Online appendix for the paper

Rewriting and narrowing for constructor systems with call-time choice semantics

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Appendix A Detailed proofs for the results

In the proofs we will use the usual notation for positions, subexpressions and repacements from (Baader and Nipkow 1998). The *set of positions* of an expression $e \in Exp$ is a set $O(e)$ of strings of positive integers defined as:

- If $e \equiv X \in \mathcal{V}$, then $O(e) = \epsilon$, where ϵ is the empty string.
- If $e \equiv h(e_1, \ldots, e_n)$ with $h \in \Sigma$, then

$$O(e) = \{\epsilon\} \cup \bigcup_{i=1}^{n} \{ip \mid p \in O(e_i)\}$$

The *subexpression of e at position $p \in O(e)$*, denoted $e|_p$, is defined as:

$$\begin{array}{rcl} e|_\epsilon & = & e \\ h(e_1, \ldots, e_n)|_{ip} & = & e_i|_p \end{array}$$

For a position $p \in O(e)$, we define the *replacement of the subexpression of e at position p by e'* —denoted $e[e']_p$— as follows:

$$\begin{array}{rcl} e[e']_\epsilon & = & e' \\ h(e_1, \ldots, e_n)[e']_{ip} & = & h(e_1, \ldots, e_i[e']_p, \ldots, e_n) \end{array}$$

When performing proofs by induction we will usually use IH to refer to the induction hypothesis of the current induction. We will use an asterisk to denote the use of a let-rewriting rule one or more times, as in (Flat*). We will also use the following auxiliary results.

A.1 Lemmas

The following lemmas are used in the proofs for the results in the article. Most of them are straightforwardly proved by induction, so we only detail the proof in the interesting cases.

Lemma 17
$\forall t \in CTerm_\perp.\ |t| = t$.

Lemma 18
$\forall t \in CTerm_\perp.\ \mathcal{P} \vdash_{CRWL_{let}} t \rightarrow t$.

Lemma 19
Given $\theta, \theta' \in LSubst_\perp$, $e \in LExp_\perp$, if $\theta \sqsubseteq \theta'$ then $e\theta \sqsubseteq e\theta'$.

Lemma 20
Given $\theta \in LSubst_\perp$, $e, e' \in LExp_\perp$, if $e \sqsubseteq e'$ then $e\theta \sqsubseteq e'\theta$.

Lemma 21
For every $e, e' \in LExp_\perp$, $\mathcal{C} \in Cntxt$, if $|e| \sqsubseteq |e'|$ then $|\mathcal{C}[e]| \sqsubseteq |\mathcal{C}[e']|$.

Proof
We proceed by induction on the structure of $\mathcal{C}$. The base case is straightforward because of the hypothesis. For the Inductive Step we have:

- $\mathcal{C} \equiv h(\ldots, \mathcal{C}', \ldots)$. Directly by IH.
- $\mathcal{C} \equiv let\ X = \mathcal{C}'\ in\ e_1$, so $\mathcal{C}[e] \equiv let\ X = \mathcal{C}'[e]\ in\ e_1$. Then:

$$\begin{array}{l} |\mathcal{C}[e]| = |let\ X = \mathcal{C}'[e]\ in\ e_1| = |e_1|[X/|\mathcal{C}'[e]|] \\ \sqsubseteq_{IH^{(*)}} |e_1|[X/|\mathcal{C}'[e']|] = |let\ X = \mathcal{C}'[e']\ in\ e_1| = |\mathcal{C}[e']| \end{array}$$

  $(*)$ By IH we have $|\mathcal{C}'[e]| \sqsubseteq |\mathcal{C}'[e']|$, therefore $[X/|\mathcal{C}'[e]|] \sqsubseteq [X/|\mathcal{C}'[e']|]$. Finally, by Lemma 19, $|e_1|[X/|\mathcal{C}'[e]|] \sqsubseteq |e_1|[X/|\mathcal{C}'[e']|]$.
- $\mathcal{C} \equiv let\ X = e_1\ in\ \mathcal{C}'$. Similar to the previous case but using Lemma 20 to obtain $|\mathcal{C}'[e]|\ [X/|e_1|] \sqsubseteq |\mathcal{C}'[e']|[X/|e_1|]$ from the IH $|\mathcal{C}'[e]| \sqsubseteq |\mathcal{C}'[e']|$.

□

Lemma 22
If $|e| = |e'|$ then $|\mathcal{C}[e]| = |\mathcal{C}[e']|$

Proof
Since $\sqsubseteq$ is a partial order, we know by reflexivity that $|e| \sqsubseteq |e'|$ and $|e'| \sqsubseteq |e|$. Then by Lemma 21 we have $|\mathcal{C}[e]| \sqsubseteq |\mathcal{C}[e']|$ and $|\mathcal{C}[e']| \sqsubseteq |\mathcal{C}[e]|$. Finally, by antisymmetry of the partial order $\sqsubseteq$ we have that $|\mathcal{C}[e]| = |\mathcal{C}[e']|$. □

Lemma 23
For all $e_1, e_2 \in LExp, X \in \mathcal{V}$, $|e_1[X/e_2]| \equiv |e_1|[X/|e_2|]$

Proof
By induction on the structure of e_1. The most interesting case is when $e_1 \equiv let\ Y = s_1\ in\ s_2$. By the variable convention $Y \notin dom([X/e_2])$ and $Y \notin vran([X/e_2])$, so:

$$\begin{array}{ll} |e_1[X/e_2]| \equiv |let\ Y = s_1[X/e_2]\ in\ s_2[X/e_2]| & \\ \equiv |s_2[X/e_2]|[Y/|s_1[X/e_2]|] & \\ \equiv_{IH} |s_2|[X/|e_2|][Y/(|s_1|[X/|e_2|])] & \\ \equiv |s_2|[Y/|s_1|][X/|e_2|] & (*) \\ \equiv |let\ Y = s_1\ in\ s_2|[X/|e_2|] \equiv |e_1|[X/|e_2|] & \end{array}$$

(*) Using Lemma 1 with the matching $[e/|s_2|, \theta/[X/|e_2|], X/Y, e'/|s_1|]$. $\square$

Lemma 24
Given $\theta \in LSubst_\perp$, $e, e' \in LExp_\perp$, if $e \sqsubseteq e'$ then $e\theta \sqsubseteq e'\theta$.

Lemma 25
For every $\sigma \in LSubst_\perp, \mathcal{C} \in Cntxt$ and $e \in LExp_\perp$ such that $(dom(\sigma) \cup vran(\sigma)) \cap BV(\mathcal{C}) = \emptyset$ we have that $(\mathcal{C}[e])\sigma \equiv \mathcal{C}\sigma[e\sigma]$.

Proof
By induction on the structure of $\mathcal{C}$. The most interesting cases are those concerning let-expressions:

- $\mathcal{C} \equiv let\ X = \mathcal{C}'\ in\ e_1$: therefore $\mathcal{C}[e] \equiv let\ X = \mathcal{C}'[e]\ in\ e_1$. Then

$$\begin{array}{c} (\mathcal{C}[e])\sigma \equiv let\ X = (\mathcal{C}'[e])\sigma\ in\ e_1\sigma \equiv_{IH}^{(*)} let\ X = \mathcal{C}'\sigma[e\sigma]\ in\ e_1\sigma \\ \equiv (let\ X = (\mathcal{C}'[])\sigma\ in\ e_1\sigma)[e\sigma] \equiv^{(**)} ((let\ X = \mathcal{C}'[]\ in\ e_1)\sigma)[e\sigma] \equiv \mathcal{C}\sigma[e\sigma] \end{array}$$

  $(*)$: by definition $BV(let\ X = \mathcal{C}'\ in\ e) = BV(\mathcal{C}')$, so $(dom(\sigma) \cup vran(\sigma)) \cap BV(\mathcal{C}) = \emptyset = (dom(\sigma) \cup vran(\sigma)) \cap BV(\mathcal{C}')$.
  $(**)$: we can apply the last step because by hypothesis we can assure that we do not need any renaming to apply $(let\ X = \mathcal{C}'[]\ in\ e_1)\sigma$.
- $\mathcal{C} \equiv let\ X = e_1\ in\ \mathcal{C}'$: therefore $\mathcal{C}[e] \equiv let\ X = e_1\ in\ \mathcal{C}'[e]$. Then

$$\begin{array}{c} (\mathcal{C}[e])\sigma \equiv let\ X = e_1\sigma\ in\ (\mathcal{C}'[e])\sigma \equiv_{IH} let\ X = e_1\sigma\ in\ \mathcal{C}'\sigma[e\sigma] \\ \equiv (let\ X = e_1\sigma\ in\ (\mathcal{C}'[])\sigma)[e\sigma] \equiv^{(*)} ((let\ X = e_1\ in\ \mathcal{C}'[])\sigma)[e\sigma] \equiv \mathcal{C}\sigma[e\sigma] \end{array}$$

  $(*)$: we can apply the last step because by hypothesis we can assure that we do not need any renaming to apply $(let\ X = e_1\ in\ \mathcal{C}'[])\sigma$.

$\square$

Lemma 26
For any $e \in Exp_\perp$, $t \in CTerm_\perp$ and program $\mathcal{P}$, if $\mathcal{P} \vdash e \twoheadrightarrow t$ then there is a derivation for $\mathcal{P} \vdash e \twoheadrightarrow t$ in which every free variable used belongs to $FV(e \twoheadrightarrow t)$.

Proof
A simple extension of the proof in (Dios-Castro and López-Fraguas 2007). □

Lemma 27
For every $CRWL_{let}$ derivation $e \rightarrow t$ there exists $e' \in LExp_\perp$ which is syntactically equivalent to e module α-conversion, and a $CRWL_{let}$ derivation for $e' \rightarrow t$ such that if $\mathcal{B}$ is the set of bound variables used in $e' \rightarrow t$ and $\mathcal{E}$ is the set of free variables used in the instantiation of extra variables in $e' \rightarrow t$ then $\mathcal{B} \cap (\mathcal{E} \cup var(t)) = \emptyset$.

Proof
By Lemma 26, if $\mathcal{F}$ is the set of free variables used in $e' \rightarrow t$, then $\mathcal{F} \subseteq FV(e' \rightarrow t)$, in fact $\mathcal{F} = FV(e' \rightarrow t)$, as $FV(e')$ and $FV(t)$ are used in the top derivation of the derivation tree for $e' \rightarrow t$. As by definition $\mathcal{E} \cup var(t) \subseteq \mathcal{F}$, if we prove $\mathcal{B} \cap \mathcal{F} = \emptyset$ then $\mathcal{B} \cap (\mathcal{E} \cup var(t)) = \emptyset$ is a trivial consequence. To prove that we will prove that for every $a \in LExp_\perp$ used in the derivation for $e' \rightarrow t$ we have $BV(a) \cap FV(a) = \emptyset$. We can build e' using α-conversion to ensure that $BV(e') \cap FV(e') = \emptyset$. This can be easily maintained as an invariant during the derivation, as the new let-bindings that appear during the derivation are those introduced in the instances of the rule used during the **OR** steps, and be can ensure by α-conversion that $BV(a) \cap FV(a) = \emptyset$ for these instances too, as α-conversion leaves the hypersemantics untouched. □

A.2 Proofs for Section 2.2

Theorem 1 (Compositionality of CRWL)
For any $\mathcal{C} \in Cntxt$, $e, e' \in Exp_\perp$

$$[\![\mathcal{C}[e]]\!] = \bigcup_{t \in [\![e]\!]} [\![\mathcal{C}[t]]\!]$$

As a consequence: $[\![e]\!] = [\![e']\!] \Leftrightarrow \forall \mathcal{C} \in Cntxt. [\![\mathcal{C}[e]]\!] = [\![\mathcal{C}[e']]\!]$

Proof
We prove that $\mathcal{C}[e] \rightarrow t \Leftrightarrow \exists s \in CTerm_\perp$ such that $e \rightarrow s$ and $\mathcal{C}[s] \rightarrow t$.

$\Rightarrow$) Induction on the size of the proof for $\mathcal{C}[e] \rightarrow t$.

Base case The base case only allows the proofs $\mathcal{C}[e] \rightarrow \perp$ using (B), $\mathcal{C}[e] \equiv X \rightarrow X$ using (RR) and $\mathcal{C}[e] \equiv c \rightarrow c$ with $c \in CS$ using (DC), that are clear. When $\mathcal{C} = [\ ]$ the proof is trivial with $s = t$ and using Lemma 18.

Inductive step Direct application of the IH.

$\Leftarrow$) By induction on the size of the proof for $\mathcal{C}[s] \rightarrow t$

Base case The base case only allows the proofs $\mathcal{C}[s] \rightarrow \perp$, $\mathcal{C}[s] \equiv X \rightarrow X$ and $\mathcal{C}[s] \equiv c \rightarrow c$ with $c \in CS$, that are clear. When $\mathcal{C} = [\ ]$ we have that $\exists s \in CTerm_\perp$ such that $e \rightarrow s$ and $s \rightarrow t$. Since $s \rightarrow t$ by Lemma 5 we have $t \sqsubseteq s$, and using Proposition 3 $e \rightarrow t$ —as $e \sqsubseteq e$ because $\sqsubseteq$ is a partial order.

Inductive step Direct application of the IH.
□

A.3 Proofs for Section 3

Theorem 3

Let $\mathcal{P}$ be a CRWL-program, $e \in Exp_\perp$ and $t \in CTerm_\perp$. Then:

$$\mathcal{P} \vdash_{CRWL} e \rightarrow t \textit{ iff } e \rightarrowtail^*_{\mathcal{P}} t$$

Proof

It is easy to see that $\rightarrowtail^*$ coincides with the relation defined by the *BRC*-proof calculus of (González-Moreno et al. 1999), that is, $\mathcal{P} \vdash_{BRC} e \rightarrow e' \;\leftrightarrow e \rightarrowtail^* e'$. But in that paper it is proved that *BRC*-derivability and CRWL-derivability (called there *GORC*-derivability) are equivalent. $\square$

A.4 Proofs for Section 4

Lemma 1 (Substitution lemma for let-expressions)

Let $e, e' \in LExp_\perp$, $\theta \in Subst_\perp$ and $X \in \mathcal{V}$ such that $X \notin dom(\theta) \cup vran(\theta)$. Then:

$$(e[X/e'])\theta \equiv e\theta[X/e'\theta]$$

Proof

By induction over the structure of e. The most interesting cases are the base cases:

- $e \equiv X$: Then $(e[X/e'])\theta \equiv (X[X/e'])\theta \equiv e'\theta \equiv X[X/e'\theta]$
  $\equiv_{X \notin dom(\theta)} X\theta[X/e'\theta] \equiv e\theta[X/e'\theta]$
- $e \equiv Y \not\equiv X$: Then $(e[X/e't])\theta \equiv (Y[X/e'])\theta \equiv Y\theta$
  $\equiv_{X \notin ran(\theta)} Y\theta[X/e'\theta] \equiv e\theta[X/e'\theta]$

$\square$

A.5 Proofs for Section 4.1

Lemma 2 (Closedness under CSubst of let-rewriting)

For any $e, e' \in LExp$, $\theta \in CSubst$ we have that $e \rightarrow^{l\,n} e'$ implies $e\theta \rightarrow^{l\,n} e'\theta$.

