications inherited by persistence.

Circuit diagram



Inertia

$$
\begin{array}{rrcl}
i: & up(X)_{t+1} & \leftarrow & up(X)_t,\ not\ down(X)_{t+1} \\
i: & down(X)_{t+1} & \leftarrow & down(X)_t,\ not\ up(X)_{t+1}
\end{array}
$$

Direct effects

$$
\begin{array}{rcl}
up(X)_t & \leftarrow & m(X,u)_t \\
down(X)_t & \leftarrow & m(X,d)_t
\end{array}
$$

Causal rules (indirect effects)

$$
\begin{array}{rrcl}
v: & on_t & \leftarrow & down(a)_t \\
w: & on_t & \leftarrow & up(a)_t,\ down(b)_t \\
 & off_t & \leftarrow & up(a)_t,\ up(b)_t
\end{array}
$$

Constraints

$$
\begin{array}{rcl}
\bot & \leftarrow & m(X,u)_t,\ m(X,d)_t \\
\bot & \leftarrow & up(X)_t,\ down(X)_t \\
\bot & \leftarrow & on_t,\ off_t
\end{array}
$$

Initial state

$$up(a)_0 \quad up(b)_0 \quad off_0$$

Fig. 2. A circuit with two switches together with a possible representation.

Suppose we perform the following sequence of actions: we first move down both switches, next switch b is moved first up and then down, and finally we move up switch a. Assume also that each action occurrence is labelled with the action name so that, for instance, moving b up in Situation 1 corresponds to the program fact $m(b,u)_1 : m(b,u)_1$. The table in Figure 3 shows the resulting temporal projection. Note how the lamp turns on in Situation 1 but only because of v, that is, moving a down. Movements of b at 2 and 3 do not affect the lamp, and its causal explanation ($down(a)$) is maintained by inertia. In Situation 4, the lamp is still on but the reason has changed. The explanation this time is that we had closed down b at 3 (and this persisted by inertia) while we have just moved a up, firing rule w.

This example also illustrates why we are not interested in providing negative justifications through default negation. This would mean to explicitly include non-occurrences of actions that might otherwise have violated inertia. For instance, the explanation for on_2 would include the fact that we did not perform $m(a,u)_2$. Including this information for one transition is perhaps not so cumbersome, but suppose that, from 2 we executed a high number of transitions without performing any action. The explanation for on_3 would *additionally* collect that we did not perform $m(a,u)_3$ either. The explanation for on_4 should also collect the negation of further possibilities: moving a up at 4; three movements of a up, down and up; moving b at 3 and both switches at 4; moving both switches at 3 and b at 4; etc. It is easy to see that negative explanations grow exponentially:

[1] Notice how strong negation would point out the cause(s) for a boolean fluent to take value false, whereas default negation represents the absence of cause.

t	0	1	2	3	4
Actions		$m(a,d)_1$, $m(b,d)_1$	$m(b,u)_2$	$m(b,d)_3$	$m(a,u)_4$
$up(a)_t$	1	0	0	0	$m(a,u)_4$
$down(a)_t$	0	$m(a,d)_1$	$m(a,d)_1 \cdot i$	$m(a,d)_1 \cdot i$	0
$up(b)_t$	1	0	$m(b,u)_2$	0	0
$down(b)_t$	0	$m(b,d)_1$	0	$m(b,d)_3$	$m(b,d)_3 \cdot i$
on_t	0	$m(a,d)_1 \cdot v$	$m(a,d)_1 \cdot iv$	$m(a,d)_1 \cdot iv$	$(\ m(b,d)_3 \cdot i * m(a,u)_4\ ) \cdot w$
off_t	1	0	0	0	0

Fig. 3. Temporal projection of a sequence of actions for program P_1.

at step t we would get the negation of *all possible plans* for making on_t false, while indeed, *nothing has actually happened* (everything persisted by inertia).

Appendix C. Example with infinite rules

Example 2
Consider the infinite program P_2 given by the ground instances of the set of rules:

$$l(s(X)): \; nat(s(X)) \leftarrow nat(X)$$
$$l(z) \; : \; nat(z)$$

defining the natural numbers with a Peano-like representation, where z stands for "zero." For each natural number n, the causal value obtained for $nat(s^n(z))$ in the least model of the program is $l(z) \cdot l(s(z)) \ldots l(s^n(z))$. Read from right to left, this value can be seen as the computation steps performed by a top-down Prolog interpreter when solving the query $nat(s^n(z))$. As a further elaboration, assume that we want to check that at least some natural number exists. For that purpose, we add the following rule to the previous program:

$$some \leftarrow nat(X) \tag{1}$$

The interesting feature of this example is that atom *some* collects an infinite number of causes from all atoms $nat(s^n(z))$ with n ranging among all the natural numbers. That is, the value for *some* is $I(some) = \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_n + \ldots$ where $\alpha_n \stackrel{\text{def}}{=} l(z) \cdot l(s(z)) \cdot \ldots \cdot l(s^n(z))$. However, it is easy to see that the fact $nat(z)$ labelled with $l(z)$ is not only sufficient to prove the existence of some natural number, but, due to the recursive definition of the natural numbers, it is also *necessary* – note how all the proofs α_i actually begin with an application of $l(z)$.

This fact is captured in our semantic by the algebraic equivalences showed in Fig **??**. From associativity and identity of '$\cdot$', the following equivalence holds:

$$\alpha_n = 1 \; \cdot \; l(z) \; \cdot \; \beta_n \qquad \text{with} \qquad \beta_i \stackrel{\text{def}}{=} l(s(z)) \; \cdot \; \ldots \; \cdot \; l(s^i(z))$$

for any $n \geq 1$. Furthermore, from absorption of '$\cdot$' w.r.t the addition, it also holds that

$$\alpha_0 + \alpha_n \quad = \quad l(z) \; + \; 1 \; \cdot \; l(z) \; \cdot \; \beta_n \quad = \quad l(z)$$

As a consequence, the previous infinite sum just collapses to $I(some) = l(z)$, reflecting the fact that, to prove the existence of a natural number, only the fact labelled as $l(z)$ is relevant.

Suppose now that, rather than defining natural numbers in a recursive way, we define them by directly asserting an infinite set of facts as follows:

$$l(s^n(z)): \; nat(s^n(z)) \tag{2}$$

for any $n \geq 0$, where $s^0(z)$ stands for z. In this variant, the causal value obtained for $nat(s^n(z))$ is simply $l(s^n(z))$, so that the dependence we had before on lower natural numbers does not exist any more. Adding rule (1) to the set of facts (2) allows us concluding $I(some) = l(z) + l(s(z)) + l(s^2(z)) + \ldots + l(s^n(z)) + \ldots$ and this infinite sum cannot be collapsed into any finite term. This reflects that we have now infinite *independent* ways to prove that some natural number exists.

This last elaboration can be more elegantly captured by replacing the infinite set of facts (2) by an auxiliary recursive predicate *aux* defined as follows:

$$aux(s(X)) \leftarrow aux(X) \qquad\qquad l(X): \; nat(X) \leftarrow aux(X)$$
$$aux(z)$$

Since rules for *aux* are unlabelled, the value of $aux(s^n(z))$ in the least model is $I(aux(s^n(z)) = 1$ so the effect of this predicate is somehow "transparent" regarding causal justifications. As a result, the value of $nat(s^n(z))$ is just $l(s^n(z))$ as before.

Appendix D. Proofs

In order to improve clarity, for any causal graph $G = \langle V, E \rangle$ we use the notation $V(G)$ and $E(G)$ to refer to V and E respectively.

Proposition 1 (*Monotonicity*)
Let G, G' be a pair of causal graph with $G \leq G'$. Then, for any causal graph H:

$$G * H \leq G' * H, \qquad G \cdot H \leq G' \cdot H \qquad \text{and} \qquad H \cdot G \leq H \cdot G'$$

Proof. First we will show that $G * H \leq G' * H$. Suppose that $E(G*H) \not\supseteq E(G'*H)$ and let (l_1, l_2) be an edge in $E(G'*H)$ but not in $E(G*H')$, i.e. $(l_1,l_2) \in E(G'*H) \backslash E(G*H')$.

Thus, since by product definition $E(G'*H) = E(G') \cup E(H)$, it follows that either $(l_1,l_2) \in E(G')$ or $(l_1,l_2) \in E(H)$. It is clear that if $(l_1,l_2) \in E(H)$ then $(l_1,l_2) \in E(G*H) = E(G) \cup E(H)$. Furthermore, since $G \leq G'$ it follows that $E(G) \supseteq E(G')$, if $(l_1,l_2) \in E(G')$ then $(l_1,l_2) \in E(G)$ and consequently $(l_1,l_2) \in E(G*H) = E(G) \cup E(H)$. That is $E(G*H) \supseteq E(G'*H)$ and then $G*H \leq G'*H$. Note that $V(G*H) \supseteq V(G'*H)$ follows directly from $E(G*H) \supseteq E(G'*H)$ and the fact that every vertex has and edge to itself.

To show that $G \cdot H \leq G' \cdot H$ (the case for $H \cdot G \leq H \cdot G'$ is analogous) we have has to show that, in addition to the previous, for every edge $(l_G, l_H) \in E(G' \cdot H)$ with $l_G \in V(G')$ and $l_H \in V(H)$ it holds that $(l_G, l_H) \in E(G \cdot H)$. Simply note that since $G \leq G'$ it follows $V(G) \supseteq V(G)'$ and then $l_G \in V(G)$. Consequently $(l_G, l_H) \in E(G \cdot H)$. □

Proposition 2 (*Distributivity*)
For every pair of sets of causal graphs S and S', it holds that

$$\left(\prod S\right) \cdot \left(\prod S'\right) = \prod \left\{ \, G \cdot G' \;\middle|\; G \in S \text{ and } G' \in S' \, \right\}.$$

Proof. For readability sake, we define two causal graphs

$$G_R \stackrel{\text{def}}{=} \left(\prod S\right) \cdot \left(\prod S'\right) \qquad \text{and} \qquad G_L \stackrel{\text{def}}{=} \prod \left\{ \, G \cdot G' \;\middle|\; G \in S \text{ and } G' \in S' \, \right\}$$

and we assume that both S and S' are not empty sets. Note that $\prod \emptyset = \mathbf{C}_{Lb} = \prod \{G_\emptyset\}$. Then, by product definition, it follows that

$$E(G_L) = \left(\bigcup \left\{ \, E(G) \;\middle|\; G \in S \, \right\} \cup \bigcup \left\{ \, E(G') \;\middle|\; G \in S' \, \right\} \cup E_L \right)^*$$
$$E(G_R) = \left(\bigcup \left\{ \, E(G) \cup E(G') \cup E_R(G,G') \;\middle|\; G \in S \text{ and } G' \in S' \, \right\} \right)^*$$

where

$$E_L = \left\{ \, (l,l') \;\middle|\; l \in \bigcup \left\{ \, V(G) \;\middle|\; G \in S \, \right\} \text{ and } l' \in \bigcup \left\{ \, V(G') \;\middle|\; G' \in S' \, \right\} \, \right\}$$
$$E_R(G,G') = \left\{ \, (l,l') \;\middle|\; l \in V(G) \text{ and } l' \in V(G') \, \right\}$$

Furthermore let $E_R = \bigcup \left\{ \, E_R(G,G') \;\middle|\; G \in S \text{ and } G' \in S' \, \right\}$. For every edge $(l,l') \in E_L$ there are a pair of c-graphs $G \in S$ and $G' \in S'$ s.t. $l \in V(G)$ and $l' \in V(G')$ and then $(l,l') \in E_R(G,G')$ and so $(l,l') \in E_R$. Moreover, for every edge $(l,l') \in E_R$ there are a pair of c-graphs $G \in S$ and $G' \in S'$ s.t. $(l,l') \in E_R(G,G')$ with $l \in V(G)$ and $l' \in V(G)'$. So that $(l,l') \in E_L$. That is $E_L = E_R$. Then

$$E(G_R) = \left(\bigcup \left\{ \, E(G) \;\middle|\; G \in S \, \right\} \cup \bigcup \left\{ \, E(G') \;\middle|\; G' \in S' \, \right\} \cup E_R \right)^*$$
$$= \left(E(G_L) \backslash E_L \cup E_R \right)^* = \left(E(G_L) \right)^* = E(G_L)$$

Consequently $G_L = G_R$. □

Proposition 3 (*Distributivity cont*)
For any causal graphs G, $G' \neq \emptyset$ and G'', it holds that

$$G \cdot G' \cdot G'' = G \cdot G' * G' \cdot G''$$

Proof. It is clear that $G \cdot G' \cdot G'' \leq G \cdot G'$ and $G \cdot G' \cdot G'' \leq G' \cdot G''$ and then $G \cdot G' \cdot G'' \leq G \cdot G' * G' \cdot G''$. Let G_1, G_2, G_L and G_R be respectively $G_1 = G \cdot G'$, $G_2 = G' \cdot G''$, $G_L = G_1 \cdot G''$, and $G_R = G_1 * G_2$. Suppose that $G_L < G_R$, i.e. $G_L \supset G_R$ and there is an edge $(v_1, v_2) \in G_L$ but $(v_1, v_2) \notin G_R$. Then $G_1 \subseteq G_R$ and $G'' \subseteq G_2 \subseteq G_R$ and one of the following conditions holds:

1. $(v_1, v_2) \in G_1 \subseteq G_R$ or $(v_1, v_2) \in G'' \subseteq G_R$ which is a contradiction with $(v_1, v_2) \notin G_R$.
2. $v_1 \in G_1$ and $v_2 \in G''$, i.e. $v_1 \in G$ and $v_2 \in G''$ or $v_1 \in G'$ and $v_2 \in G''$. Furthermore, if the last it is clear that $(v_1, v_2) \in G' \cdot G'' = G_2 \subseteq G_R$ which is a contradiction with $(v_1, v_2) \notin G_R$.

Thus it must be that $v_1 \in G$ and $v_2 \in G''$. But then, since $G' \neq \emptyset$ there is some $v' \in G'$ and consequently there are edges $(v_1, v') \in G \cdot G' = G_1 \subseteq G_R$ and $(v', v_2) \in G' \cdot G'' = G_2 \subseteq G_R$. Since G_R is closed transitively, $(v_1, v_2) \in G_R$ which is a contradiction with the assumption that $(v_1, v_2) \notin G_R$. That is, $G_L = G \cdot G' \cdot G'' = G \cdot G' * G' \cdot G'' = G_R$.

Proposition 4
For every causal graph $G = \langle V, E \rangle$ it holds that $G = \prod \{ \, l \cdot l' \mid (l, l') \in E \, \}$.

Proof. Let G' be a causal graphb s.t. $G' = \prod \{ \, l \cdot l' \mid (l, l') \in E \, \}$. Then for every edge $(l, l') \in E(G)$ it holds that $(l, l') \in E(l \cdot l')$ and then $(l, l') \in E(G') = \bigcup \{ \, E(l \cdot l') \mid (l, l') \in E \, \}$, i.e. $E(G) \subseteq E(G')$. Furthermore for every $(l, l') \in E(G')$ there is $l_i \cdot l_j$ s.t. $(l, l') \in l_i \cdot l_j$ and $(l_i, l_j) \in E(G)$. Then, since $E(l_i \cdot l_j) = \{(l_i, l_j)\}$ it follows that $(l, l') \in E(G)$, i.e. $E(G) \supseteq E(G')$. Consequently $G = G' = \prod \{ \, l \cdot l' \mid (l, l') \in E \, \}$. □

Proposition 5 (*Infimum*)
Any set of causal graphs S has a $\leq$-infimum given by their product $\prod S$.