Proof

We prove that $e \rightarrow^l e'$ implies $e\theta \rightarrow^l e'\theta$ by a case distinction over the rule of the let-rewriting calculus applied:

(Fapp) Assume $f(t_1, \ldots, t_n) \rightarrow^l r$, using $(f(p_1, \ldots, p_n) \rightarrow e) \in \mathcal{P}$ and $\sigma \in CSubst$ such that $\forall i. p_i\sigma = t_i$ and $e\sigma = r$. But since $\sigma\theta \in CSubst$ and $\forall i. p_i\sigma\theta = t_i\theta$ then we can perform a (Fapp) step $f(t_1, \ldots, t_n)\theta \equiv f(t_1\theta, \ldots, t_n\theta) \rightarrow^l e\sigma\theta \equiv r\theta$.

(LetIn) Easily since $X \notin dom(\theta)$ because X is fresh.

(Bind) Assume $let\ X = t\ in\ e \rightarrow^l e[X/t]$ and some $\theta \in CSubst$. Then $t \in CTerm$ by the conditions of (Bind), hence $t\theta \in CTerm$ too and we can perform a (Bind) step $(let\ X = t\ in\ e)\theta \equiv let\ X = t\theta\ in\ e\theta \rightarrow^l e\theta[X/t\theta]$. Besides $X \notin (dom(\theta) \cup vran(\theta))$ by the variable convention, and so $e\theta[X/t\theta] \equiv e[X/t]\theta$ by Lemma 1, so are done.

(Elim) Easily as $X \notin FV(e_2\theta)$ because $X \notin vran(\theta)$ by the variable convention.

(Flat) Similar to the previous case since $Y \notin FV(e_3\theta)$.

(Contx) Assume $\mathcal{C}[e] \to^l \mathcal{C}[e']$ because $e \to^{l\,\prime} e'$ by one of the previous rules, and some $\theta \in CSubst$. Then we have already proved that $e\theta \to^l e'\theta$. Besides by the variable convention we have $BV(\mathcal{C}) \cap (dom(\theta) \cup vran(\theta)) = \emptyset$, hence by Lemma 25 $(\mathcal{C}[e])\theta \equiv \mathcal{C}\theta[e\theta]$. Furthermore, if $e \to^l e'$ was a (Fapp) step using $\sigma \in CSubst$ to build the instance of the program rule $(f(\overline{p})\sigma \to r\sigma)$, then $vran(\sigma|_{\setminus var(\overline{p})}) \cap BV(\mathcal{C}) = \emptyset$ by the conditions of (Contx), and therefore $vran((\sigma\theta)|_{\setminus var(\overline{p})}) \cap BV(\mathcal{C}) = \emptyset$. But as $\sigma\theta$ is the substitution used in the (Fapp) step $e\theta \to^l e'\theta$, then $\mathcal{C}\theta[e\theta] \to^l \mathcal{C}\theta[e'\theta]$ by (Contx). On the other hand, if $e \to^l e'$ was not a (Fapp) step then $\mathcal{C}\theta[e\theta] \to^l \mathcal{C}\theta[e'\theta]$ too, and finally we can apply Lemma 25 again to get $\mathcal{C}\theta[e'\theta] \equiv (\mathcal{C}[e'])\theta$.

The proof for $e \to^{l\,n} e'$ proceeds straightforwardly by induction on the length n of the derivation. □

Proposition 2 (Termination of $\to^{lnf}$)
Under any program we have that $\to^{lnf}$ is terminating.

Proof
We define for any $e \in LExp$ the size (k_1, k_2, k_3), where

$k_1 \equiv$ *number of subexpressions in e to which (LetIn) is applicable.*
$k_2 \equiv$ *number of* lets *in e.*
$k_3 \equiv$ *sum of the levels of nesting of all let-subexpressions in e.*

Sizes are lexicographically ordered. We prove now that application of *(LetIn), (Bind), (Elim), (Flat)* in any context (hence, also the application of (Contxt)) decreases the size, what proves termination of $\to^{lnf}$. The effect of each rule in the size is summarized as follows (in each case, we stop at the decreasing component):

(LetIn):	$(<, \_, \_)$
(Bind):	$(=, <, \_)$
(Elim):	$(\leq, <, \_)$
(Flat):	$(=, =, <)$

□

Lemma 3 (Peeling lemma)
For any $e, e' \in LExp$ if $e \downarrow^{lnf} e'$ —i.e, e' is a $\to^{lnf}$ normal form for e— then e' has the shape $e' \equiv let\ \overline{X = f(\overline{t})}\ in\ e''$ such that $e'' \in \mathcal{V}$ or $e'' \equiv h(\overline{t'})$ with $h \in \Sigma$, $\overline{f} \subseteq FS$ and $\overline{t}, \overline{t'} \subseteq CTerm$.
Moreover if $e \equiv h(e_1, \ldots, e_n)$ with $h \in \Sigma$, then

$$e \equiv h(e_1, \ldots, e_n) \to^{lnf^*} let\ \overline{X = f(\overline{t})}\ in\ h(t_1, \ldots, t_n) \equiv e'$$

under the conditions above, and verifying also that $t_i \equiv e_i$ whenever $e_i \in CTerm$.

Proof
We prove it by contraposition: if an expression e does not have that shape, e is not a $\to^{lnf}$ normal form. We define the set of expressions which are not cterms as:

$$\begin{array}{rl} nt ::= & c(\ldots, nt, \ldots) \\ & \mid f(\overline{e}) \\ & \mid let\ X = e_1\ in\ e_2 \end{array}$$

We also define the set of expressions which do not have the presented shape recursively as:

$$\begin{array}{rl} ne ::= & h(\ldots, nt, \ldots) \\ & \mid let\ X = f(\overline{t})\ in\ ne \\ & \mid let\ X = f(\ldots, nt, \ldots)\ in\ e \\ & \mid let\ X = c(\overline{e})\ in\ e \\ & \mid let\ X = (let\ Y = e'\ in\ e'')\ in\ e \end{array}$$

We prove by induction on the structure of an expression ne that it is always possible to perform a $\to^{lnf}$ step:

Base case:

- $ne \equiv h(\ldots, nt, \ldots)$: there are various cases depending on nt:
  - — at some depth the non-cterm will contain a subexpression $c'(\ldots, nt', \ldots)$ where nt' is a function application $f(\overline{e})$ or a let-rooted expression $let\ X = e_1\ in\ e_2$. Therefore we can apply the rule (Contx) with (LetIn) in that position.
  - — $f(\overline{e})$: we can apply the rule (LetIn) and perform the step

$$h(\ldots, f(\overline{e}), \ldots) \to^{lnf} let\ X = f(\overline{e})\ in\ h(\ldots, X, \ldots)$$

  - — $let\ X = e_1\ in\ e_2$: the same as the previous case.
- $let\ X = f(\ldots, nt, \ldots)\ in\ e$: we can perform a (Contx) with (LetIn) step in $f(\ldots, nt, \ldots)$ as in the previous $h(\ldots, nt, \ldots)$ case.
- $let\ X = c(\overline{e})\ in\ e$: if $\overline{e}$ are cterms $\overline{t}$, then $c(\overline{t})$ is a cterm and we can perform a (Bind) step $let\ X = c(\overline{t})\ in\ e \to^{lnf} e[X/c(\overline{t})]$. If $\overline{e}$ contains any expression ne then we can perform a (Contx) with (LetIn) step as in the previous $h(\ldots, nt, \ldots)$ case.
- $let\ X = (let\ Y = e'\ in\ e'')\ in\ e$: by the variable convention we can assume that $Y \notin FV(e)$, so we can perform a (Flat) step $let\ X = (let\ Y = e'\ in\ e'')\ in\ e \to^{lnf} let\ Y = e'\ in\ let\ X = e''\ in\ e$.

Inductive step:

- $let\ X = f(\overline{t})\ in\ ne$: by IH we have that $ne \to^{lnf} ne'$, so by the rule (Contx) we can perform a step $let\ X = f(\overline{t})\ in\ ne \to^{lnf} let\ X = f(\overline{t})\ in\ ne'$.

Notice that if the original expression has the shape $h(e_1, \ldots, e_n)$ the arguments e_i which are cterms remain unchanged in the same position. The reason is that no rule can affect them: the only rule applicable at the top is (LetIn), and it can not place them in a let binding outside $h(\ldots)$; besides cterms do not match with the left-hand side of any rule, so they can not be rewritten by any rule. □

Lemma 4 (Growing of shells)
Under any program $\mathcal{P}$ and for any $e, e' \in LExp$

i) $e \to^{l^*} e'$ implies $|e| \sqsubseteq |e'|$
ii) $e \to^{lnf^*} e'$ implies $|e| \equiv |e'|$

Proof for Lemma 4
We prove the lemma for one step ($e \rightarrow^l e'$ and $e \rightarrow^{lnf} e'$) by a case distinction over the rule of the let-rewriting calculus applied:

(Fapp) The step is $f(t_1, \ldots, t_n) \rightarrow^l r$, and $|f(t_1, \ldots, t_n)| = \perp \sqsubseteq |r|$.
(LetIn) The equality $|h(e_1, \ldots, e, \ldots, e_n)| = |let\ X = e\ in\ h(e_1, \ldots, X, \ldots, e_n)|$ follows easily by a case distinction on h.
(Bind) The step is $let\ X = t\ in\ e \rightarrow^l e[X/t]$, so $|let\ X = t\ in\ e| = |e|[X/|t|] = |e[X/t]|$ by Lemma 23.
(Elim) The step is $let\ X = e_1\ in\ e_2 \rightarrow^l e_2$ with $X \notin FV(e_2)$. Then $|let\ X = e_1\ in\ e_2| = |e_2|[X/|e_1|] = |e_2|$. Since the variables in the shell of an expression is a subset of the variables in the original expression, we can conclude that if $X \notin FV(e_2)$ then $X \notin FV(|e_2|)$.
(Flat) The step is $let\ X = (let\ Y = e_1\ in\ e_2)\ in\ e_3 \ \rightarrow^l \ let\ Y = e_1\ in\ (let\ X = e_2\ in\ e_3)$ with $Y \notin FV(e_3)$. By the variable convention we can assume that $X \notin FV(let\ Y = e_1\ in\ e_2)$ —in particular $X \notin FV(e_1)$. Then:

$$\begin{array}{l} |let\ Y = e_1\ in\ (let\ X = e_2\ in\ e_3)| \\ = |let\ X = e_2\ in\ e_3|[Y/|e_1|] \\ = (|e_3|[X/|e_2|])[Y/|e_1|] \end{array}$$

Notice that $X \notin dom([Y/|e_1|])$ and $X \notin vran([Y/|e_1|]) = FV(|e_1|)$ because $X \notin FV(e_1)$ and $FV(|e_1|) \subseteq FV(e_1)$. Therefore we can use Lemma 1:

$$\begin{array}{ll} (|e_3|[X/|e_2|])[Y/|e_1|] & \\ = (|e_3|[Y/|e_1|])[X/(|e_2|[Y/|e_1|])] & \text{By Lemma 1} \\ = |e_3|[X/(|e_2|[Y/|e_1|])] & Y \notin FV(e_3)\text{, so } Y \notin FV(|e_3|) \\ = |e_3|[X/|let\ Y = e_1\ in\ e_2|] & \\ = |let\ X = (let\ Y = e_1\ in\ e_2)\ in\ e_3| & \end{array}$$

(Contx) The step is $\mathcal{C}[e] \rightarrow^l \mathcal{C}[e']$ with $e \rightarrow^l e'$ using any of the previous rules. Then we have $|e| \sqsubseteq |e'|$, and by Lemma 21 $\mathcal{C}[e] \sqsubseteq \mathcal{C}[e']$. If the step is $\mathcal{C}[e] \rightarrow^{lnf} \mathcal{C}[e']$ then rule (Fapp) has not been used in the reduction $e \rightarrow^{lnf} e'$ and by the previous rules we have $|e| = |e'|$. In that case by Lemma 22 we have $\mathcal{C}[e] = \mathcal{C}[e']$.

The extension of this result to $\rightarrow^{l^*}$ and $\rightarrow^{lnf^*}$ is a trivial induction over the number of steps of the derivation. □

A.6 Proofs for Section 4.2

Theorem 4 (CRWL vs. CRWL$_{let}$)
For any program $\mathcal{P}$ without lets, and any $e \in Exp_\perp$:

$$[\![e]\!]^{\mathcal{P}}_{CRWL} = [\![e]\!]^{\mathcal{P}}_{CRWL_{let}}$$

Proof
As any calculus rule from CRWL is also a rule from CRWL$_{let}$, then any CRWL-proof is also a CRWL$_{let}$-proof, therefore $[\![e]\!]_{CRWL} \subseteq [\![e]\!]_{CRWL_{let}}$. For the other inclusion, assume no let-binding is present in the program and let $e \in Exp$. Then, for any

$t \in CTerm_\perp$, as the rules of CRWL$_{let}$ do not introduce any let-binding and the rule (Let) is only used for let-rooted expressions, the CRWL$_{let}$-proof $\mathcal{P} \vdash_{CRWL_{let}} e \rightarrowtriangle t$ will be also a CRWL-proof for $\mathcal{P} \vdash_{CRWL_{let}} e \rightarrowtriangle t$, hence $[\![e]\!]_{CRWL_{let}} \subseteq [\![e]\!]_{CRWL}$ too. $\square$

The following Lemma is used to prove point *iii)* of Lemma 5. Notice that this Lemma uses the notions of hyperdenotation ($[\![\![\ ]\!]\!]$) and hyperinclusion ($\Subset$) presented in the final part of Section 4.2.

Lemma 28
Under any program $\mathcal{P}$ and for any $e \in LExp_\perp$ we have that $[\![\![e]\!]\!] \Subset \lambda\theta.(|e\theta|\Uparrow)\Downarrow$.

Proof
We will use the following equivalent characterization of $(e\Uparrow)\Downarrow$:

$$(e\Uparrow)\Downarrow = \{e_1 \in LExp_\perp \mid \exists e_2 \in LExp_\perp.\ e \sqsubseteq e_2 \wedge e_1 \sqsubseteq e_2\}$$

note that $\{e_2 \in LExp_\perp \mid e \sqsubseteq e_2\}$ is precisely the set $e\Uparrow$. Besides note that:

$$\begin{array}{l} [\![\![e]\!]\!] \Subset \lambda\theta.(|e\theta|\Uparrow)\Downarrow \\ \Leftrightarrow \forall\theta \in CSubst_\perp.\ [\![e\theta]\!] \subseteq (|e\theta|\Uparrow)\Downarrow \\ \Leftrightarrow \forall\theta \in CSubst_\perp, t \in CTerm_\perp.\ e\theta \rightarrowtriangle t \\ \qquad \Rightarrow t \in (|e\theta|\Uparrow)\Downarrow \\ \Leftrightarrow \forall\theta \in CSubst_\perp, t \in CTerm_\perp.\ e\theta \rightarrowtriangle t \\ \qquad \Rightarrow \exists t' \in CTerm_\perp.\ |e\theta| \sqsubseteq t' \wedge t \sqsubseteq t' \end{array}$$

where $t' \in CTerm_\perp$ is implied by $|e\theta| \sqsubseteq t'$. To prove this last formulation first consider the case when $t \equiv \perp$. Then we are done with $t' \equiv |e\theta|$ because then $|e\theta| \sqsubseteq |e\theta| \equiv t'$ and $t \equiv \perp \sqsubseteq |e\theta| \equiv t'$.

For the other case we proceed by induction on the structure of e. Regarding the base cases:

- If $e \equiv \perp$ then $t \equiv \perp$ and we are in the previous case.
- If $e \equiv X \in \mathcal{V}$ then $e\theta \equiv \theta(X) \rightarrowtriangle t$, and as $\theta \in CSubst_\perp$ then $\theta(X) \in CTerm_\perp$ which implies $t \sqsubseteq \theta(X)$ by Lemma 5. But then we can take $t' \equiv \theta(X)$ for which $t \sqsubseteq \theta(X) \equiv t'$ and $|e\theta| \equiv |\theta(X)| \equiv \theta(X)$ —by Lemma 17 since $\theta(X) \in CTerm_\perp$—, and $\theta(X) \sqsubseteq \theta(X) \equiv t'$.
- If $e \equiv c \in DC$ then either $t \equiv \perp$ and we are in the previous case, or $t \equiv c$. But then we can take $t' \equiv c$ for which $|e\theta| \equiv c \sqsubseteq c \equiv t'$, and $t \equiv c \sqsubseteq c \equiv t'$.
- If $e \equiv f \in FS$ then $|e\theta| \equiv |f| \equiv \perp$, and so $|e\theta|\Uparrow = CTerm_\perp$ and $(|e\theta|\Uparrow)\Downarrow = CTerm_\perp \supseteq [\![e\theta]\!]$, so we are done.

Concerning the inductive steps:

- If $e \equiv f(e_1, \ldots, e_n)$ for $f \in FS$ then $|e\theta| \equiv \perp$ and we proceed like in the case for $e \equiv f$.
- If $e \equiv c(e_1, \ldots, e_n)$ for $c \in DC$ then either $t \equiv \perp$ and we are in the previous case, or $t \equiv c(t_1, \ldots, t_n)$ such that $\forall i.\ e_i\theta \rightarrowtriangle t_i$. But then by IH we get $\forall i.\ \exists t'_i.\ |e_i\theta| \sqsubseteq t'_i \wedge t_i \sqsubseteq t'_i$, so we can take $t' \equiv c(t'_1, \ldots, t'_n)$ for which $|e\theta| \equiv c(|e_1\theta|, \ldots, |e_n\theta|) \sqsubseteq c(t'_1, \ldots, t'_n) \equiv t'$ and $t \equiv c(t_1, \ldots, t_n) \sqsubseteq c(t'_1, \ldots, t'_n) \equiv t'$.

- If $e \equiv let\ X = e_1\ in\ e_2$ then either $t \equiv \perp$ and we are in the previous case, or we have the following proof:

$$\frac{e_1\theta \rightarrowtriangle t_1 \quad e_2\theta[X/t_1] \rightarrowtriangle t}{e\theta \equiv let\ X = e_1\theta\ in\ e_2\theta \rightarrowtriangle t}\ Let$$

Then by IH over e_1 we get that $\exists t'_1.\ |e_1\theta| \sqsubseteq t'_1 \wedge t_1 \sqsubseteq t'_1$. Hence $[X/t_1] \sqsubseteq [X/t'_1]$ so by Proposition 5 we have that $e_2\theta[X/t_1] \rightarrowtriangle t$ implies $e_2\theta[X/t'_1] \rightarrowtriangle t$. But then we can apply the IH over e_2 with $\theta[X/t'_1]$ to get some $t' \in CTerm_\perp$ such that $t \sqsubseteq t'$ and $|e_2\theta[X/t'_1]| \sqsubseteq t'$, which implies:

$$\begin{array}{ll} t' \sqsupseteq |e_2\theta[X/t'_1]| & \\ \equiv |e_2\theta|[X/|t'_1|] & \text{by Lemma 23} \\ \equiv |e_2\theta|[X/t'_1] & \text{by Lemma 17 as } t'_1 \in CTerm_\perp \\ \sqsupseteq |e_2\theta|[X/|e_1\theta|] & \text{as } |e_1\theta| \sqsubseteq t'_1 \\ \equiv |let\ X = e_1\theta\ in\ e_2\theta| \equiv |e\theta| & \end{array}$$

□

Lemma 5
For any program $e \in LExp_\perp$, $t, t' \in CTerm_\perp$:

1. $t \rightarrowtriangle t'$ iff $t' \sqsubseteq t$.
2. $|e| \in [\![e]\!]$.
3. $[\![e]\!] \subseteq (|e|\uparrow)\downarrow$, where for a given $E \subseteq LExp_\perp$ its upward closure is $E\uparrow = \{e' \in LExp_\perp \mid \exists e \in E.\ e \sqsubseteq e'\}$, its downward closure is $E\downarrow = \{e' \in LExp_\perp \mid \exists e \in E.\ e' \sqsubseteq e\}$, and those operators are overloaded for let-expressions as $e\uparrow = \{e\}\uparrow$ and $e\downarrow = \{e\}\downarrow$.

Proof

1. Easily by induction on the structure of t.
2. Straightforward by induction on the structure of e. In the case of let expressions, the proof uses $|e| \in CTerm_\perp$ and Proposition 4 in order to apply the CRWL$_{let}$ rule (Let).
3. By Lemma 28 we have that $[\![e]\!] \Subset \lambda\theta.(|e\theta|\uparrow)\downarrow$. By definition of hyperinclusion —Definition 8— we know that $[\![e]\!]\epsilon \subseteq (\lambda\theta.(|e\theta|\uparrow)\downarrow)\epsilon$, so $[\![e]\!]\epsilon = [\![e\epsilon]\!] \equiv [\![e]\!] \subseteq (|e|\uparrow)\downarrow \equiv (|e\epsilon|\uparrow)\downarrow = (\lambda\theta.(|e\theta|\uparrow)\downarrow)\epsilon$.

   □

Proposition 3 (Polarity of CRWL$_{let}$)
For any program $e, e' \in LExp_\perp$, $t, t' \in CTerm_\perp$, if $e \sqsubseteq e'$ and $t' \sqsubseteq t$ then $e \rightarrowtriangle t$ implies $e' \rightarrowtriangle t'$ with a proof of the same size or smaller—where the size of a CRWL$_{let}$-proof is measured as the number of rules of the calculus used in the proof.

Proof
By induction on the size of the CRWL-derivation. All the cases are straightforward except the (Let) rule:

(Let) We have the derivation:

$$\frac{e_1 \to t_1 \quad e_2[X/t_1] \to t}{e \equiv let\ X = e_1\ in\ e_2 \to t}\ (Let)$$

Since $e \sqsubseteq e'$ then $e' \equiv let\ X = e'_1\ in\ e'_2$ with $e_1 \sqsubseteq e'_1$ and $e_2 \sqsubseteq e'_2$. As $e_1 \sqsubseteq e'_1$ and $t_1 \sqsubseteq t_1$ —because $\sqsubseteq$ is reflexive— then by IH we have $e'_1 \to t_1$. We know that $e_2 \sqsubseteq e'_2$ so by Lemma 24 we have $e_2[X/t_1] \sqsubseteq e'_2[X/t_1]$ and by IH $\mathcal{P} \vdash_{CRWL_{let}} e'_2[X/t_1] \to t'$ such that $t' \sqsubseteq t$. Therefore:

$$\frac{e'_1 \to t_1 \quad e'_2[X/t_1] \to t'}{e' \equiv let\ X = e'_1\ in\ e'_2 \to t'}\ (Let)$$

□

Proposition 4 (Closedness under c-substitutions)
For any $e \in LExp_\perp$, $t \in CTerm_\perp$, $\theta \in CSubst_\perp$, $t \in [\![e]\!]$ implies $t\theta \in [\![e\theta]\!]$.

Proof
By induction on the size of the CRWL$_{let}$-proof. All the cases are straightforward except the (Let) rule:

(Let) In this case the expression is $e \equiv let\ X = e_1\ in\ e_2$ so we have a derivation

$$\frac{e_1 \to t_1 \quad e_2[X/t_1] \to t}{let\ X = e_1\ in\ e_2 \to t}\ (Let)$$

By IH we have that $e_1\theta \to t_1\theta$ and $(e_2[X/t_1])\theta \to t\theta$. By the variable convention we assume that $X \notin dom(\theta) \cup vran(\theta)$, so by Lemma 1 $e_2[X/t_1]\theta \equiv e_2\theta[X/t_1\theta]$ and $e_2\theta[X/t_1\theta] \to t\theta$. Then we can construct the proof:

$$\frac{e_1\theta \to t_1\theta \quad e_2\theta[X/t_1\theta] \to t\theta}{let\ X = e_1\theta\ in\ e_2\theta \to t\theta}\ (Let)$$

□

Theorem 5 (Weak Compositionality of CRWL$_{let}$)
For any $\mathcal{C} \in Cntxt$, $e \in LExp_\perp$

$$[\![\mathcal{C}[e]]\!] = \bigcup_{t \in [\![e]\!]} [\![\mathcal{C}[t]]\!] \qquad \text{if } BV(\mathcal{C}) \cap FV(e) = \emptyset$$

As a consequence, $[\![let\ X = e_1\ in\ e_2]\!] = \bigcup_{t_1 \in [\![e_1]\!]} [\![e_2[X/t_1]]\!]$.

Proof
We prove that $\mathcal{C}[e] \to t \Leftrightarrow \exists s \in CTerm_\perp$ such that $e \to s$ and $\mathcal{C}[s] \to t$.

$\Rightarrow$) By induction on the size of the proof for $\mathcal{C}[e] \to t$. The proof proceeds in a similar way to the proof for Theorem 1, page 4, so we only have to prove the (Let) case:

(Let) There are two cases depending on the context $\mathcal{C}$ (since $\mathcal{C} \neq [\ ]$):

- $\mathcal{C} \equiv let\ X = C'\ in\ e_2$) Straightforward.

- $\mathcal{C} \equiv let\ X = e_1\ in\ \mathcal{C}'$) The proof is

$$\frac{e_1 \twoheadrightarrow t_1 \quad \mathcal{C}'[e][X/t_1] \twoheadrightarrow t}{\mathcal{C}[e] \equiv let\ X = e_1\ in\ \mathcal{C}'[e] \twoheadrightarrow t}\ (Let)$$

  We assume that $X \notin var(t_1)$ by the variable convention, since X is bound in $\mathcal{C}$ and we can rename it freely. Moreover, we assume also that $X \notin BV(\mathcal{C}')$ because X is bound in $\mathcal{C}$, so we could rename the bound occurrences in $\mathcal{C}'$. Therefore $(dom([X/t_1] \cup vran([X/t_1])) \cap BV(\mathcal{C}') = \emptyset$ and $\mathcal{C}'[e][X/t_1] \equiv (\mathcal{C}'[X/t_1])[e[X/t_1]]$ by Lemma 25. Since $BV(\mathcal{C}) \cap FV(e) = \emptyset$ by the premise and $X \in BV(\mathcal{C})$ then $X \notin FV(e)$, so $(\mathcal{C}'[X/t_1])[e[X/t_1]] \equiv \mathcal{C}'[X/t_1][e]$. Then by IH $\exists s \in CTerm_\perp$ such that $e \twoheadrightarrow s$ and $\mathcal{C}'[X/t_1][s] \twoheadrightarrow t$. Therefore we can build:

$$\frac{e_1 \twoheadrightarrow t_1 \quad \mathcal{C}'[s][X/t_1] \equiv^{(*)} \mathcal{C}'[X/t_1][s] \twoheadrightarrow t}{\mathcal{C}[s] \equiv let\ X = e_1\ in\ \mathcal{C}'[s] \twoheadrightarrow t}\ (Let)$$

  (*) Using Lemma 25 as above and the assumption that $X \notin var(s)$ by the variable convention, since X is bound in $\mathcal{C}$ and we can rename it freely.

$\Leftarrow$) By induction on the size of the proof for $\mathcal{C}[s] \twoheadrightarrow t$. As before, the proof proceeds in a similar way to the proof for Theorem 1, page 4, so we only have to prove the (Let) case:

(Let) If we use (Let) then there are two cases depending on the context $\mathcal{C}$ (since $\mathcal{C} \neq [\ ]$):

- $\mathcal{C} = let\ X = \mathcal{C}'\ in\ e_2$) Straighforward.
- $\mathcal{C} = let\ X = e_1\ in\ \mathcal{C}'$) then we have $e \twoheadrightarrow s$ and

$$\frac{e_1 \twoheadrightarrow t_1 \quad \mathcal{C}'[s][X/t_1] \twoheadrightarrow t}{\mathcal{C}[s] \equiv let\ X = e_1\ in\ \mathcal{C}'[s] \twoheadrightarrow t}\ (Let)$$

  By the same reasoning as in the second case of the (Let) rule of the $\Rightarrow$) part of this theorem, $\mathcal{C}'[s][X/t_1] \equiv \mathcal{C}'[X/t_1][s]$. Then by IH $\mathcal{C}'[X/t_1][e] \twoheadrightarrow t$. Again by the same reasoning we have $\mathcal{C}'[e][X/t_1] \equiv \mathcal{C}'[X/t_1][e]$, so we can build the proof:

$$\frac{e_1 \twoheadrightarrow t_1 \quad \mathcal{C}'[e][X/t_1] \equiv \mathcal{C}'[X/t_1][e] \twoheadrightarrow t}{\mathcal{C}[e] \equiv let\ X = e_1\ in\ \mathcal{C}'[e] \twoheadrightarrow t}\ (Let)$$

This ends the proof of the main part of the theorem. With respect to the consequence $[\![let\ X = e_1\ in\ e_2]\!]_{CRWL_{let}} = \bigcup_{t_1 \in [\![e_1]\!]_{CRWL_{let}}} [\![e_2[X/t_1]]\!]_{CRWL_{let}}$ we have:

$$\begin{array}{lll}
& [\![let\ X = e_1\ in\ e_2]\!]_{CRWL_{let}} & \\
& = [\![(let\ X = [\ ]\ in\ e_2)[e_1]]\!]_{CRWL_{let}} & \\
& = \bigcup\limits_{t_1 \in [\![e_1]\!]_{CRWL_{let}}} [\![let\ X = t_1\ in\ e_2]\!]_{CRWL_{let}} & \text{by Theorem 5} \\
& = \bigcup\limits_{t_1 \in [\![e_1]\!]_{CRWL_{let}}} [\![e_2[X/t_1]]\!]_{CRWL_{let}} & \text{by Proposition 8}
\end{array}$$

In the last step we replace $let\ X = t_1\ in\ e_2$ by $e_2[X/t_1]$ which is a (Bind) step of $\to^{lnf}$, so by Proposition 8 it preserves the denotation. $\square$

For Proposition 5, in this Appendix we prove a generalization of the statement appearing in Section 4.2 (page 21). However, it is easy to check that Proposition 5 in Section 4.2 follows easily from points *2* and *3* here.