Proof. By definition $\prod S$ is the causal graph whose vertices and edges are respectively the sets $V(\prod S) = \bigcup \{ \, V(G) \mid G \in S \, \}$ and $E(\prod S) = \bigcup \{ \, E(G) \mid G \in S \, \}$. It is easy to see that $\prod S$ is the supremum of the subgraph relation, so that, since for every pair of causal graphs $G \leq G'$ iff $G \supseteq G'$, it follows that infimum of S w.r.t. $\leq$. □

Proof of Theorem 1. Let $\mathbf{F}$ be the set of filters over the lower semilattice $\langle \mathbf{C}_{Lb}, * \rangle$. Stumme was showed in (Stumme 1997) that the concept lattice $\underline{\mathfrak{B}}\langle \mathbf{F}, \mathbf{V}_{Lb}, \Delta \rangle$ (with $F \Delta I \Leftrightarrow F \cap I \neq \emptyset$) is isomorphic to the free completely distributive complete lattice generated by the partial lattice $\langle \mathbf{C}_{Lb}, +, * \rangle$ where $+$ and $*$ are two partial functions corresponding with the supremum and infimum. In our particular, case for every set of causal graphs S its infimum is defined as $\prod S$ and the supremum is defined as $G \in S$ such that $G' \leq G$ for all $G' \in S$, when such G exists and undefined otherwise. Thus $\mathbf{V}_{Lb}$ is the set of ideals over the partial lattice $\langle \mathbf{C}_{Lb}, +, * \rangle$, i.e. every $I \in \mathbf{V}_{Lb}$ is also closed under defined suprema. He also show that the elements of such lattice are described as pairs

$$\left\{ \, (\mathbf{F}_t, \mathbf{I}_t) \mid \mathbf{F}_t \subseteq \mathbf{F}, \; \mathbf{I}_t \subseteq \mathbf{V}_{Lb}, \; \mathbf{F}_t^I = \mathbf{I}_t \text{ and } \mathbf{F}_t = \mathbf{I}_t^I \, \right\}$$

where

$$\mathbf{F}_t^I = \left\{ I \in \mathbf{I} \;\middle|\; \forall F \in \mathbf{F}_t : F \cap I \neq \emptyset \right\}$$
$$\mathbf{I}_t^I = \left\{ F \in \mathbf{F} \;\middle|\; \forall I \in \mathbf{I}_t : F \cap I \neq \emptyset \right\}$$

That is, every element is in the form $\langle \mathbf{I}_t^I, \mathbf{I}_t \rangle$. Furthermore infima and suprema can be described as follows:

$$\bigwedge_{t \in T} (\mathbf{I}_t^I, \mathbf{I}_t) = \Big(\bigcap_{t \in T} \mathbf{I}_t^I, \Big(\bigcup_{t \in T} \mathbf{I}_t \Big)^{II} \Big)$$
$$\bigvee_{t \in T} (\mathbf{I}_t^I, \mathbf{I}_t) = \Big(\Big(\bigcup_{t \in T} \mathbf{I}_t^I \Big)^{II}, \bigcap_{t \in T} \mathbf{I}_t \Big)$$

We will show that $\varepsilon_I : \underline{\mathfrak{B}} \longrightarrow \mathbf{V}_{Lb}$ given by $(\mathbf{I}_t^I, \mathbf{I}_t) \mapsto \bigcap \mathbf{I}_t$ is an isomorphism between $\langle \underline{\mathfrak{B}}, \vee, \wedge \rangle$ and $\langle \mathbf{V}_{Lb}, \cup, \cap \rangle$. Note that, since $\prod \emptyset = G_\emptyset$ it holds that the empty set is not close under defined infimum and then it is not a filter, i.e. $\emptyset \notin \mathbf{F}$, and then for every filter $F \in \mathbf{F}$ it holds that $G_\emptyset \in F$. Thus if $\mathbf{I}_t = \emptyset$ follows that $\mathbf{I}_t^I = \mathbf{F}$ and then $\mathbf{I}_t^{II} = \{ I \in \mathbf{V}_{Lb} \mid G_\emptyset \in I \} = \mathbf{C}_{Lb} \neq \mathbf{I}_t$. That is, $\langle \emptyset^I, \emptyset \rangle \notin \underline{\mathfrak{B}}$.

We will show that for every ideal $I_t \in \mathbf{I}$ and for every set of ideals $\mathbf{I}_t \subseteq \mathbf{I}$ s.t. $I_t = \bigcap \mathbf{I}_t$ it holds that

$$(\mathbf{I}_t^I, \mathbf{I}_t) \in \underline{\mathfrak{B}} \langle \mathbf{F}, \mathbf{I}, \Delta \rangle \Longleftrightarrow \mathbf{I}_t = \left\{ I \in \mathbf{I} \;\middle|\; I_t \subseteq I \right\} \tag{3}$$

and consequently ε_I is a bijection between and $\underline{\mathfrak{B}}$ and $\mathbf{V}_{Lb}$.

Suppose that $(\mathbf{I}_t^I, \mathbf{I}_t) \in \underline{\mathfrak{B}} \langle \mathbf{F}, \mathbf{I}, \Delta \rangle$. For every $I \in \mathbf{I}_t$ it holds that $I_t \subseteq I$. So suppose there is $I \in \mathbf{I}$ s.t. $I_t \subseteq I$ and $I \notin \mathbf{I}_t$. Then there is $F \in \mathbf{I}_t^I$ s.t. $I \cap F = \emptyset$ and for every element $I' \in \mathbf{I}_t$ it holds that $I' \cap F \neq \emptyset$. Pick a causal graph G s.t. $G = \prod \{ G' \mid G' \in I' \cap F \text{ and } I' \in \mathbf{I}_t \}$. Since for every G' it holds $G' \in F$ and $G \leq G'$ follows that $G \in F$ (F is close under infimum) and $G \in I'$ (every I' is close under $\leq$). That is, for every $I' \in \mathbf{I}_t$ it holds that $G \in I' \cap F$ and then, since $I_t = \bigcap \mathbf{I}_t$, it also holds that $G \in I_t \cap F$ and since $I_t \subseteq I$ also $G \in I \cap F$ which contradict that $I \cap F = \emptyset$. So that $I \in \mathbf{I}_t$ and it holds that

$$(\mathbf{I}_t^I, \mathbf{I}_t) \in \underline{\mathfrak{B}} \langle \mathbf{F}, \mathbf{I}, \Delta \rangle \Longrightarrow \mathbf{I}_t = \left\{ I \in \mathbf{I} \;\middle|\; I_t \subseteq I \right\}$$

Suppose that $\mathbf{I}_t = \{ I \in \mathbf{I} \mid I_t \subseteq I \}$ but $(\mathbf{I}_t^I, \mathbf{I}_t) \in \underline{\mathfrak{B}} \langle \mathbf{F}, \mathbf{I}, \Delta \rangle$, i.e. $\mathbf{I}_t \neq \mathbf{I}_t^{II}$. Note that $\mathbf{I}_t \subseteq \mathbf{I}_t^{II}$ because otherwise there are $I \in \mathbf{I}_t$ and $F \in \mathbf{I}_t^I$ s.t. $I \cap F = \emptyset$ which is a contradiction with the fact that for every $F \in \mathbf{I}_t^I$ and $I \in \mathbf{I}_t$ it holds that $F \cap I \neq \emptyset$.

So, there is $I \in \mathbf{I}_t^{II}$ s.t. $I \notin \mathbf{I}_t$, i.e. for every $F \in \mathbf{I}_t^I$ it holds that $F \cap I \neq \emptyset$ but $I_t \not\subseteq I$. Pick $G \in I_t \backslash I$ and $F = \{ G' \mid G \leq G' \}$. It is clear that $F \in \mathbf{F}$ and $F \cap I_t \neq \emptyset$ because $G \in I_t$, so that $F \in \mathbf{I}_t^I$. Furthermore $F \cap I = \emptyset$, because $G \notin I$, which is a contradiction with the assumption. Thus

$$(\mathbf{I}_t^I, \mathbf{I}_t) \in \underline{\mathfrak{B}} \langle \mathbf{F}, \mathbf{I}, \Delta \rangle \Longleftarrow \mathbf{I}_t = \left\{ I \in \mathbf{I} \;\middle|\; I_t \subseteq I \right\}$$

Now, we will show that $(\mathbf{I}_1^I, \mathbf{I}_1) \vee (\mathbf{I}_2^I, \mathbf{I}_2) = (\mathbf{I}_3^I, \mathbf{I}_3)$ iff $I_1 \cup I_2 = I_3$. From the above statement follows that

$$\begin{aligned} \mathbf{I}_1 \cap \mathbf{I}_2 &= \{ I \in \mathbf{I} \mid I_1 \subseteq I \text{ and } I_2 \subseteq I \} = \\ &= \{ I \in \mathbf{I} \mid I_1 \cup I_2 \subseteq I \} \\ \mathbf{I}_3 &= \{ I \in \mathbf{I} \mid I_3 \subseteq I \} \end{aligned}$$

That is, $\mathbf{I}_1 \cap \mathbf{I}_2 = \mathbf{I}_3$ iff $I_1 \cup I_2 = I_3$ and by definition of $\vee$ the first is equivalent to $(\mathbf{I}_1^I, \mathbf{I}_1) \vee (\mathbf{I}_2^I, \mathbf{I}_2) = (\mathbf{I}_3^I, \mathbf{I}_3)$.