Proposition 5 (Monotonicity for substitutions of CRWL$_{let}$)
For any program $e \in LExp_\perp$, $t \in CTerm_\perp$, $\sigma, \sigma' \in LSubst_\perp$

1. If $\forall X \in \mathcal{V}, s \in CTerm_\perp$ given $\sigma(X) \rightarrowtriangle s$ with size K we also have $\sigma'(X) \rightarrowtriangle s$ with size $K' \leq K$, then $e\sigma \rightarrowtriangle t$ with size L implies $e\sigma' \rightarrowtriangle t$ with size $L' \leq L$.
2. If $\sigma \sqsubseteq \sigma'$ then $e\sigma \rightarrowtriangle t$ implies $e\sigma' \rightarrowtriangle t$ with a proof of the same size or smaller.
3. If $\sigma \trianglelefteq \sigma'$ then $[\![e\sigma]\!] \subseteq [\![e\sigma']\!]$.

Proof

1. If $e \equiv X \in \mathcal{V}$, assume $X\sigma \rightarrowtriangle t$, then $X\sigma' \rightarrowtriangle t$ with a proof of the same size or smaller, by hypothesis. Otherwise we proceed by induction on the structure of the proof $e\sigma \rightarrowtriangle t$.

   Base cases

   (B) Then $t \equiv \perp$ and $e\sigma' \rightarrowtriangle \perp$ with a proof of size 1 just applying rule (B).

   (RR) Then $e \in \mathcal{V}$ and we are in the previous case.

   (DC) Then $e \equiv c \in CS^0$, as $e \notin \mathcal{V}$, hence $e\sigma \equiv c \equiv e\sigma'$ and every proof for $e\sigma \rightarrowtriangle t$ is a proof for $e\sigma' \rightarrowtriangle t$.

   Inductive steps

   (DC) Then $e \equiv c(e_1, \ldots, e_n)$, as $e \notin \mathcal{V}$, and we have:

$$\frac{e_1\sigma \rightarrowtriangle t_1 \quad \ldots \quad e_n\sigma \rightarrowtriangle t_n}{e\sigma \equiv c(e_1\sigma, \ldots, e_n\sigma) \rightarrowtriangle c(t_1, \ldots, t_n) \equiv t}\ (DC)$$

   By IH or the proof of the other cases $\forall i \in \{1, \ldots, n\}$ we have $e_i\sigma' \rightarrowtriangle t_i$ with a proof of the same size or smaller, so we can built a proof for $e\sigma' \equiv c(e_1\sigma', \ldots, e_n\sigma') \rightarrowtriangle c(t_1, \ldots, t_n) \equiv t$ using (DC), with a size equal or smaller than the size of the starting proof.

   (OR) Similar to the previous case.

   (Let) Then $e \equiv let\ X = e_1\ in\ e_2$, as $e \notin \mathcal{V}$, and we have:

$$\frac{e_1\sigma \rightarrowtriangle t_1 \quad e_2\sigma[X/t_1] \rightarrowtriangle t}{let\ X = e_1\sigma\ in\ e_2\sigma \rightarrowtriangle t}\ (Let)$$

By IH we have $e_1\sigma \rightarrowtriangle t_1$. By the variable convention we assume that $X \notin dom(\sigma) \cup vran(\sigma)$ and $X \notin dom(\sigma') \cup vran(\sigma')$. Then it is easy to check that $\forall Y \in \mathcal{V}, s, t \in CTerm_\perp$, given $Y(\sigma[X/t]) \rightarrowtriangle s$ with size K we also have $Y(\sigma'[X/t]) \rightarrowtriangle s$ with size $K' \leq K$. Then by IH we have $e_2\sigma'[X/t_1] \rightarrowtriangle t$. Therefore we can construct a proof with a size equal or smaller than the starting one:

$$\frac{e_1\sigma' \rightarrowtriangle t_1 \quad e_2\sigma'[X/t_1] \rightarrowtriangle t}{let\ X = e_1\sigma'\ in\ e_2\sigma' \rightarrowtriangle t}\ (Let)$$

2. By induction on the size of the CRWL$_{let}$-proof. The cases for classical CRWL appear in (Vado-Vírseda 2002), so we only have to prove the case for the (Let) rule:

(Let) In this case the expression is $e \equiv let\ X = e_1\ in\ e_2$ so we have a proof

$$\frac{e_1\sigma \rightarrowtriangle t_1 \quad e_2\sigma[X/t_1] \rightarrowtriangle t}{let\ X = e_1\sigma\ in\ e_2\sigma \rightarrowtriangle t}\ (Let)$$

By IH we have that $e_1\sigma \rightarrowtriangle t_1$. By the variable convention we can assume that $BV(e) \cap (dom(\sigma) \cup vran(\sigma)) = \emptyset$ and $BV(e) \cap (dom(\sigma') \cup vran(\sigma')) = \emptyset$. With the previous properties it is easy to see that $\sigma[X/t_1] \sqsubseteq \sigma'[X/t_1]$, so by IH $e_2\sigma'[X/t_1] \rightarrowtriangle t$. Therefore we can build the proof:

$$\frac{e_1\sigma' \rightarrowtriangle t_1 \quad e_2\sigma'[X/t_1] \rightarrowtriangle t}{let\ X = e_1\sigma'\ in\ e_2\sigma' \rightarrowtriangle t}\ (Let)$$

3. By induction on the structure of e:

$e \equiv X \in \mathcal{V}$ **-** In this case $[\![X\sigma]\!]_{CRWL_{let}} \subseteq [\![X\sigma']\!]_{CRWL_{let}}$ because by the hypothesis $\sigma \trianglelefteq \sigma'$.

$e \equiv h(e_1, \ldots, e_n)$ **-** Applying Theorem 5 with $\mathcal{C} \equiv h([\,], e_2\sigma, \ldots, e_n\sigma)$ we have $[\![h(e_1\sigma, \ldots, e_n\sigma)]\!]_{CRWL_{let}} = [\![\mathcal{C}[e_1\sigma]]\!]_{CRWL_{let}} = \bigcup_{t \in [\![e_1\sigma]\!]_{CRWL_{let}}} [\![\mathcal{C}[t]]\!]_{CRWL_{let}}$ because $BV(\mathcal{C}) = \emptyset$. On the other hand, by Theorem 5 we also know that

$$\begin{array}{rcl} [\![h(e_1\sigma', e_2\sigma, \ldots, e_n\sigma)]\!]_{CRWL_{let}} & = & [\![\mathcal{C}[e_1\sigma']]\!]_{CRWL_{let}} \\ & = & \bigcup_{t \in [\![e_1\sigma']\!]_{CRWL_{let}}} [\![\mathcal{C}[t]]\!]_{CRWL_{let}} \end{array}$$

Since by IH we have $[\![e_1\sigma]\!]_{CRWL_{let}} \subseteq [\![e_1\sigma']\!]_{CRWL_{let}}$ it is easy to check that

$$\bigcup_{t \in [\![e_1\sigma]\!]_{CRWL_{let}}} [\![\mathcal{C}[t]]\!]_{CRWL_{let}} \subseteq \bigcup_{t \in [\![e_1\sigma']\!]_{CRWL_{let}}} [\![\mathcal{C}[t]]\!]_{CRWL_{let}}$$

so $[\![h(e_1\sigma, e_2\sigma, \ldots, e_n\sigma)]\!]_{CRWL_{let}} \subseteq [\![h(e_1\sigma', e_2\sigma, \ldots, e_n\sigma)]\!]_{CRWL_{let}}$. Using the same reasoning in the rest of subexpressions $e_i\sigma$ we can prove:
$[\![h(e_1\sigma', e_2\sigma, \ldots, e_n\sigma)]\!]_{CRWL_{let}} \subseteq [\![h(e_1\sigma', e_2\sigma', e_3\sigma \ldots, e_n\sigma)]\!]_{CRWL_{let}}$
$[\![h(e_1\sigma', e_2\sigma', e_3\sigma \ldots, e_n\sigma)]\!]_{CRWL_{let}} \subseteq [\![h(\ldots, e_3\sigma', e_4\sigma \ldots, e_n\sigma)]\!]_{CRWL_{let}}$
$\ldots$
$[\![\ldots, e_{n-1}\sigma', e_n\sigma)]\!]_{CRWL_{let}} \subseteq [\![h(e_1\sigma', \ldots, e_n\sigma')]\!]_{CRWL_{let}}$
Then by transitivity of $\subseteq$ we have:

$[\![h(e_1,\ldots,e_n)\sigma]\!]_{CRWL_{let}} \equiv [\![h(e_1\sigma,\ldots,e_n\sigma)]\!]_{CRWL_{let}} \subseteq$
$[\![h(e_1\sigma',\ldots,e_n\sigma')]\!]_{CRWL_{let}} \equiv [\![h(e_1,\ldots,e_n)\sigma']\!]_{CRWL_{let}}$.

$e \equiv let\ X = e_1\ in\ e_2$ - As Theorem 5 states, $[\![let\ X = e_1\sigma\ in\ e_2\sigma]\!]_{CRWL_{let}} = \bigcup_{t_1\in[\![e_1\sigma]\!]_{CRWL_{let}}} [\![e_2\sigma[X/t_1]]\!]_{CRWL_{let}}$. By the Induction Hypothesis we have that $[\![e_1\sigma]\!]_{CRWL_{let}} \subseteq [\![e_1\sigma']\!]_{CRWL_{let}}$. Due to the variable convention we assume that $X \notin dom(\sigma) \cup vran(\sigma)$ and $X \notin dom(\sigma') \cup vran(\sigma')$, so it is easy to check that $\sigma[X/t] \trianglelefteq \sigma'[X/t]$ for any $t \in CTerm$. Then by the Induction Hypothesis we know that $[\![e_2\sigma[X/t]]\!]_{CRWL_{let}} \subseteq [\![e_2\sigma'[X/t]]\!]_{CRWL_{let}}$. Therefore

$$\begin{array}{rcl}
[\![(let\ X = e_1\ in\ e_2)\sigma]\!]_{CRWL_{let}} & = & \bigcup_{t_1\in[\![e_1\sigma]\!]_{CRWL_{let}}} [\![e_2\sigma[X/t_1]]\!]_{CRWL_{let}} \\
 & \subseteq & \bigcup_{t_1\in[\![e_1\sigma']\!]_{CRWL_{let}}} [\![e_2\sigma'[X/t_1]]\!]_{CRWL_{let}} \\
 & = & [\![let\ X = e_1\sigma'\ in\ e_2\sigma']\!]_{CRWL_{let}} \\
 & = & [\![(let\ X = e_1\ in\ e_2)\sigma']\!]_{CRWL_{let}}
\end{array}$$

□

Theorem 6 (Compositionality of hypersemantics)
For all $\mathcal{C} \in Cntxt$, $e \in LExp_\perp$

$$[\![\![\mathcal{C}[e]]\!]\!] = [\![\![\mathcal{C}]\!]\!][\![\![e]\!]\!]$$

As a consequence: $[\![\![e]\!]\!] = [\![\![e']\!]\!] \Leftrightarrow \forall\mathcal{C} \in Cntxt.[\![\![\mathcal{C}[e]]\!]\!] = [\![\![\mathcal{C}[e']]\!]\!]$.

Proof
By induction over the structure of contexts. The base case is $\mathcal{C} = []$, so $[\![\![\mathcal{C}[e]]\!]\!] = [\![\![e]\!]\!] = [\![\![[]]\!]\!][\![\![e]\!]\!] = [\![\![\mathcal{C}]\!]\!][\![\![e]\!]\!]$, as $[\![\![[]]\!]\!]$ is the identity function by definition. Regarding the inductive step:

- $\mathcal{C} = h(e_1,\ldots,\mathcal{C}',\ldots,e_n)$: Then

$$\begin{array}{ll}
[\![\![\mathcal{C}]\!]\!][\![\![e]\!]\!] = \lambda\theta. \bigcup_{t\in[\![\![\mathcal{C}']\!]\!][\![\![e]\!]\!]\theta} [\![h(e_1\theta,\ldots,t,\ldots,e_n\theta)]\!] & \\
= \lambda\theta. \bigcup_{t\in[\![\![\mathcal{C}'[e]]\!]\!]\theta} [\![h(e_1\theta,\ldots,t,\ldots,e_n\theta)]\!] & \text{by IH} \\
= \lambda\theta. \bigcup_{t\in[\![(\mathcal{C}'[e])\theta]\!]} [\![h(e_1\theta,\ldots,t,\ldots,e_n\theta)]\!] & \text{by definition} \\
= \lambda\theta.[\![h(e_1\theta,\ldots,(\mathcal{C}'[e])\theta,\ldots,e_n\theta)]\!] & \text{by Lemma 5} \\
= \lambda\theta.[\![(\mathcal{C}[e])\theta]\!] = [\![\![\mathcal{C}[e]]\!]\!] &
\end{array}$$

- $\mathcal{C} = let\ X = \mathcal{C}'\ in\ s$: Then

$$\begin{array}{ll}
[\![\![\mathcal{C}]\!]\!][\![\![e]\!]\!] = \lambda\theta. \bigcup_{t\in[\![\![\mathcal{C}']\!]\!][\![\![e]\!]\!]\theta} [\![let\ X = t\ in\ s\theta]\!] & \text{by definition} \\
= \lambda\theta. \bigcup_{t\in[\![\![\mathcal{C}']\!]\!][\![\![e]\!]\!]\theta} [\![s\theta[X/t]]\!] & \text{by rule (Bind)}^{(*)} \\
= \lambda\theta. \bigcup_{t\in[\![\![\mathcal{C}'[e]]\!]\!]\theta} [\![s\theta[X/t]]\!] & \text{by IH} \\
= \lambda\theta. \bigcup_{t\in[\![(\mathcal{C}'[e])\theta]\!]} [\![s\theta[X/t]]\!] & \text{by definition} \\
= \lambda\theta.[\![let\ X = (\mathcal{C}'[e])\theta\ in\ s\theta]\!] & \text{by Lemma 5} \\
= [\![\![\mathcal{C}[e]]\!]\!] &
\end{array}$$

(*): by Proposition 8 $[\![let\ X = t\ in\ s\theta]\!] = [\![s\theta[X/t]]\!]$ since $let\ X = t\ in\ s\theta \to^{lnf} s\theta[X/t]$.

- $\mathcal{C} = let\ X = s\ in\ \mathcal{C}'$: Then

$$\begin{aligned}
[\![\mathcal{C}]\!][\![e]\!] &= \lambda\theta. \bigcup_{t\in[\![s]\!]\theta} [\![\mathcal{C}']\!][\![e]\!](\theta[X/t]) \\
&= \lambda\theta. \bigcup_{t\in[\![s]\!]\theta} [\![\mathcal{C}'[e]]\!](\theta[X/t]) && \text{by IH} \\
&= \lambda\theta. \bigcup_{t\in[\![s]\!]\theta} [\![(\mathcal{C}'[e])(\theta[X/t])]\!] && \text{by definition} \\
&= \lambda\theta. \bigcup_{t\in[\![s\theta]\!]} [\![(\mathcal{C}'[e])(\theta[X/t])]\!] && \text{by definition} \\
&= \lambda\theta. \bigcup_{t\in[\![s\theta]\!]} [\![((\mathcal{C}'[e])\theta)[X/t]]\!] \\
&= \lambda\theta.[\![let\ X = s\theta\ in\ (\mathcal{C}'[e])\theta]\!] && \text{by Lemma 5} \\
&= [\![\mathcal{C}[e]]\!]
\end{aligned}$$

□

Proposition 6
Consider two sets A, B, and let $\mathcal{F}$ be the set of functions $A \to \mathcal{P}(B)$. Then:

i) $\Subset$ is indeed a partial order on $\mathcal{F}$, and Δf is indeed a decomposition of $f \in \mathcal{F}$, i.e., $\biguplus\ (\Delta f) = f$.

ii) Monotonicity of hyperunion wrt. inclusion: for any $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathcal{F}$

$$\mathcal{I}_1 \subseteq \mathcal{I}_2 \text{ implies } \biguplus\ \mathcal{I}_1 \Subset \biguplus\ \mathcal{I}_2$$

iii) Distribution of unions: for any $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathcal{F}$

$$\biguplus\ (\mathcal{I}_1 \cup \mathcal{I}_2) = (\biguplus\ \mathcal{I}_1) \uplus (\biguplus\ \mathcal{I}_2)$$

iv) Monotonicity of decomposition wrt. hyperinclusion: for any $f_1, f_2 \in \mathcal{F}$

$$f_1 \Subset f_2 \text{ implies } \Delta f_1 \subseteq \Delta f_2$$

Proof

i) The binary relation $\Subset$ is a partial order on $\mathcal{F}$ because:
   - It is reflexive, as for any function f and any $x \in A$ we have that $f(x) = f(x)$, and thus $f(x) \subseteq f(x)$, therefore $f \Subset f$.
   - It is transitive because given some functions f_1, f_2, f_3 such that $f_1 \Subset f_2$ and $f_2 \Subset f_3$, then for any $x \in A$ we have $f_1(x) \subseteq f_2(x) \subseteq f_3(x)$ by definition of $\Subset$, hence $f_1 \Subset f_3$.
   - It is antisymmetric wrt. extensional function equality, because for any pair of hypersemantics f_1, f_2 such that $f_1 \Subset f_2$ and $f_2 \Subset f_1$ and any $x \in A$ we have that $f_1(x) \subseteq f_2(x)$ and $f_2(x) \subseteq f_1(x)$ by definition of $\Subset$, hence $f_1(x) = f_2(x)$ by antisymmetry of $\subseteq$ and $f_1 = f_2$.

   In order to prove that Δf is indeed a decomposition of $f \in \mathcal{F}$ we first perform a little massaging by using the definitions of $\biguplus$ and Δ.

$$\biguplus\ (\Delta f) = \biguplus\ \{\hat{\lambda}a.\{b\} \mid a \in A, b \in f(a)\} = \lambda x \in A. \bigcup_{a\in A} \bigcup_{b\in f(a)} (\hat{\lambda}a.\{b\})x$$

Now we will use the fact that $\Subset$ is a partial order, and therefore it is anti-symmetric, so mutual inclusion by $\Subset$ implies equality.

- $\underline{f \Subset \biguplus\ (\Delta f)}$: Given arbitraries $a \in A$, $b \in f(a)$ then

$$\begin{aligned}
&(\biguplus\ (\Delta f))a = \bigcup_{x \in A} \bigcup_{y \in f(x)} (\hat{\lambda} x.\{y\})a \\
&\supseteq \bigcup_{y \in f(a)} (\hat{\lambda} a.\{y\})a && \text{as } a \in A \\
&= \bigcup_{y \in f(a)} \{y\} \ni b && \text{as } b \in f(a)
\end{aligned}$$

- $\underline{\biguplus\ (\Delta f) \Subset f}$: Given arbitraries $a \in A$, $b \in (\biguplus\ (\Delta f))a$ then we have that $b \in \bigcup_{x \in A} \bigcup_{y \in f(x)} (\hat{\lambda} x.\{y\})a$, therefore $\exists x \in A, y \in f(x)$ such that $b \in (\hat{\lambda} x.\{y\})a$. But then $a \equiv x$ —otherwise $(\hat{\lambda} x.\{y\})a = \emptyset$— and $y \equiv b$ —because $b \in (\hat{\lambda} x.\{y\})a = \{y\}$—, and so $y \in f(x)$ implies $b \in f(a)$.

ii) Given an arbitrary $a \in A$ then

$$\begin{aligned}
&(\biguplus\ \mathcal{I}_1)a = \bigcup_{f \in \mathcal{I}_1} f(a) && \text{by definition of } \biguplus \\
&\subseteq \bigcup_{f \in \mathcal{I}_2} f(a) && \text{as } \mathcal{I}_1 \subseteq \mathcal{I}_2 \\
&= (\biguplus\ \mathcal{I}_2)a && \text{by definition of } \biguplus
\end{aligned}$$

iii)

$$\begin{aligned}
&\biguplus\ (\mathcal{I}_1 \cup \mathcal{I}_2) = \lambda a. \bigcup_{f \in (\mathcal{I}_1 \cup \mathcal{I}_2)} f(a) && \text{by definition of } \biguplus \\
&= \lambda a. \bigcup_{f \in \mathcal{I}_1} f(a) \ \cup \bigcup_{f \in \mathcal{I}_2} f(a) \\
&= \lambda a.(\biguplus\ \mathcal{I}_1)a \cup (\biguplus\ \mathcal{I}_2)a && \text{by definition of } \biguplus \\
&= (\biguplus\ \mathcal{I}_1) \uplus (\biguplus\ \mathcal{I}_2) && \text{by definition of } \uplus
\end{aligned}$$

iv) Suppose an arbitrary $\hat{\lambda} a.\{b\} \in \Delta f_1$ with $a \in A$ and $b \in f_1(a)$ by definition. Since $f_1 \Subset f_2$ then $f_1(a) \subseteq f_2(a)$. Therefore $b \in f_2(a)$ and $\hat{\lambda} a.\{b\} \in \Delta f_2$.

□

Proposition 7 (Distributivity under context of hypersemantics union)

$$[\![\mathcal{C}]\!](\biguplus\ H) = \biguplus_{\varphi \in H} [\![\mathcal{C}]\!]\varphi$$

Proof

We proceed by induction on the structure of $\mathcal{C}$. Regarding the base case, then $\mathcal{C} = []$ and so:

$$\begin{aligned}
&[\![\mathcal{C}]\!](\biguplus\ H) = \biguplus\ H && \text{by definition of } [\![\mathcal{C}]\!] \\
&= \biguplus_{\varphi\, \in\, H} \varphi \\
&= \biguplus_{\varphi\, \in\, H} [\![\mathcal{C}]\!]\varphi && \text{by definition of } [\![\mathcal{C}]\!]
\end{aligned}$$

For the inductive step we have several possibilities.

- $\mathcal{C} \equiv h(e_1, \ldots, \mathcal{C}', \ldots, e_n)$: then

$$
\begin{aligned}
&[\![\mathcal{C}]\!](\Cup H) = \lambda\theta. \bigcup_{t \in [\![\mathcal{C}']\!](\Cup H)\theta} [\![h(e_1\theta, \ldots, t, \ldots, e_n\theta)]\!] && \text{by definition of } [\![\mathcal{C}]\!] \\
&= \lambda\theta. \bigcup_{t \in ((\Cup \{[\![\mathcal{C}']\!]\varphi \mid \varphi \in H\})\theta)} [\![h(e_1\theta, \ldots, t, \ldots, e_n\theta)]\!] && \text{by IH} \\
&= \lambda\theta. \bigcup_{t \in (\bigcup_{\varphi \in H} [\![\mathcal{C}']\!]\varphi\theta)} [\![h(e_1\theta, \ldots, t, \ldots, e_n\theta)]\!] && \text{by definition of } \Cup \\
&= \lambda\theta. \bigcup_{\varphi \in H} \bigcup_{t \in [\![\mathcal{C}']\!]\varphi\theta} [\![h(e_1\theta, \ldots, t, \ldots, e_n\theta)]\!] && \\
&= \lambda\theta. \bigcup_{\varphi \in H} [\![\mathcal{C}]\!]\varphi\theta && \text{by definition of } [\![\mathcal{C}]\!] \\
&= \Cup_{\varphi \,\in\, H} [\![\mathcal{C}]\!]\varphi && \text{by definition of } \Cup
\end{aligned}
$$

- $\mathcal{C} \equiv let\ X = \mathcal{C}'\ in\ e$: then

$$
\begin{aligned}
&[\![\mathcal{C}]\!](\Cup H) = \lambda\theta. \bigcup_{t \in [\![\mathcal{C}']\!](\Cup H)\theta} [\![let\ X = t\ in\ e\theta]\!] && \text{by definition of } [\![\mathcal{C}]\!] \\
&= \lambda\theta. \bigcup_{t \in ((\Cup \{[\![\mathcal{C}']\!]\varphi \mid \varphi \in H\})\theta)} [\![let\ X = t\ in\ e\theta]\!] && \text{by IH} \\
&= \lambda\theta. \bigcup_{t \in (\bigcup_{\varphi \in H} [\![\mathcal{C}']\!]\varphi\theta)} [\![let\ X = t\ in\ e\theta]\!] && \text{by definition of } \Cup \\
&= \lambda\theta. \bigcup_{\varphi \in H} \bigcup_{t \in [\![\mathcal{C}']\!]\varphi\theta} [\![let\ X = t\ in\ e\theta]\!] && \\
&= \lambda\theta. \bigcup_{\varphi \in H} [\![\mathcal{C}]\!]\varphi\theta && \text{by definition of } [\![\mathcal{C}]\!] \\
&= \Cup_{\varphi \,\in\, H} [\![\mathcal{C}]\!]\varphi && \text{by definition of } \Cup
\end{aligned}
$$

- $\mathcal{C} \equiv let\ X = e\ in\ \mathcal{C}'$: then

$$
\begin{aligned}
&[\![\mathcal{C}]\!](\Cup H) = \lambda\theta. \bigcup_{t \in [\![e]\!]\theta} [\![\mathcal{C}']\!](\Cup H)(\theta[X/t]) && \text{by definition of } [\![\mathcal{C}]\!] \\
&= \lambda\theta. \bigcup_{t \in [\![e]\!]\theta} (\Cup \{[\![\mathcal{C}']\!]\varphi \mid \varphi \in H\})(\theta[X/t]) && \text{by IH} \\
&= \lambda\theta. \bigcup_{t \in [\![e]\!]\theta} \bigcup_{\varphi \in H} [\![\mathcal{C}']\!]\varphi(\theta[X/t]) && \text{by definition of } \Cup \\
&= \lambda\theta. \bigcup_{\varphi \in H} \bigcup_{t \in [\![e]\!]\theta} [\![\mathcal{C}']\!]\varphi(\theta[X/t]) && \text{as } H \text{ is independent from } t \\
&= \lambda\theta. \bigcup_{\varphi \in H} [\![\mathcal{C}]\!]\varphi\theta && \text{by definition of } [\![\mathcal{C}]\!] \\
&= \Cup_{\varphi \,\in\, H} [\![\mathcal{C}]\!]\varphi && \text{by definition of } \Cup
\end{aligned}
$$

□

A.7 Proofs for Section 4.3

Theorem 9 (Hyper-Soundness of let-rewriting)
For all $e, e' \in LExp$, if $e \to^{l^*} e'$ then $[\![e']\!] \Subset [\![e]\!]$.