Finally we will show that $(\mathbf{I}_1^I, \mathbf{I}_1) \wedge (\mathbf{I}_2^I, \mathbf{I}_2) = (\mathbf{I}_3^I, \mathbf{I}_3)$ iff $I_1 \cap I_2 = I_3$. It holds that

$$(\mathbf{I}_1 \cup \mathbf{I}_2)^{II} = \Big(\big\{ \, I \in \mathbf{I} \; \big| \; I_1 \subseteq I \text{ or } I_2 \subseteq I \, \big\} \Big)^{II} =$$
$$= \Big(\big\{ \, I \in \mathbf{I} \; \big| \; I_1 \cap I_2 \subseteq I \, \big\} \Big)^{II}$$
$$\mathbf{I}_3 = \big\{ \, I \in \mathbf{I} \; \big| \; I_3 \subseteq I \, \big\}$$

Since ε_I is a bijection, it holds that $(\mathbf{I}_1 \cup \mathbf{I}_2)^{II} = \mathbf{I}_3$ iff $I_1 \cap I_2 = I_3$.

Thus $\varepsilon_I : \underline{\mathfrak{B}} \longrightarrow \mathbf{V}_{Lb}$ is an isomorphism between $\langle \underline{\mathfrak{B}}, \vee, \wedge \rangle$ and $\langle \mathbf{V}_{Lb}, \cup, \cap \rangle$, i.e. $\langle \mathbf{V}_{Lb}, \cup, \cap \rangle$ is isomorphic to the free completely distributive lattice generated by $\langle \mathbf{C}_{Lb}, * \rangle$.

Let's check now that $\downarrow : \mathbf{C}_{Lb} \longrightarrow \mathbf{V}_{Lb}$ defined as is an injective homomorphism. Stumme has already showed that $\varepsilon_p : \mathbf{C}_{Lb} \longrightarrow \underline{\mathfrak{B}}$ given by

$$\varepsilon_p(G) \mapsto \Big(\big\{ \, F \in \mathbf{F} \; \big| \; G \in F \, \big\}, \big\{ \, I \in \mathbf{I} \; \big| \; G \in I \, \big\} \Big)$$

is an injective homomorphism between the partial lattice $\langle \mathbf{C}_{Lb}, +, * \rangle$ and $\langle \underline{\mathfrak{B}}, \vee, \wedge \rangle$. So that $\varepsilon_I \circ \varepsilon_p$ is an injective homomorphism between $\langle \mathbf{C}_{Lb}, +, * \rangle$ and $\langle \mathbf{V}_{Lb}, \cup, \cap \rangle$ given by

$$\varepsilon_I \circ \varepsilon_p(G) \mapsto \bigcap \big\{ \, I \in \mathbf{V}_{Lb} \; \big| \; G \in I \, \big\}$$

Note that for any causal graph G and $G' \in \mathbf{C}_{Lb}$ s.t. $G' \leq G$ it holds that $G' \in \varepsilon_I \circ \varepsilon_p(G)$ that is $\downarrow G \subseteq \varepsilon_I \circ \varepsilon_p(G)$. Furthermore for every causal graph G it holds that $\varepsilon(G)$ is an ideal, i.e. $\downarrow G \in \mathbf{V}_{Lb}$ and it is clear that $G \in \downarrow G$ so that, $\varepsilon_I \circ \varepsilon_p$ is a intersection of which one element is $\downarrow G$, thus $\varepsilon_I \circ \varepsilon_p(G) \subseteq \downarrow G$. That is $\downarrow G = \varepsilon_I \circ \varepsilon_p(G)$ and consequently it is an injective homomorphism between $\langle \mathbf{C}_{Lb}, +, * \rangle$ $\langle \mathbf{V}_{Lb}, \cup, \cap \rangle$.

Lemma 0.1
Let P be a positive (and possible infinite) logic program over signature $\langle At, Lb \rangle$. Then, (i) the least fix point of T_P, $lfp(T_P)$ is the least model of P, and (ii) $lfp(T_P) = T_P \uparrow^{\omega} (\mathbf{0})$. □

Proof. Since the set of causal values forms a lattice causal logic programs can be translated to *Generalized Annotated Logic Programming* (GAP). GAP is a general a framework for multivalued logic programming where the set of truth values must to form an upper semilattice and rules (*annotated clauses*) have the following form:

$$H : \rho \leftarrow B_1 : \mu_1 \;\&\; \ldots \;\&\; B_n : \mu_n \tag{4}$$

where $L_0, \ldots, L_m$ are literals, ρ is an *annotation* (may be just a truth value, an *annotation variable* or a *complex annotation*) and $\mu_1, \ldots, \mu_n$ are values or annotation variables. A complex annotation is the result to apply a total continuous function to a tuple of annotations. Thus a positive program P is encoded in a GAP program, GAP(P) rewriting each rule $R \in \Pi$ of the form

$$t : H \leftarrow B_1 \wedge \ldots \wedge B_n \tag{5}$$

as a rule GAP(R) in the form (4) where $\mu_1, \ldots, \mu_n$ are annotation that capture the causal values of each body literal and ρ is a complex annotation defined as $\rho = (\mu_1 * \ldots * \mu_n) \cdot t$.

Thus we will show that a causal interpretation $I \models \Pi$ if and only if $I \models^r$ GAP(P) where $\models^r$ refers to the GAP restricted semantics.

For any program P and interpretation I, by definition, $I \models P$ (resp. $I \models^r \text{GAP}(P)$) iff $I \models R$ (resp. $I \models^r \text{GAP}(R)$) for every rule $R \in P$. Thus it is enough to show that for every rule R it holds that $I \models R$ iff $I \models^r \text{GAP}(R)$.

By definition, for any rule R of the form of (5) and an interpretation I, $I \models R$ if and only if $\big(I(B_1) * \ldots * I(B_n)\big) \cdot t \leq I(H)$ whereas for any rule GAP(R) in the form of (4), $I \models^r GAP(R)$ iff for all $\mu_i \leq I(B_i)$ implies that $\rho = (\mu_1 * \ldots * \mu_n) \cdot t \leq I(H)$.

For the only if direction, take $\mu_i = I(B_i)$, then $\rho = (\mu_1 * \ldots * \mu_n) \cdot t = (I(B_1) * \ldots * I(B_n)) \cdot t$ and then $\rho \leq I(H)$ implies $\big(I(B_1) * \ldots * I(B_n)\big) \cdot t \leq I(H)$, i.e. $I \models^r GAP(R)$ implies $I \models R$. For the if direction, take $\mu_i \leq I(B_i)$ then, since product an applications are monotonic operations, it follows that $(\mu_1 * \ldots * \mu_n) \cdot t \leq (I(B_1) * \ldots * I(B_n)) \cdot t \leq I(H)$, That is, $I \models R$ also implies $I \models^r GAP(R)$. Consequently $I \models R$ iff $I \models^r GAP(R)$.

Thus, from Theorem 1 in (Kifer and Subrahmanian 1992), it follows that the operator T_P is monotonic.

To show that the operator T_P is also continuous we need to show that for every causal program P the translation GAP(P) is an *acceptable* program. Indeed since in a program GAP(P) all body atoms are v-annotated it is *acceptable*. Thus from Theorem 3 in (Kifer and Subrahmanian 1992), it follows that $T_P \uparrow^{\omega} (\mathbf{0}) = lfp(T_P)$ and this is the least model of P.