Proof
We first prove the theorem for a single step of $\to^l$. We proceed assumming some

$\theta \in CSubst_\perp$ such that $e'\theta \rightarrowtriangle t$ and then proving $e\theta \rightarrowtriangle t$. The case where $t \equiv \perp$ holds trivially using the rule **B**, so we will prove the rest by a case distinction on the rule of the let-rewriting calculus applied:

(Fapp) Assume $f(t_1, \ldots, t_n) \to^l r$ with $(f(p_1, \ldots, p_n) \to e) \in \mathcal{P}$, $\sigma \in CSubst$, such that $\forall i.p_i\sigma \equiv t_i$ and $e\sigma \equiv r$, and $\theta \in CSubts_\perp$ such that $r\theta \rightarrowtriangle t$. Then as $\sigma\theta \in CSubts_\perp, \forall i.p_i\sigma\theta \equiv t_i\theta$ and $e\sigma\theta \equiv r\theta$ we can use the (OR) rule to build the following proof:

$$\dfrac{\dfrac{\text{Lemma 18}}{t_1\theta \rightarrowtriangle t_1\theta} \quad \ldots \quad \dfrac{\text{Lemma 18}}{t_n\theta \rightarrowtriangle t_n\theta} \quad r\theta \rightarrowtriangle t}{f(t_1\theta, \ldots, t_n\theta) \rightarrowtriangle t}\ (OR)$$

(LetIn) Assume $h(\ldots, e, \ldots) \to^l let\ X = e\ in\ h(\ldots, X, \ldots)$ by (LetIn) and $\theta \in CSubts_\perp$ such that $(let\ X = e\ in\ h(\ldots, X, \ldots))\theta \rightarrowtriangle t$. This proof must be of the shape of:

$$\dfrac{e\theta \rightarrowtriangle t_1 \qquad h(d_1\theta, \ldots, X\theta, \ldots, d_n\theta)[X/t_1] \rightarrowtriangle t}{let\ X = e\theta\ in\ h(d_1\theta, \ldots, X\theta, \ldots, d_n\theta) \rightarrowtriangle t}\ (Let)$$

for some $d_1, \ldots, d_n \in LExp, t_1 \in CTerm_\perp$. Besides $X \notin (dom(\theta) \cup vran(\theta))$ by the variable convention[5], hence $X\theta \equiv X$ and so $h(d_1\theta, \ldots, X\theta, \ldots, d_n\theta)[X/t_1] \equiv h(d_1\theta, \ldots, t_1, \ldots, d_n\theta)$, as X is fresh by the conditions in (LetIn) and so it does not appear in any d_i. Now we have two possibilities:

a) $h \equiv c \in DC$: Then $h(d_1\theta, \ldots, t_1, \ldots, d_n\theta) \rightarrowtriangle t$ must proved by (DC):

$$\dfrac{d_1\theta \rightarrowtriangle s_1 \ \ldots\ t_1 \rightarrowtriangle t'_1 \ \ldots\ d_n\theta \rightarrowtriangle s_n}{c(d_1\theta, \ldots, t_1, \ldots, d_n\theta) \rightarrowtriangle c(s_1, \ldots, t'_1, \ldots, s_n) \equiv t}\ (DC)$$

for some $s_1, \ldots, s_n, t'_1 \in CTerm_\perp$. Then $t_1 \rightarrowtriangle t'_1$ implies $t'_1 \sqsubseteq t_1$ by Lemma 5, hence $e\theta \rightarrowtriangle t_1$ implies $e\theta \rightarrowtriangle t'_1$ by Proposition 3, and we can build the following proof:

$$\dfrac{d_1\theta \rightarrowtriangle s_1 \ \ldots\ e\theta \rightarrowtriangle t'_1 \ \ldots\ d_n\theta \rightarrowtriangle s_n}{h(\ldots, e, \ldots)\theta \equiv c(d_1\theta, \ldots, e\theta, \ldots, d_n\theta) \rightarrowtriangle c(s_1, \ldots, t'_1, \ldots, s_n) \equiv t}$$

b) $h \equiv f \in FS$: Then $h(d_1\theta, \ldots, t_1, \ldots, d_n\theta) \rightarrowtriangle t$ must be proved by (OR):

$$\dfrac{d_1\theta \rightarrowtriangle s_1\sigma \quad \ldots \quad t_1 \rightarrowtriangle t'_1\sigma \quad \ldots \quad d_n\theta \rightarrowtriangle s_n\sigma \quad r\sigma \rightarrowtriangle t}{f(d_1\theta, \ldots, t_1, \ldots, d_n\theta) \rightarrowtriangle t}\ (OR)$$

for some $s_1\sigma, \ldots, s_n\sigma, t'_1\sigma \in CTerm_\perp$, $(f(s_1, \ldots, t'_1, \ldots s_n) \to r) \in \mathcal{P}$, $\sigma \in CSubst_\perp$. Then we can prove $e\theta \rightarrowtriangle t'_1\sigma$ like in the previous case, to build the following proof:

$$\dfrac{d_1\theta \rightarrowtriangle s_1\sigma \quad \ldots \quad e\theta \rightarrowtriangle t'_1\sigma \quad \ldots \quad d_n\theta \rightarrowtriangle s_n\sigma \quad r\sigma \rightarrowtriangle t}{h(\ldots, e, \ldots)\theta \equiv f(d_1\theta, \ldots, e\theta, \ldots, d_n\theta) \rightarrowtriangle t}\ (OR)$$

[5] Actually, to prove this theorem properly, we cannot restrict the substitution to fulfill these restrictions, so in fact we rename the bound variables in an α-conversion fashion and use the equivalence $e[X/e'] \equiv e[X/Y][Y/e']$ (with Y the new bound variable), to use the hypothesis. This will be done implicitly when needed during the remaining of the proof.

(Bind) Assume $let\ X = t_1\ in\ e \to^l e[X/t_1]$ by (Bind) and $\theta \in CSubst_\perp$ such that $(e[X/t_1])\theta \rightarrowtriangle t$. Then $X \notin (dom(\theta) \cup vran(\theta))$ by the variable convention, so we can apply Lemma 1 (Substitution lemma) to get $e\theta[X/t_1\theta] \equiv (e[X/t_1])\theta$. Besides $t_1 \in CTerm$ and $\theta \in CSubst_\perp$ by hypothesis, hence $t_1\theta \in CTerm_\perp$ and we can build the following proof:

$$\dfrac{\dfrac{\text{Lemma 18}}{t_1\theta \rightarrowtriangle t_1\theta} \qquad e\theta[X/t_1\theta] \equiv (e[X/t_1])\theta \rightarrowtriangle t}{let\ X = t_1\theta\ in\ e\theta \rightarrowtriangle t}\ (Let)$$

(Elim) Assume $let\ X = e_1\ in\ e_2 \to^l e_2$ by (Elim) and $\theta \in CSubts_\perp$ such that $e_2\theta \rightarrowtriangle t$. Then $X \notin vran(\theta)$ by the variable convention and $X \notin FV(e_2)$ by the condition of (Elim), hence $e_2\theta[X/\perp] \equiv e_2\theta$ and we can build the following proof:

$$\dfrac{\dfrac{}{e_1\theta \rightarrowtriangle \perp}\ (B) \qquad e_2\theta[X/\perp] \equiv e_2\theta \rightarrowtriangle t}{let\ X = e_1\theta\ in\ e_2\theta \rightarrowtriangle t}\ (Let)$$

(Flat) Assume $let\ X = (let\ Y = e_1\ in\ e_2)\ in\ e_3 \to^l let\ Y = e_1\ in\ (let\ X = e_2\ in\ e_3)$ by (Flat) and $\theta \in CSubts_\perp$ such that $(let\ Y = e_1\ in\ (let\ X = e_2\ in\ e_3))\theta \rightarrowtriangle t$. This proof must be must be of the shape of:

$$\dfrac{e_1\theta \rightarrowtriangle t_1 \qquad \dfrac{e_2\theta[Y/t_1] \rightarrowtriangle t_2 \quad e_3\theta[Y/t_1][X/t_2] \rightarrowtriangle t}{(let\ X = e_2\theta\ in\ e_3\theta)[Y/t_1] \rightarrowtriangle t}\ (Let)}{let\ Y = e_1\theta\ in\ (let\ X = e_2\theta\ in\ e_3\theta) \rightarrowtriangle t}\ (Let)$$

for some $t_1, t_2 \in CTerm_\perp$. Besides $Y \notin vran(\theta)$ by the variable convention and $Y \notin FV(e_3)$ by the condition of (Flat), hence $e_3\theta[Y/t_1] \equiv e_3\theta$ and we can build the following proof:

$$\dfrac{\dfrac{\dfrac{Hypothesis}{e_1\theta \rightarrowtriangle t_1} \quad \dfrac{Hypothesis}{e_2\theta[Y/t_1] \rightarrowtriangle t_2}}{let\ Y = e_1\theta\ in\ e_2\theta \rightarrowtriangle t_2}\ (Let) \qquad e_3\theta[X/t_2] \equiv e_3\theta[Y/t_1][X/t_2] \rightarrowtriangle t}{let\ X = (let\ Y = e_1\theta\ in\ e_2\theta)\ in\ e_3\theta \rightarrowtriangle t}\ (Let)$$

(Contx) By the proof of the other cases, $[\![e']\!] \Subset [\![e]\!]$, but then $[\![\mathcal{C}[e']]\!] \Subset [\![\mathcal{C}[e]]\!]$ by Lemma 7, and we are done.

The proof for several steps is a trivial induction on the length of the derivation $e \to^{l^*} e'$. $\square$

Proposition 8 (The $\to^{lnf}$ relation preserves hyperdenotation)
For all $e, e' \in LExp$, if $e \to^{lnf^*} e'$ then $[\![e]\!] = [\![e']\!]$—and therefore $[\![e]\!] = [\![e']\!]$.

Proof
We first prove the lemma for one step of $\to^{lnf}$ by case distinction over the rule applied to reduce e to e'. By Theorem 9 we already have that $\forall e, e' \in LExp$ if $e \to^{lnf} e'$ then $[\![e']\!] \Subset [\![e]\!]$, so all that is left is proving that $[\![e]\!] \Subset [\![e']\!]$ also, and finally applying the transitivity of $\Subset$, as it is a partial order by Lemma 6-i. We proceed assumming some $\theta \in CSubst_\perp$ such that $e\theta \rightarrowtriangle t$ and then proving $e'\theta \rightarrowtriangle t$. The case where $t \equiv \perp$ holds trivially using the rule (B), so we will prove the other by a case distinction on the rule of the let calculus applied:

(LetIn) Assume $h(d_1, \ldots, e, \ldots, d_n) \to^l let\ X = e\ in\ h(d_1, \ldots, X, \ldots, d_n)$ by the (LetIn) rule and $\theta \in CSubts_\perp$ such that

$$h(d_1, \ldots, e, \ldots, d_n)\theta \equiv h(d_1\theta, \ldots, e\theta, \ldots, d_n\theta) \rightarrowtriangle t$$

Then by the compositionality of Theorem 5 we have that $\exists t_1 \in \llbracket e\theta \rrbracket$ such that $h(d_1\theta, \ldots, t_1, \ldots, d_n\theta) \rightarrowtriangle t$. Besides X is fresh and $X \notin (dom(\theta) \cup vran(\theta))$ by the variable convention, hence

$$(let\ X = e\ in\ h(d_1, \ldots, X, \ldots, d_n))\theta \equiv let\ X = e\theta\ in\ h(d_1\theta, \ldots, X, \ldots, d_n\theta)$$

and

$$h(d_1\theta, \ldots, X, \ldots, d_n\theta)[X/t_1] \equiv h(d_1\theta, \ldots, t_1, \ldots, d_n\theta)$$

and so we can do:

$$\dfrac{\dfrac{hypothesis}{e\theta \rightarrowtriangle t_1} \quad \dfrac{hypothesis}{h(d_1\theta, \ldots, X, \ldots, d_n\theta)[X/t_1] \equiv h(d_1\theta, \ldots, t_1, \ldots, d_n\theta) \rightarrowtriangle t}}{(let\ X = e\ in\ h(d_1, \ldots, X, \ldots, d_n))\theta \equiv let\ X = e\theta\ in\ h(d_1\theta, \ldots, X, \ldots, d_n\theta) \rightarrowtriangle t}\ (Let)$$

(Bind) Assume $let\ X = t_1\ in\ e \to^l e[X/t_1]$ by (Bind) and $\theta \in CSubst_\perp$ such that $(let\ X = t_1\ in\ e)\theta \equiv let\ X = t_1\theta\ in\ e\theta \rightarrowtriangle t$. Then it must be with a proof of the following shape:

$$\dfrac{t_1\theta \rightarrowtriangle t_1' \quad e\theta[X/t_1'] \rightarrowtriangle t}{let\ X = t_1\theta\ in\ e\theta \rightarrowtriangle t}\ (Let)$$

But $\theta \in CSubst_\perp$ and $t_1 \in CTerm$ implies $t_1\theta \in CTerm_\perp$, and so $t_1\theta \rightarrowtriangle t_1'$ implies $t_1' \sqsubseteq t_1\theta$ by Lemma 5-1. Hence $[X/t_1'] \sqsubseteq [X/t_1\theta]$ and so $e\theta[X/t_1'] \rightarrowtriangle t$ implies $e\theta[X/t_1\theta] \rightarrowtriangle t$ by the monoticity of Proposition 5. Besides $X \notin (dom(\theta) \cup vran(\theta))$ by the variable convention, and so we can apply Lemma 1 (substitution lemma) to get $(e[X/t_1])\theta \equiv e\theta[X/t_1\theta]$, so we are done.

(Elim) Assume $let\ X = e_1\ in\ e_2 \to^l e_2$ by (Elim) and $\theta \in CSubts_\perp$ such that $(let\ X = e_1\ in\ e_2)\theta \equiv let\ X = e_1\theta\ in\ e_2\theta \rightarrowtriangle t$. Then it must be with a proof of the following shape:

$$\dfrac{e_1\theta \rightarrowtriangle t_1 \quad e_2\theta[X/t_1] \rightarrowtriangle t}{let\ X = e_1\theta\ in\ e_2\theta \rightarrowtriangle t}\ (Let)$$

Then $X \notin vran(\theta)$ by the variable convention and $X \notin FV(e_2)$ by the condition of (Elim), hence $e_2\theta \equiv e_2\theta[X/t_1] \rightarrowtriangle t$, so we are done.

(Flat) Straightforward since $e_3\theta[Y/t_1] \equiv e_3\theta$ because $Y \notin vran(\theta)$ by the variable convention and $Y \notin FV(e_3)$ by the condition of (Flat).

(Contx) By the proof of the other cases, $\llbracket e \rrbracket \Subset \llbracket e' \rrbracket$, but then $\llbracket \mathcal{C}[e] \rrbracket \Subset \llbracket \mathcal{C}[e'] \rrbracket$ by Lemma 7, and we are done.

□

The following lemmas —Lemmas 29, 30, 31 and 32— will be used to prove Lemma 8.

Lemma 29

Let linear $e, e_1 \in Exp$ such that $e\theta \sqsubseteq e_1$ for $\theta \in Subst_\perp$. Then $\exists \theta' \in Subst$ such that $e\theta' \equiv e_1$ and $\theta \sqsubseteq \theta'$.

Proof
By induction on the structure of e. For the base case ($e \equiv X \in \mathcal{V}$) we define a function $rep_\perp : Exp_\perp \to Exp \to Exp$ $rep_\perp(e, e')$ that replaces the occurrences of $\perp$ in e by the expression e'. We define this function recursively on the structure of e:

- $rep_\perp(\perp, e') = e'$
- $rep_\perp(Z, e') = Z$
- $rep_\perp(h(e_1, \ldots, e_n), e') = h(rep_\perp(e_1, e'), \ldots, rep_\perp(e_n, e'))$

It is easy to check that $rep_\perp(e, e') = e''$ implies $e \sqsubseteq e''$. Then we define $\theta' \in Subst$ as:

$$\theta'(Y) = \begin{cases} e_1 & if\ X \equiv Y \\ rep_\perp(\theta(Y), Y) & if\ Y \in dom(\theta) \setminus \{X\} \end{cases}$$

Trivially $e\theta' \equiv X\theta' \equiv e_1$ and $\theta \sqsubseteq \theta'$ because $e\theta \sqsubseteq e_1$ by the premise and $\theta(Y) \sqsubseteq rep_\perp(\theta(Y), Y)$.