Lemma 0.2

Given a positive and completely labelled program P, for every atom p and integer $k \geq 1$,

$$T_P \uparrow^{k} (\mathbf{0})(p) = \sum_{R \in \Psi} \sum_{f \in R} \prod \left\{ f\big(T_P \uparrow^{k-1} (\mathbf{0})(q)\big) \;\middle|\; q \in body(R) \right\} \cdot label(R)$$

where Ψ is the set of rules $\Psi = \left\{ R \in \Pi \;\middle|\; head(R) = p \right\}$ and R is the set of choice functions $R = \left\{ f \;\middle|\; f(S) \in S \right\}$.

Proof. By definition of $T_P \uparrow^{k} (\mathbf{0})(p)$ it follows that

$$T_P \uparrow^{k} (\mathbf{0})(p) = \sum \left\{ \big(T_P \uparrow^{k-1} (\mathbf{0})(q_1) * \ldots * T_P \uparrow^{k-1} (\mathbf{0})(q_1)\big) \cdot label(R) \;\middle|\; R \in P \text{ with } head(R) = p \right\}$$

then, applying distributive of application w.r.t. to the sum and and rewriting the sum and the product aggregating properly, it follows that

$$T_P \uparrow^{k} (\mathbf{0})(p) = \sum_{R \in \Psi} \prod \left\{ T_P \uparrow^{k-1} (\mathbf{0})(q) \;\middle|\; q \in body(R) \right\} \cdot label(R)$$

Furthermore for any atom q the causal value $T_P \uparrow^{k-1} (\mathbf{0})(q)$ can be expressed as the sum of all c-graphs in it and then

$$T_P \uparrow^{k} (\mathbf{0})(p) = \sum_{R \in \Psi} \prod \left\{ \sum_{f \in R} f\big(T_P \uparrow^{k-1} (\mathbf{0})(q)\big) \;\middle|\; q \in body(R) \right\} \cdot abel(R)$$

and applying distributivity of products over sums it follows that

$$T_P \uparrow^{k} (\mathbf{0})(p) = \sum_{R \in \Psi} \sum_{f \in R} \prod \left\{ f\big(T_P \uparrow^{k-1} (\mathbf{0})(q)\big) \;\middle|\; q \in body(R) \right\} \cdot l_R \qquad \square$$

Lemma 0.3
Given a positive and completely labelled program P and a causal graph G, for every atom p and integer $k \geq 1$, it holds that $G \in T_P \!\uparrow^k (\mathbf{0})(p)$ iff there is a rule $l : p \leftarrow q_1, \dots, q_m$ and causal graphs $G_{q_1}, \dots, G_{q_m}$ respectively in $T_P \!\uparrow^{k-1} (\mathbf{0})(q_i)$ and $G \leq \big(G_{q_1} * \ldots * G_{q_m}\big) \cdot l$.

Proof. From Lemma 0.2 it follows that $G \in T_P \!\uparrow^k (\mathbf{0})(p)$ iff

$$G \in value\Big(\sum_{R \in \Psi} \sum_{f \in R} \prod \big\{ \, f\big(T_P \!\uparrow^{k-1} (\mathbf{0})(q)\big) \;\big|\; q \in body(R) \, \big\} \cdot label(R) \Big)$$

iff

$$G \in \bigcup_{R \in \Psi} \bigcup_{f \in R} value\Big(\prod \big\{ \, \downarrow f\big(T_P \!\uparrow^{k-1} (\mathbf{0})(q)\big)\Big) \;\big|\; q \in body(R) \, \big\} \cdot label(R)$$

iff there is $R \in \Phi$, with $head(R) = p$ and a choice function $f \in \Psi$ s.t.

$$G \in value\Big(\prod \big\{ \, f\big(T_P \!\uparrow^{k-1} (\mathbf{0})(q)\big) \;\big|\; q \in body(R) \, \big\} \cdot label(R)\Big)$$

Let $R = l : p \leftarrow q_1, \dots, q_m$ and $f\big(T_P \!\uparrow^{k-1} (\mathbf{0})(q_i)\big) = G_{q_i}$. Then the above can be rewritten as $G \leq \big(G_{q_1} * \ldots * G_{q_m}\big) \cdot l$. □

Definition 1
Given a causal graph $G = \langle V, E \rangle$, we define the *restriction* of G to a set of vertex $V' \subseteq V$ as the casual graph $G' = \langle V', E' \rangle$ where $E' = \{ \, (l_1, l_2) \in E \;\big|\; l_1 \in V' \text{ and } l_2 \in V' \, \}$, and we define the *reachable restriction* of G to a set of vertex $V' \subseteq V$, in symbols $G^{V'}$, as the restriction of G to the set of vertex V'' from where some vertex $l \in V'$ is reachable $V'' = \{ \, l' \in V \;\big|\; (l', l) \in E^* \text{ for some } l \in V' \, \}$. When $V' = l$ is a singleton we write G^l.

Lemma 0.4
Let P be a positive, completely labelled program, p and q be atoms, G be a causal graph, R be a causal rule s.t. $head(R) = q$ and $label(R) = l$ is a vertex in G and $k \in \{1, \dots, \omega\}$ be an ordinal. If $G \in T_P \!\uparrow^k (\mathbf{0})(p)$, then $G^l \in T_P \!\uparrow^k (\mathbf{0})(q)$. □

Proof. In case that $k = 0$ the lemma statement holds vacuous. Otherwise assume as induction hypothesis that the lemma statement holds for $k-1$. From Lemma 0.3, since $G \in T_P \!\uparrow^k (\mathbf{0})(p)$, there is a rule $R_p = (l_p : p \leftarrow p_1, \dots, p_m)$ and c-graph $G_{p_1}, \dots, G_{p_m}$ s.t. each $G_{p_i} \in T_P \!\uparrow^{k-1} (\mathbf{0})(p_i)$ and $G \leq (G_{p_1} * \ldots * G_{p_m}) \cdot l_p$.

If $l = l_p$ then, since P is uniquely labelled, $R = R_p$, $G^l = G$ and by assumption $G \in T_P \!\uparrow^k (\mathbf{0})(p)$. Otherwise $l \in G_{p_i}$ for some G_{p_i} and in its turn $G_{p_i} \in T_P \!\uparrow^{k-1} (\mathbf{0})(p_i)$. By induction hypothesis $G^l \in T_P \!\uparrow^{k-1} (\mathbf{0})(q)$ and since $T_P \!\uparrow^{k-1} (\mathbf{0})(q) \subseteq T_P \!\uparrow^k (\mathbf{0})(q)$ it follows that $G^l \in T_P \!\uparrow^k (\mathbf{0})(q)$.

In case that $k = \omega$, by definition $T_P \!\uparrow^\omega (\mathbf{0})(p) = \sum_{i<\omega} T_P \!\uparrow^i (\mathbf{0})(p)$ and the same for atom q. Thus, if $G \in T_P \!\uparrow^\omega (\mathbf{0})(p)$ there is some $i < \omega$ s.t. $G \in T_P \!\uparrow^i (\mathbf{0})(p)$, and as we already show, $G^l \in T_P \!\uparrow^i (\mathbf{0})(q)$ and consequently $G^l \in T_P \!\uparrow^\omega (\mathbf{0})(q)$. □

Lemma 0.5
Let P be a positive, completely labelled program, p be an atom and G be a causal graph and $k \geq 1$ be an integer. If G is maximal in $T_P \!\uparrow^k (\mathbf{0})(p)$ then

1. there is a causal rule $R = (l : p \leftarrow p_1, \dots, p_m)$ and there are causal graphs $G_{p_1}, \dots, G_{p_m}$ s.t. each $G_{p_i} \in \max T_P \uparrow^{k-1} (\mathbf{0})(p_i)$ and $G_p = (G_{p_1} * \dots * G_{p_m}) \cdot l$ and
2. l is not a vertex of any G_{p_i}. □

Proof. From Lemma 0.3 it follows that $G \in T_P \uparrow^{k} (\mathbf{0})(p)$ iff there is a rule $R = (l : p \leftarrow q_1, \dots, q_m)$ and causal graphs $G'_{q_1}, \dots, G'_{q_m}$ s.t. each $G_{p_i} \in \max T_P \uparrow^{k-1} (\mathbf{0})(p_i)$ and $G = \big(G'_{q_1} * \dots * G'_{q_m}\big) \cdot l$. Let $G_{q_1}, \dots, G_{q_m}$ be causes such that each $G_{q_i} \in \max T_P \uparrow^{k-1} (\mathbf{0})(q_i)$ and $G'_{q_i} \leq G_{q_i}$ and let G' be the c-graph $G' = \big(G_{q_1} * \dots * G_{q_m}\big) \cdot l$. By product and application monotonicity it holds that $G \leq G'$ and, again from Lemma 0.3, it follows that $G' \in T_P \uparrow^{k} (\mathbf{0})(p)$. Thus, since G is maximal, it must to be that $G = G'$ and consequently $G = \big(G_{q_1} * \dots * G_{q_m}\big) \cdot l$ where each G_{q_i} is maximal.

Suppose that l is a vertex of G_{p_i} for some G_{p_i}. From Lemma 0.4, if follows that $G^{l}_{p_i} \in T_P \uparrow^{k} (\mathbf{0})(p)$. Furthermore, since $G_{p_i} \supseteq G^{l}_{p_i}$, it follows that $G_{p_i} \leq G^{l}_{p_i}$ and, since l is a label ($l \neq 1$), it follows that $G < G_{p_i}$ and so that $G < G^{l}_{p_i}$ which contradicts the assumption that $G \in \max T_P \uparrow^{k} (\mathbf{0})(p)$. □

Definition 2

Given a causal graph G we define $height(G)$ as the length of the maximal simple (no repeated vertices) path in G.

Lemma 0.6

Let P be a positive, completely labelled program, p be an atom, $k \in \{1, \dots, \omega\}$ be an ordinal and G be a causal graph. If $G \in \max T_P \uparrow^{k} (\mathbf{0})(p)$ and $height(G) = h \leq k$ then $G \in T_P \uparrow^{h} (\mathbf{0})(p)$.

Proof. In case that $h = 0$, from Lemma 0.5, it follows that if $G \in \max T_P \uparrow^{k} (\mathbf{0})(p)$ there is a causal rule $R = (l : p \leftarrow p_1, \dots, p_m)$ and c-graphs.... Furthermore, since P is completely labelled, it follows that $l \neq 1$ and then $G < l < 1$. Since 1 is the only c-graph whose *height* is 0 the lemma statement holds vacuous.

In case that $h > 0$, we proceed by induction assuming as hypothesis that the lemma statement holds for any $h' < h$. From Lemma 0.5, there is a causal rule $l : p \leftarrow p_1, \dots, p_m$, and there are causal graphs $G_{p_1}, \dots, G_{p_m}$ s.t. each $G_{p_i} \in \max T_P \uparrow^{k-1} (\mathbf{0})(p_i)$, $G = G_R \cdot l$ and $l \notin V(G_{p_i})$ for any G_{p_i} where $G_R = G_{p_1} * \dots * G_{p_m}$.

If $m = 0$ then $G = 1 \cdot l = l$, $height(l) = 1$ and $l \in \max T_P \uparrow^{k} (\mathbf{0})(p)$ for any $k \geq 1$. Otherwise, since any path in G_{p_i} is also a path G_p, it is clear that $height(G_{p_i}) = h'_{p_i} \leq h$ for any G_{p_i}. Suppose that $h'_{p_i} = h$ for some G_{p_i}. Then there is a simple path $l_1, \dots, l_h$ of length h in G_{p_i} and, since $G = G_R \cdot l$, there is an edge $(l_h, l) \in E(G)$. That is $l_1, \dots, l_h, l$ is a walk of length $h+1$ in G and, since $l \notin V(G_{p_i})$, it follows that $l_i \neq l$ with $1 \leq i \leq h$. So that $l_1, \dots, l_h, l$ is a simple path of length $h+1$ which contradicts the assumption that $height(G) = h$. Thus $height(G_{p_i}) = h'_{p_i} < h$ for any G_{p_i} and then, by induction hypothesis, $G_{p_i} \in \max T_P \uparrow^{h'_{p_i}} (\mathbf{0})(p_i)$.

Let $h' = \max\{\, h'_{p_i} \mid 1 \leq i \leq m \,\} < h$. Since the T_P operator is monotonic and $h'_{p_i} \leq h'$ for any p_i, it follows that $T_P \uparrow^{h'_{p_i}} (\mathbf{0})(p_i) \leq T_P \uparrow^{h'} (\mathbf{0})(p_i)$ and then there are casual graphs $G'_{p_1}, \dots, G'_{p_m}$ such that each $G'_{p_i} \in \max T_P \uparrow^{h'} (\mathbf{0})(p_i)$, $G_{p_i} \leq G'_{p_i}$ and $G' = G_R \cdot l$ where $G'_R = G'_{p_1} * \dots * G'_{p_m}$. By product and application monotonicity, it follows that $G \leq G'$, and, from Lemma 0.3, it follows that $G' \in T_P \uparrow^{h'+1} (\mathbf{0})(p)$. Since $h'+1 \leq h$ it follows that $G' \in T_P \uparrow^{h} (\mathbf{0})(p)$ and since $G \leq G'$ it follows that $G \in T_P \uparrow^{h} (\mathbf{0})(p)$.

Suppose that $G \notin \max T_P\uparrow^h(\mathbf{0})(p)$. Then there is $G'' \in \max T_P\uparrow^h(\mathbf{0})(p)$ s.t. $G < G''$ and then, since $h \leq k$, it follows that $G'' \in T_P\uparrow^k(\mathbf{0})(p)$ which, since $G < G''$, contradicts the assumption that $G \in \max T_P\uparrow^k(\mathbf{0})(p)$. Thus, if $G \in \max T_P\uparrow^k(\mathbf{0})(p)$ and $height(G) = h \leq k$ it follows that $G \in T_P\uparrow^h(\mathbf{0})(p)$.

In case that $k = \omega$, by definition $T_P\uparrow^\omega(\mathbf{0})(p) = \sum_{i<\omega} T_P\uparrow^i(\mathbf{0})(p)$. Thus, if $G \in \max T_P\uparrow^\omega(\mathbf{0})(p)$ and $height(G) = h$ then there is some $i < \omega$ s.t. $G \in \max T_P\uparrow^i(\mathbf{0})(p)$ and $h \leq i$, and as we already show, then $G \in T_P\uparrow^h(\mathbf{0})(p)$. □

Lemma 0.7
Let P,Q two positive causal logic programs such that Q is the result of replacing label l in P by some u (a label or 1) then $T_Q\uparrow^k(\mathbf{0})(p) = T_P\uparrow^k(\mathbf{0})(p)[l \mapsto u]$ for any atom p and $k \in \{1,\ldots,\omega\}$.

Proof. In case that $n = 0$, $T_Q\uparrow^k(\mathbf{0})(p) = 0$ and $T_P\uparrow^k(\mathbf{0})(p) = 0$ and $0 = 0[l \mapsto u]$. That is $T_Q\uparrow^k(\mathbf{0})(p) = T_P\uparrow^k(\mathbf{0})(p)[l \mapsto u]$.

We proceed by induction on k assuming that $T_Q\uparrow^{k-1}(\mathbf{0})(p) = T_P\uparrow^{k-1}(\mathbf{0})(p)[l \mapsto u]$ for any atom p and we will show that $T_Q\uparrow(\mathbf{0})(p) = T_P\uparrow^k(\mathbf{0})(p)[l \mapsto u]$.

Pick $G \in T_P\uparrow^k(\mathbf{0})(p)$ then, from Lemma 0.3, there is a rule $l' : p \leftarrow q_1,\ldots,q_m$ and causal graphs $G_{q_1}, \ldots, G_{q_m}$ each one respectively in $T_P\uparrow^{k-1}(\mathbf{0})(q_i)$ s.t. $G \leq G_R = (G_{q_1} * \ldots * G_{q_m}) \cdot l'$. Thus, by induction hypothesis, for every atom q_i and c-graph $G_{q_i} \in T_P\uparrow^{k-1}(\mathbf{0})(q)$ it holds that $G_{q_i}[l \mapsto u] \in T_Q\uparrow^{k-1}(\mathbf{0})(q_i)$.