Regarding the inductive step —$e \equiv h(e_1, \ldots, e_n)$— we know that

$$e\theta \equiv h(e_1\theta, \ldots, e_n\theta) \sqsubseteq e_1 \equiv h(e'_1, \ldots, e'_n)$$

so $e_i\theta \sqsubseteq e'_i$. Then by IH $\exists \theta'_i \in Subst$ such that $e_i\theta'_i \equiv e'_i$ and $\theta \sqsubseteq \theta'_i$. Then we define θ' as:

$$\theta'(Y) = \begin{cases} \theta'_1(Y) & if\ Y \in var(e_1) \\ \theta'_2(Y) & if\ Y \in var(e_2) \\ \ldots & \\ \theta'_n(Y) & if\ Y \in var(e_n) \\ rep_\perp(\theta(Y), Y) & if\ Y \in dom(\theta) \setminus var(e) \end{cases}$$

The substitution θ' is well defined because e is linear. Then $e\theta' \equiv h(e_1\theta', \ldots, e_n\theta') \equiv h(e_1\theta'_1, \ldots, e_n\theta'_n) = h(e'_1, \ldots, e'_n) \equiv e_1$ and $\theta \sqsubseteq \theta'$ by IH and the fact that $\theta(Y) \sqsubseteq rep_\perp(\theta(Y), Y)$. $\square$

Lemma 30
For any $e \in LExp_\perp$, $FV(|e|) \subseteq FV(e)$.

Proof
Straightforward by induction on the structure of e. $\square$

Lemma 31
Given $e \in LExp$, $\theta \in LSubst_\perp$, $|e\theta| = |e|\hat{\theta}$ where $\hat{\theta}$ is defined as $X\hat{\theta} = |X\theta|$

Proof
By induction on the structure of e. We have two base cases:

- $e \equiv X \in \mathcal{V}$. Then $|e\theta| \equiv |X\theta| = X\hat{\theta} = |X|\theta \equiv |e|\hat{\theta}$.
- $e \equiv f(e_1, \ldots, e_n)$. Then $|e\theta| \equiv |f(e_1, \ldots, e_n)\theta| = |f(e_1\theta, \ldots, e_n\theta)| = \perp = \perp\ \hat{\theta} = |f(e_1, \ldots, e_n)|\hat{\theta} \equiv |e|\hat{\theta}$.

Regarding the inductive step we have:

- $e \equiv c(e_1, \ldots, e_n)$. Straightforward.
- $e \equiv let\ X = e_1\ in\ e_2$. Then $|e\theta| = |(let\ X = e_1\ in\ e_2)\theta| = |let\ X = e_1\theta\ in\ e_2\theta| = |e_2\theta|[X/|e_1\theta|]$. By IH we have that $|e_1\theta| = |e_1|\hat{\theta}$ and $|e_2\theta| = |e_2|\hat{\theta}$, so $|e_2\theta|[X/|e_1\theta|] = |e_2\theta| = (|e_2|\hat{\theta})[X/|e_1|\hat{\theta}]$. By the variable convention we can assume that $X \notin dom(\theta) \cup vran(\theta)$, and since $dom(\hat{\theta}) = dom(\theta)$ and $vran(\hat{\theta}) \subseteq vran(\theta)$ —using Lemma 30— we can use Lemma 1 and obtain $(|e_2|\hat{\theta})[X/|e_1|\hat{\theta}] = (|e_2|[X/|e_1|])\hat{\theta}$. Finally, $(|e_2|[X/|e_1|])\hat{\theta} = |let\ X = e_1\ in\ e_2|\hat{\theta} = |e|\hat{\theta}$.

□

Lemma 32
Given $e \in LExp$, $\theta \in LSubst_\perp$, if $|e| =\perp$ then $|e\theta| =\perp$.

Proof
By induction on the structure of e. Notice that e cannot be a variable X or an applied constructor symbol $c(e_1, \ldots, e_n)$ because in those cases $|e| \neq\perp$. The base case $e \equiv f(e_1, \ldots, e_n)$ is straightforward. Regarding the inductive step we have $e \equiv let\ X = e_1\ in\ e_2$ such that $|let\ X = e_1\ in\ e_2| = |e_2|[X/|e_1|] =\perp$. Then $|e\theta| = |(letX = e_1\ in\ e_2)\theta| = |let\ X = e_1\theta\ in\ e_2\theta| = |e_2\theta|[X/|e_1\theta|]$. By Lemma 23 $|e_2\theta|[X/|e_1\theta|] = |(e_2\theta)[X/e_1\theta]|$, and since $X \notin dom(\theta) \cup vran(\theta)$ by the variable convention then we can apply Lemma 1 and $|(e_2\theta)[X/e_1\theta]| = |(e_2[X/e_1])\theta|$. Finally by Lemma 31 $|(e_2[X/e_1])\theta| = |e_2[X/e_1]|\hat{\theta}$, and by Lemma 23 $|e_2[X/e_1]|\hat{\theta} = (|e_2|[X/|e_1|])\hat{\theta} =\perp \hat{\theta} =\perp$. □

Lemma 8 (Completeness lemma for let-rewriting)
For all $e \in LExp$ and $t \in CTerm_\perp$ such that $t \not\equiv\perp$,

$$e \twoheadrightarrow t \text{ implies } e \to^{l^*} let\ \overline{X = a}\ in\ t'$$

for some $t' \in CTerm$ and $\overline{a} \subseteq LExp$ in such a way that $t \sqsubseteq |let\ \overline{X = a}\ in\ t'|$ and $|a_i| =\perp$ for every $a_i \in \overline{a}$. As a consequence, $t \sqsubseteq t'[\overline{X/\perp}]$.

Proof
By induction on the size s of the $CRWL_{let}$-proof, that we measure as the number of $CRWL_{let}$ rules applied. Concerning the base cases:

(B) This contradicts the hypothesis because then $t \equiv\perp$, so we are done. In the rest of the proof we will assume that $t \not\equiv\perp$ because otherwise we would be in this case.
(RR) Then we have $X \twoheadrightarrow X$. But then $X \to^{l\,0} X$ and $X \sqsubseteq X \equiv |X|$, so we are done with $\overline{X} = \emptyset$.
(DC) Then we have $c \twoheadrightarrow c$. But then $c \to^{l\,0} c$ and $c \sqsubseteq c \equiv |c|$, so we are done with $\overline{X} = \emptyset$.

Now we treat the inductive step:

(DC) Then we have $e \equiv c(e_1, \dots, e_n)$ and the $CRWL_{let}$-proof has the shape:

$$\frac{e_1 \rightarrowtriangle t_1, \dots, e_n \rightarrowtriangle t_n}{c(e_1, \dots, e_n) \rightarrowtriangle c(t_1, \dots, t_n)} \ (DC)$$

In the general case some t_i will be equal to $\perp$ and some others will be different. For the sake of simplicity we consider the case when $n = 2$ with $t_1 = \perp$ and $t_2 \not\equiv \perp$, the proof can be easily extended to the general case. Then we have $c(e_1, e_2) \rightarrowtriangle c(\perp, t_2)$, so by IH over the second argument we get $e_2 \to^{l^*} let\ \overline{X_2 = a_2}\ in\ t'_2$ with $t'_2 \in CTerm$, $|a_{2_i}| =\perp$ for every $a_{2_i} \in \overline{a_2}$ and $|let\ \overline{X_2 = a_2}\ in\ t'_2| = t'_2[\overline{X_2/\perp}] \sqsupseteq t_2$. So:

$$\begin{array}{ll}
c(e_1, e_2) \to^{l^*} c(e_1, let\ \overline{X_2 = a_2}\ in\ t'_2) & \text{by IH} \\
\to^{l} let\ Y = (let\ \overline{X_2 = a_2}\ in\ t'_2)\ in\ c(e_1, Y) & \text{by (LetIn)} \\
\to^{l^*} let\ \overline{X_2 = a_2}\ in\ let\ Y = t'_2\ in\ c(e_1, Y) & \text{by (Flat*)} \\
\to^{l} let\ \overline{X_2 = a_2}\ in\ c(e_1, t'_2) & \text{by (Bind)}
\end{array}$$

Then there are several possible cases:

a) $e_1 \equiv f_1(\overline{e_1})$: Then $let\ \overline{X_2 = a_2}\ in\ c(f_1(\overline{e_1}), t'_2) \to^{l} let\ \overline{X_2 = a_2}\ in\ let\ Z = f_1(\overline{e_1})\ in\ c(Z, t'_2)$, by (LetIn). So we are done as $|a_{2_i}| =\perp$ for every a_{2_i} by the IH, $|f_1(\overline{e_1})| =\perp$ and $|let\ \overline{X_2 = a_2}\ in\ let\ Z = f_1(\overline{e_1})\ in\ c(Z, t'_2)| = c(Z, t'_2)[\overline{X_2/\perp}, Z/\perp] \sqsupseteq c(\perp, t_2)$ because $t'_2[\overline{X_2/\perp}] \sqsupseteq t_2$ by the IH, and Z is fresh and so it does not appear in t'_2

b) $e_1 \equiv t'_1 \in CTerm$: Then we are done as $|a_{2_i}| =\perp$ for every $a_{2_i} \in \overline{a_2}$ by the IH, and $|let\ \overline{X_2 = a_2}\ in\ c(t'_1, t'_2)| = c(t'_1, t'_2)[\overline{X_2/\perp}] \sqsupseteq c(\perp, t_2)$, because $t'_2[\overline{X_2/\perp}] \sqsupseteq t_2$ by the IH

c) $e_1 \equiv c_1(\overline{e_1}) \notin CTerm$ with $c_1 \in CS$: Then by Lemma 3 we have the derivation $c_1(\overline{e_1}) \to^{l^*} let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ c_1(\overline{t_1})$. But then:

$$\begin{array}{ll}
let\ \overline{X_2 = a_2}\ in\ c(c_1(\overline{e_1}), t'_2) & \\
\to^{l^*} let\ \overline{X_2 = a_2}\ in\ c(let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ c_1(\overline{t_1}), t'_2) & \text{Lemma 3} \\
\to^{l} let\ \overline{X_2 = a_2}\ in\ let\ Y = (let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ c_1(\overline{t_1}))\ in\ c(Y, t'_2) & \text{by (LetIn)} \\
\to^{l^*} let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ let\ Y = c_1(\overline{t_1})\ in\ c(Y, t'_2) & \text{by (Flat*)} \\
\to^{l} let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ c(c_1(\overline{t_1}), t'_2) & \text{by (Bind)}
\end{array}$$

In the last step notice that Y is fresh and it cannot appear in t'_2. Then we are done as $|f_i(\overline{t'_i})| =\perp$, $|a_{2_i}| =\perp$ for every $a_{2_i} \in \overline{a_2}$ by the IH, and $|let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ c(c_1(\overline{t_1}), t'_2)| = c(c_1(\overline{t_1}), t'_2)[\overline{X_1/\perp}][\overline{X_2/\perp}] \sqsupseteq c(\perp, t_2)$ because $t'_2[\overline{X_2/\perp}] \sqsupseteq t_2$ by the IH, and no variable in $\overline{X_1}$ appears in t'_2 by α-conversion, as those are bound variables which were present in $c_1(\overline{e_1})$ or that appeared after applying Lemma 3 to it, and this expression was placed in a position parallel to the position of t'_2.

d) $e_1 \equiv let\ X = e_{11}\ in\ e_{12}$: Then by Lemma 3 $let\ X = e_{11}\ in\ e_{12} \to^{l^*} let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ e''$ where $e'' \in \mathcal{V}$ or $e'' \equiv h_1(\overline{t_1})$. Then:

$$\begin{array}{ll}
let\ \overline{X_2 = a_2}\ in\ c(let\ X = e_{11}\ in\ e_{12}, t'_2) & \\
\to^{l^*}\ let\ \overline{X_2 = a_2}\ in\ c(let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ e'', t'_2) & \text{by Lemma 3} \\
\to^{l}\ let\ \overline{X_2 = a_2}\ in\ let\ Y = (let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ e'')\ in\ c(Y, t'_2) & \text{by (LetIn)} \\
\to^{l^*}\ let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ let\ Y = e''\ in\ c(Y, t'_2) & \text{by (Flat}^*\text{)}
\end{array}$$

Then we have two possibilities depending on e'':

i) $e'' \equiv Z \in \mathcal{V}$: Then we can do:

$$\begin{array}{ll}
let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ let\ Y = Z\ in\ c(Y, t'_2) & \\
\to^{l}\ let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ c(Z, t'_2) & \text{by (Bind)}
\end{array}$$

Then we are done as $|f_1(\overline{t'_1})| =\perp$, $|a_{2_i}| =\perp$ for every $a_{2_i} \in \overline{a_2}$ by IH, and $|let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ c(Z, t'_2)| = c(Z, t'_2)[\overline{X_1/\perp}][\overline{X_2/\perp}] \sqsupseteq c(\perp, t_2)$, as $t'_2[\overline{X_2/\perp}] \sqsupseteq t_2$ by IH, and no variable in $\overline{X_1}$ appears in t'_2 by α-conversion, like in the case *c)*.

ii) $e'' \equiv h_1(\overline{t_1})$: there are two possible cases:

A) $h_1 = f_1 \in FS$: We are done as $|f_1(\overline{t'_1})| =\perp$, $|a_{2_i}| =\perp$ for every $a_{2_i} \in \overline{a_2}$ by IH, $|f_1(\overline{t_1})| =\perp$, and $|let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ let\ Y = f_1(\overline{t_1})\ in\ c(Y, t'_2)| = c(Y, t'_2)[Y/\perp][\overline{X_1/\perp}][\overline{X_2/\perp}] \sqsupseteq c(\perp, t_2)$, as by IH $t'_2[\overline{X_2/\perp}] \sqsupseteq t_2$, Y is fresh and so it does not appear in t'_2, and no variable in $\overline{X_1}$ appears in t'_2 as in the case *i)*.