Let $G_R[l \mapsto u]$ be a c-graph defined as $G_R[l \mapsto u] = \big(G_{q_1}[l \mapsto u] * \ldots * G_{q_m}[l \mapsto u]\big) \cdot l'[l \mapsto u]$. Then, since $G \leq G_R$, it follows that $G[l \mapsto u] \leq G_R[l \mapsto u]$ and then, again from Lemma 0.3, it follows that $G[l \mapsto u] \in T_Q\uparrow^n(\mathbf{0})(p)$. That is $T_P\uparrow^k(\mathbf{0})(p)[l \mapsto u] \subseteq T_Q\uparrow^k(\mathbf{0})(p)$.

Pick $G \in T_Q\uparrow^k(\mathbf{0})(p)$ then, from Lemma 0.3, there is a rule there is a rule $l' : p \leftarrow q_1,\ldots,q_m$ and c-graphs $G_{q_1}, \ldots, G_{q_m}$ respectively in $T_P\uparrow^{k-1}(\mathbf{0})(q_i)$ s.t. $G \leq G_R$ where $G_R = (G_{q_1} * \ldots * G_{q_m}) \cdot l'$.

By induction hypothesis, for every atom q_i and graph G_{q_i} it holds that if $G_{q_i} \in T_Q\uparrow^{k-1}(\mathbf{0})(q_i)$ then $G_{q_i} \in T_P\uparrow^{k-1}(\mathbf{0})(q_i)[l \mapsto u]$. Thus, it follows that there is a graph $G'_{q_i} \in T_P\uparrow^{k-1}(\mathbf{0})(q_i)$ such that $G'_{q_i}[l \mapsto u] = G_{q_i}$. Let G'_R be a graph s.t. $G'_R = \big(G'_{q_1} * \ldots * G'_{q_m}\big) \cdot l'$. From Lemma 0.3 for every causal graph $G' \leq G'_R$ it holds that $G' \in T_P\uparrow^k(\mathbf{0})(p)$. Since $G'_R[l \mapsto u] = G_R$ and $G \leq G_R$ it follows that $G \leq G_R[l \mapsto u]$ and, since $G_R \in T_P\uparrow^k(\mathbf{0})(p)$, it follows that $G \in T_P\uparrow^k(\mathbf{0})(p)[l \mapsto u]$. Consequently $T_P\uparrow^k(\mathbf{0})(p)[l \mapsto u] \supseteq T_Q\uparrow^n(\mathbf{0})(p)$ and then $T_P\uparrow^k(\mathbf{0})(p)[l \mapsto u] = T_Q\uparrow^n(\mathbf{0})(p)$.

In case that $k = \omega$, by definition $T_P\uparrow^\omega(\mathbf{0})(p)[l \mapsto u] = \sum_{i<\omega} T_P\uparrow(\mathbf{0})(i)p[l \mapsto u]$ and as we alerady show $T_P\uparrow(\mathbf{0})(i)p[l \mapsto u] = T_Q\uparrow^i(\mathbf{0})(p)$ for all integer $i < \omega$, so that, their sum is also equal and consequently $T_P\uparrow^\omega(\mathbf{0})(p)[l \mapsto u] = T_Q\uparrow^\omega(\mathbf{0})(p)$.

Proof of Theorem 2. Let P' be a positive, completely labelled causal program with the same rules as P. From Lemma 0.1 it follows that (*i*) $lfp(T_P)$ and $lfp(T_{P'})$ are respectively the least model of the programs P and P', and (*ii*) $lfp(T_P) = T_P\uparrow^\omega(\mathbf{0})$ and $lfp(T_{P'}) = T_{P'}\uparrow^\omega(\mathbf{0})$.

Futhermore, it is clear that if P is an infinite program, i.e. $n = \omega$, then $T_{P'}\uparrow^n(\mathbf{0}) = T_{P'}\uparrow^\omega(\mathbf{0})$. Otherwise, by definition it holds that $T_{P'}\uparrow^n(\mathbf{0}) \leq T_{P'}\uparrow^\omega(\mathbf{0})$. Suppose $T_{P'}\uparrow^n(\mathbf{0}) < T_{P'}\uparrow^\omega(\mathbf{0})$. Then there is some atom p and c-graph $G \in T_{P'}\uparrow^\omega(\mathbf{0})(p)$ such that $G \notin T_{P'}\uparrow^n(\mathbf{0})(p)$. The longest simple path in G must be smaller than the number of its vertices and this must be smaller than the number of labels of the program which in its turn is equal to the number of rules n, i.e. $height(G) = h \leq n$. From Lemma 0.6 it follows that $G \in T_{P'} \uparrow^h (\mathbf{0})(p)$ and since $h \leq n$ it follows that $T_{P'} \uparrow^h (\mathbf{0})(p) \subseteq T_{P'} \uparrow^n (\mathbf{0})(p)$ and so that $G \in T_{P'} \uparrow^n (\mathbf{0})(p)$ which is a contradiction with the assumption that $G \in T_P \uparrow^\omega (\mathbf{0})(p)$ but $G \notin T_P \uparrow^n (\mathbf{0})(p)$. Thus $T_{P'} \uparrow^n (\mathbf{0}) = T_{P'} \uparrow^\omega (\mathbf{0})$.

Furthermore, from Lemma 0.7, $T_P \uparrow^k (\mathbf{0})(p) = T_{P'} \uparrow^k (\mathbf{0})(p)[l'_1 \mapsto l_1] \dots [l'_n \mapsto l_n]$ for $k \in \{n, \omega\}$ and where $l'_1, \dots, l'_n$ are the labels of rules of P' and $l_1, \dots, l_n$ are the correspondent labels of such rules in P. Thus, since $T_{P'} \uparrow^n (\mathbf{0}) = T_{P'} \uparrow^\omega (\mathbf{0})$, it follows that $T_P \uparrow^n (\mathbf{0}) = T_P \uparrow^\omega (\mathbf{0})$.

Lemma 0.8
For any proof $\pi(p)$ it holds that

$$graph\Big(\frac{\pi(q_1), \dots, \pi(q_m)}{p}\,(l)\Big) = \big(graph(\pi(q_1)) * \dots * graph(\pi(q_1))\big) \cdot l$$

Proof. We proceed by structural induction assuming that for every proof in the antecedent $\pi(q_i)$ and every label $l' \in V(graph(\pi(q_i)))$ there is an edge $(l', label(\pi(q_i))) \in E(graph(\pi(q_i)))$.

By definition $graph(\pi(p)) = G^*_{\pi(p)}$ is the reflexive and transitive closure of $G_{\pi(p)}$ and then

$$graph(\pi(p)) = \Big(\bigcup \big\{\, graph(\pi(q_i)) \;\big|\; 1 \leq i \leq m \,\big\} \cup \big\{\, (label(\pi(q_i)), l) \;\big|\; 1 \leq i \leq m \,\big\}\Big)^*$$

Thus, $graph(\pi(p)) \geq \prod \big\{\, graph(\pi(q_i)) \;\big|\; 1 \leq i \leq m \,\big\} \cdot l$ and remain to show that for every atom q_i and label $l' \in V(graph(\pi(q_i)))$ the edge $(l', l) \in E(graph(\pi(p)))$. Indeed, since by induction hypothesis there is an edge $(l', label(\pi(q_i))) \in E(graph(\pi(q_i))) \subseteq E(graph(\pi(p))$, the fact that the edge $(label(\pi(q_i), l) \in E(graph(\pi(p)))$ and since $graph(\pi(p))$ is closed transitively, it follows that $(l', l) \in E(graph(\pi(p)))$. □

Lemma 0.9
Let P be a positive, completely labelled program and $\pi(p)$ be a proof for p w.r.t. P. Then it holds that $graph(\pi_p) \in T_P \uparrow^h (\mathbf{0})(p)$ where h is the height of $\pi(p)$ which is recursively defined as

$$height(\pi) = 1 + \max\{\, height(\pi') \;\big|\; \pi' \text{ is a sub-proof of } \pi \,\}$$

Proof. In case that $h = 1$ the antecedent of $\pi(p)$ is empty, i.e.

$$\pi(p) = \frac{\top}{p}\,(l)$$

where l is the label of the fact $(l : p)$. Then $graph(\pi(p)) = l$. Furthermore, since the fact $(l : p)$ is in the program P, it follows that $l \in T_P \uparrow^1 (\mathbf{0})(p)$.

In the remain cases, we proceed by structural induction assuming that for every natural number $h \leq n-1$, atom p and proof $\pi(p)$ of p w.r.t. P whose $height(\pi(p)) = h$ it holds that $graph(\pi(p)) \in T_P \uparrow^h (\mathbf{0})(p)$ and we will show it in case that $h = n$.

Since $height(\pi_p) > 1$ it has a non empty antecedent, i.e.

$$\pi(p) = \frac{\pi(q_1), \dots, \pi(q_m)}{p}\,(l)$$

where l is the label of the rule $l : p \leftarrow q_1, \dots, q_m$. By *height* definition, for each q_i it holds that $height(\pi(q_i)) \leq n-1$ and so that, by induction hypothesis, $graph(\pi(q_i)) \in T_P \uparrow^{h-1} (\mathbf{0})(q_i)$. Thus, from Lemmas 0.3 and 0.8, it follows respectively that

$$\prod\{\ graph(\pi(q_i)) \mid 1 \leq i \leq m\ \} \cdot l \in T_P \uparrow^{h} (\mathbf{0})(p)$$
$$graph(\pi(p)) = \prod\{\ graph(\pi(q_i)) \mid 1 \leq i \leq m\ \} \cdot l$$

That is, $graph(\pi(p)) \in T_P \uparrow^{h} (\mathbf{0})(p)$. □

Lemma 0.10
Let P be a positive, completely labelled program and $\pi(p)$ be a proof of p w.r.t. P. For every atom p and maximal causal graph $G \in T_P \uparrow^{\omega} (\mathbf{0})(p)$ there is a non-redundant proof $\pi(p)$ for p w.r.t. P s.t. $graph(\pi(p)) = G$.

Proof. From Lemma 0.5 for any maximal graph $G \in T_P \uparrow^{k} (\mathbf{0})(p)$, there is a rule $l : p \leftarrow q_1, \dots, q_m$ and maximal graphs $G_{q_1} \in T_P \uparrow^{h-1} (\mathbf{0})(q_1), \dots, G_{q_m} \in T_P \uparrow^{k-1} (\mathbf{0})(q_m)$ s.t.

$$G = (G_{q_1} * \dots * G_{q_m}) \cdot l$$

Furthermore, we assume as induction hypothesis that for every atom q_i there is a non redundant proof $\pi(q_i)$ for q_i w.r.t. P s.t. $graph(\pi(q_i)) = G_{q_i}$. Then $\pi(p)$ defined as

$$\pi(p) = \frac{\pi(q_1), \dots, \pi(q_m)}{p}\,(l)$$

is a proof for p w.r.t. P which holds $graph(\pi(p)) = G_p$ (from Lemma 0.8) and $height(\pi(p) \leq h$. Furthermore, suppose that $\pi(p)$ is redundant, i.e. there is a proog π' for p w.r.t P such that $graph(\pi(p)) < graph(\pi')$. Let $h = height(\pi')$. Then, from Lemma 0.9, it follows that $graph(\pi') \in T_P \uparrow^{h} (\mathbf{0})(p)$ and then $graph(\pi') \in T_P \uparrow^{\omega} (\mathbf{0})(p)$ which contradicts the hypothesis that G is maximal in $T_P \uparrow^{\omega} (\mathbf{0})(p)$. □

Proof of Theorem 3. From Theorem 2 it follows that the least model I is equal to $T_P \uparrow^{\omega} (\mathbf{0})$. For the only if direction, from Lemma 0.10, it follows that for every maximal c-graph $G \in I(p) = T_P \uparrow^{\omega} (\mathbf{0})(p)$ there is a non-redundant proof $\pi(p)$ for p w.r.t P s.t. $G = graph(\pi(p))$. That is, $\pi(p) \in \Pi_p$ and then $G = graph(\pi(p)) \in graph(\Pi_p)$. For the if direction, from Lemma 0.9, for every $G \in graph(\Pi_p)$, i.e. $G = graph(\pi(p))$ for some non-redundant proof $\pi(p)$ for p w.r.t. P, it holds that $G \in T_P \uparrow^{\omega} (\mathbf{0})(p)$ and so that $G \in I(p)$. Furthermore, suppose that G is not maximal, i.e. there is a maximal c-graph $G' \in I(p)$ s.t. $G < G'$ and a proof π' for p w.r.t. P s.t. $graph(\pi') = G'$ which contradicts that $\pi(p)$ is non-redundant. □

Lemma 0.11
Let t be a causal term. Then $value(t[l \mapsto u]) = value(t)[l \mapsto u]$.

Proof. We proceed by structural induction. In case that t is a label. If $t = l$ then $value(l[l \mapsto u] = value(u) =\downarrow u = value(l)[l \mapsto u]$. If $t = l' \neq l$ then $value(l'[l \mapsto u] = value(l') =\downarrow l' = value(l')[l \mapsto u]$. In case that $t = \prod T$ it follows that $value(\prod T[l \mapsto u]) = \bigcap\{\ value(t'[l \mapsto u]) \mid t' \in T\ \})$ and by induction hypothesis $value(t'[l \mapsto u]) = value(t')[l \mapsto u]$. Then $value(\prod T[l \mapsto u]) = \bigcap\{\ value(t')[l \mapsto u] \mid t' \in T\ \}) = value(\prod T)[l \mapsto u]$. The cases for $t = \sum T$ is analogous. In case that $t = t_1 \cdot t_2$ it follows that $value(t[l \mapsto u]) = value(t_1[l \mapsto u]) \cdot value(t_2[l \mapsto u]) = value(t_1)[l \mapsto u] \cdot value(t_2)[l \mapsto u] = value(t)[l \mapsto u]$

Proof of Theorem 4. From Theorem 2, models I and I' are respectively equal to $T_P \uparrow^{\omega} (\mathbf{0})$ and $T_{P'} \uparrow^{\omega} (\mathbf{0})$. Furthermore, from Lemma 0.7, it follows that $T_{P'} \uparrow^{\omega} (\mathbf{0})(p) = T_P \uparrow^{\omega} (\mathbf{0})(p)[l \mapsto u]$ for any atom p. Lemma 0.11 shows that the replacing can be done in any causal term without operate it.

Proof of Theorem 5. It is clear that if every rule in P is unlabelled, i.e. $P = P'$, then their least model assigns 0 to every $false$ atom and 1 to every $true$ atom, so that their least models coincide with the classical one, i.e. $I = I'$ and then $I^{cl} = I = I'$. Otherwise, let P_n be a program where n rules are labelled. We can build a program P_{n-1} removing one label l and, from Theorem 4, it follows that $I_{n-1} = I_n[l \rightarrow 1]$. By induction hypothesis the corresponding classical interpretation of least model of P_{n-1} coincides with the least model of the unlabelled program, i.e. $I^{cl}_{n-1} = I'$, and then $I_n[l \mapsto 1]^{cl} = I^{cl}_{n-1} = I'$. Furthermore, for every atom p and c-graph G it holds that $G \in I_n(p)$ iff $G[l \mapsto 1] \in I_n[l \mapsto 1](p)$. Simple remain to note that $value(z) = \emptyset$, so that $I_n(p) = 0$ iff $I_n[l \mapsto 1](p) = 0$ and consequently $I^{cl}_n = I_n[l \mapsto 1]^{cl} = I'$. □

Proof of Theorem 6. By definition I and I^{cl} assigns 0 to the same atoms, so that $P^I = P^{I^{cl}}$. Furthermore let Q (instead of P' for clarity) be the unlabelled version of P. Then $Q^{I^{cl}}$ is the unlabelled version of P^I. (1) Let I be a stable model of P and J be the least model of $Q^{I^{cl}}$. Then, I is the least model of P^I and, from Theorem 5, it follows that $I^{cl} = J$, i.e. I^{cl} is a stable model of Q. (2) Let I' is a stable model of Q and I be the least model of $P^{I'}$. Since I' is a stable model of Q, by definition it is the least model of $Q^{I'}$, furthermore, since $Q^{I'}$ is the unlabelled version of $P^{I'}$ it follows, from Theorem 5, that $I^{cl} = I'$. Note that $P^I = P^{I^{cl}} = P^{I'}$. Thus I is a stable model of P. □