B) $h_1 = c_1 \in DC$: Then we can do a (Bind) step:

$$\begin{array}{l}
let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ let\ Y = c_1(\overline{t_1})\ in\ c(Y, t'_2) \\
\to^{l}\ let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = a_1}\ in\ c(c_1(\overline{t_1}), t'_2)
\end{array}$$

Then we are done as $|f_1(\overline{t'_1})| =\perp$, $|a_{2_i}| =\perp$ for every $a_{2_i} \in \overline{a_2}$ by IH, and

$$\begin{array}{rl}
& |let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'_1})}\ in\ c(c_1(\overline{t_1}), t'_2)| \\
= & c(c_1(\overline{t_1}), t'_2)[\overline{X_1/\perp}][\overline{X_2/\perp}] \\
\sqsupseteq & c(\perp, t_2)
\end{array}$$

as $t'_2[\overline{X_2/\perp}] \sqsupseteq t_2$ by IH, and no variable in $\overline{X_1}$ appears in t'_2, as we saw in *i)*.

(OR) If f has no arguments ($n = 0$) then we have:

$$\frac{r\theta \twoheadrightarrow t}{f \twoheadrightarrow t}\ (OR)$$

with $(f \twoheadrightarrow r) \in \mathcal{P}$ and $\theta \in CSubst_\perp$. Let us define $\theta' \in CSubst$ as the substitution which is equal to θ except that every $\perp$ introduced by θ is replaced with some constructor symbol or variable. Then $\theta \sqsubseteq \theta'$, so by Proposition 5 we have $r\theta' \twoheadrightarrow t$ with a proof of the same size. But then applying the IH to this proof we get $r\theta' \to^{l^*} let\ \overline{X = a}\ in\ t'$ under the conditions of the lemma. Hence $f \to^{l} r\theta' \to^{l^*} let\ \overline{X = a}\ in\ t'$ applying (Fapp) in the first step, and we are done.

If $n > 0$, we will proceed as in the case for (DC), doing a preliminary version for $f(e_1, e_2) \twoheadrightarrow t$ which can be easily extended for the general case. Then we have:

$$\frac{e_1 \twoheadrightarrow \perp \quad e_2 \twoheadrightarrow t_2 \quad r\theta \twoheadrightarrow t}{f(e_1, e_2) \twoheadrightarrow t} \; (OR)$$

such that $t_2 \not\equiv \perp$, and with $(f(p_1, p_2) \to r) \in \mathcal{P}$, $\theta \in CSubst_\perp$, such that $p_1\theta = \perp$ and $p_2\theta = t_2$. Then applying the IH to $e_2 \twoheadrightarrow t_2$ we get that $e_2 \to^{l^*} let\ \overline{X_2 = a_2}\ in\ t'_2$ such that $|a_{2_i}| = \perp$ for every a_{2_i} and $|let\ \overline{X_2 = a_2}\ in\ t'_2| = t'_2[\overline{X_2/\perp}] \sqsupseteq t_2$. Then we can do:

$$\begin{array}{ll} f(e_1, e_2) \to^{l^*} f(e_1, let\ \overline{X_2 = a_2}\ in\ t'_2) & \text{by IH} \\ \to^{l} let\ Y = (let\ \overline{X_2 = a_2}\ in\ t'_2)\ in\ f(e_1, Y) & \text{by (LetIn)} \\ \to^{l^*} let\ \overline{X_2 = a_2}\ in\ let\ Y = t'_2\ in\ f(e_1, Y) & \text{by (Flat*)} \\ \to^{l} let\ \overline{X_2 = a_2}\ in\ f(e_1, t'_2) & \text{by (Bind)} \end{array}$$

Then applying Lemma 3 we get

$$f(e_1, t'_2) \to^{l^*} let\ \overline{X_1 = f_1(\overline{t'})}\ in\ f(t'_1, t'_2)$$

Now as $t'_2[\overline{X_2/\perp}] \sqsupseteq t_2$ then $(t'_1, t'_2) \sqsupseteq (\perp, t_2)$, so by Lemma 29 there must exist $\theta' \in CSubst$ such that $\theta \sqsubseteq \theta'$ and $(p_1, p_2)\theta' \equiv (t'_1, t'_2)$. Then by Proposition 5, as $r\theta \twoheadrightarrow t$ then $r\theta' \twoheadrightarrow t$ with a proof of the same size. As $\theta' \in CSubst$ and $e \in LExp$ (because it is part of the program) then $r\theta' \in LExp$ and we can apply the IH to that proof getting that $r\theta' \to^{l^*} let\ \overline{X = a}\ in\ t'$ such that $|a_i| = \perp$ for every a_i and $|let\ \overline{X = a}\ in\ t'| = t'[\overline{X/\perp}] \sqsupseteq t$. Then we can do:

$$\begin{array}{ll} let\ \overline{X_2 = a_2}\ in\ f(e_1, t'_2) & \\ \to^{l^*} let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'})}\ in\ f(t'_1, t'_2) & \text{by Lemma 3} \\ \equiv let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'})}\ in\ f(p_1, p_2)\theta' & \\ \to^{l} let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'})}\ in\ r\theta' & \text{by (Fapp)} \\ \to^{l^*} let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = f_1(\overline{t'})}\ in\ let\ \overline{X = a}\ in\ t' & \text{by } 2^{nd} \text{ IH} \end{array}$$

Then $|a_{2_i}| = \perp$ for every $a_{2_i} \in \overline{a_2}$ by IH, $|f_1(\overline{t'})| = \perp$ and $|a_i| = \perp$ for every a_i by IH. Besides the variables in $\overline{X_1} \cup \overline{X_2}$ either belong to $BV(e_1) \cup BV(e_2)$ or are fresh, hence none of them may appear in t (by Lemma 27 over $f(e_1, e_2) \twoheadrightarrow t$ or by freshness). So $t'[\overline{X/\perp}] \sqsupseteq t$ implies that $\forall p \in O(t')$ such that $t'|_p = Y$ for some $Y \in \overline{X_1} \cup \overline{X_2}$ then $t|_p = \perp$. But then $|let\ \overline{X_2 = a_2}\ in\ let\ \overline{X_1 = a_1}\ in\ let\ \overline{X = a}\ in\ t'| \equiv t'[\overline{X/\perp}][\overline{X_1/\perp}][\overline{X_2/\perp}] \sqsupseteq t$.

(Let) Then $e \equiv let\ X = e_1\ in\ e_2$ and we have a proof of the following shape:

$$\frac{e_1 \twoheadrightarrow t_1 \quad e_2[X/t_1] \twoheadrightarrow t}{let\ X = e_1\ in\ e_2 \twoheadrightarrow t} \; (Let)$$

Then we have two possibilities:

a) $t_1 \equiv \perp$: Then $e_2[X/t_1] \equiv e_2[X/\perp] \sqsubseteq e_2$. Hence, as $e_2[X/t_1] \twoheadrightarrow t$ and $[X/t_1] \sqsubseteq \epsilon$, by Proposition 5 we get $e_2\epsilon \equiv e_2 \twoheadrightarrow t$ with a proof of the same size or smaller, and so by IH we get $e_2 \to^{l^*} let\ \overline{X = a}\ in\ t'$, with $t' \in CTerm$, $|a_i| \equiv \perp$ for every a_i and $|let\ \overline{X = a}\ in\ t'| \equiv t'[\overline{X/\perp}] \sqsupseteq t$, and we can do:

$$let\ X = e_1\ in\ e_2 \to^{l^*} let\ X = e_1\ in\ let\ \overline{X = a}\ in\ t'$$

Besides $X \notin var(t)$ by Lemma 27 over $let\ X = e_1\ in\ e_2 \twoheadrightarrow t$, and then $t'[\overline{X/\perp}] \sqsupseteq t$ implies $\forall p \in O(t')$ such that $t'|_p \equiv X$ then $t|_p \equiv \perp$, and we have several possible cases:

i) $e_1 = f_1(\overline{e_1})$: Then we are donde because $|\overline{a}| \equiv \overline{\perp}$ by IH, $|f_1(\overline{e_1})| \equiv \perp$ and $|let\ X = f_1(\overline{e_1})\ in\ let\ \overline{X = a}\ in\ t'| \equiv t'[\overline{X/\perp}][X/\perp] \sqsupseteq t$, as $t'[\overline{X/\perp}] \sqsupseteq t$ and $\forall p \in O(t')$ such that $t'|_p \equiv X$ then $t|_p \equiv \perp$, as we saw above.

ii) $e_1 = t'_1 \in CTerm$: But then

$$let\ X = t'_1\ in\ let\ \overline{X = a}\ in\ t' \to^l let\ \overline{X = a[X/t'_1]}\ in\ t'[X/t'_1] \quad \text{by (Bind)}$$

and we are done because $|\overline{a}| \equiv \overline{\perp}$ by IH, and so $|\overline{a}[X/t'_1]| \equiv \overline{\perp}$ by Lemma 32. Besides, as in *i)*, $t'[\overline{X/\perp}] \sqsupseteq t$ combined with the fact that $\forall p \in O(t')$ such that $t'|_p \equiv X$ we have $t|_p \equiv \perp$, implies that $|let\ \overline{X = a[X/t'_1]}\ in\ t'[X/t'_1]| \equiv t'[X/t'_1][\overline{X/\perp}] \sqsupseteq t$.

iii) $e_1 = c_1(\overline{e_1}) \notin CTerm$ with $c_1 \in CS$: Then by Lemma 3 we have $c_1(\overline{e_1}) \to^{l^*} let\ \overline{X_1 = f_1(\overline{t_1})}\ in\ c_1(\overline{t_1})$, hence

$$\begin{array}{ll} let\ X = c_1(\overline{e_1})\ in\ let\ \overline{X = a}\ in\ t' & \\ \to^{l^*} let\ X = (let\ \overline{X_1 = f_1(\overline{t_1})}\ in\ c_1(\overline{t_1}))\ in\ let\ \overline{X = a}\ in\ t' & \text{by Lemma 3} \\ \to^{l^*} let\ \overline{X_1 = f_1(\overline{t_1})}\ in\ let\ X = c_1(\overline{t_1})\ in\ let\ \overline{X = a}\ in\ t' & \text{by (Flat}^*\text{)} \\ \to^{l} let\ \overline{X_1 = f_1(\overline{t_1})}\ in\ let\ \overline{X = a[X/c_1(\overline{t_1})]}\ in\ t'[X/c_1(\overline{t_1})] & \text{by (Bind)} \end{array}$$

As by IH $|\overline{a}| \equiv \overline{\perp}$ then $|\overline{a[X/c_1(\overline{t_1})]}| \equiv \overline{\perp}$ by Lemma 32. At this point we have to check that $|let\ \overline{X_1 = a_1}\ in\ let\ \overline{X = a[X/c_1(\overline{t_1})]}\ in\ t'[X/c_1(\overline{t_1})]| \equiv t'[X/c_1(\overline{t_1})][\overline{X/\perp}][\overline{X_1/\perp}] \sqsupseteq t$. The variables in $\overline{X_1}$ either belong to $BV(c_1(\overline{e_1}))$ or are fresh, hence by α-conversion none of them may appear in t', because in $let\ X = c_1(\overline{e_1})\ in\ let\ \overline{X = a}\ in\ t'$ the expression t' has no access to the variables bound in $c_1(\overline{e_1})$. Hence $t'[X/c_1(\overline{t_1})][\overline{X/\perp}][\overline{X_1/\perp}] \equiv t'[X/t''][\overline{X/\perp}]$, for some $t'' \in CTerm_\perp$. But then, as in *ii)*, $t'[\overline{X/\perp}] \sqsupseteq t$ combined with the fact that $\forall p \in O(t')$ such that $t'|_p \equiv X$ we have $t|_p \equiv \perp$, implies that $t'[X/t''][\overline{X/\perp}] \sqsupseteq t$.

iv) $e_1 \equiv let\ Y = e_{11}\ in\ e_{12}$: Then by Lemma 3 we have $let\ Y = e_{11}\ in\ e_{12} \to^{l^*} let\ \overline{X_1 = f_1(\overline{t_1})}\ in\ h_1(\overline{t_1})$, and so

$$\begin{array}{ll} let\ X = (let\ Y = e_{11}\ in\ e_{12})\ in\ let\ \overline{X = a}\ in\ t' & \\ \to^{l^*} let\ X = (let\ \overline{X_1 = f_1(\overline{t_1})}\ in\ h_1(\overline{t_1}))\ in\ let\ \overline{X = a}\ in\ t' & \text{by Lemma 3} \\ \to^{l^*} let\ \overline{X_1 = f_1(\overline{t_1})}\ in\ let\ X = h_1(\overline{t_1})\ in\ let\ \overline{X = a}\ in\ t' & \text{by (Flat}^*\text{)} \end{array}$$

Then either $h \in CS$ and we are like in *iii)* before the final (Bind) step, or $h \in FS$ and $|h_1(\overline{t_1})| = \perp$ and $|\overline{a}| = \overline{\perp}$ (by IH), and $|let\ \overline{X_1 = a_1}\ in\ let\ X = h_1(\overline{t_1})\ in\ let\ \overline{X = a}\ in\ t'| \equiv t'[\overline{X/\perp}][X/\perp][\overline{X_1/\perp}] \equiv t'[\overline{X/\perp}][X/\perp]$ because $\overline{X_1} \cap var(t') = \emptyset$, as we saw in *iii)*. But then, as in *ii)*, $t'[\overline{X/\perp}] \sqsupseteq t$ combined with the fact that $\forall p \in O(t')$ such that $t'|_p \equiv X$ we have $t|_p \equiv \perp$, implies that $t'[\overline{X/\perp}][X/\perp] \sqsupseteq t$.

b) $t_1 \not\equiv \perp$: Then by IH we get $e_1 \to^{l^*} let\ \overline{X_1 = a_1}\ in\ t'_1$, with $t'_1 \in CTerm$, $|a_{1_i}| \equiv \perp$ for every a_{1_i} and $|let\ \overline{X_1 = a_1}\ in\ t'_1| \equiv t'_1[\overline{X_1/\perp}] \sqsupseteq t_1$. Hence $t_1 \sqsubseteq t'_1$

and so $e_2[X/t_1] \sqsubseteq e_2[X/t_1']$, but then $e_2[X/t_1] \twoheadrightarrow t$ implies $e_2[X/t_1'] \twoheadrightarrow t$ with a proof of the same size or smaller, by Proposition 3. Therefore we may apply the IH to that proof to get $e_2[X/t_1'] \to^{l^*} let\ \overline{X = a}\ in\ t'$, with $t' \in CTerm$, $|a_i| \equiv \perp$ for every a_i and $|let\ \overline{X = a}\ in\ t'| \equiv t'[\overline{X/\perp}] \sqsupseteq t$. But then we can do:

$$
\begin{array}{ll}
let\ X = e_1\ in\ e_2 \to^{l^*} let\ X = (let\ \overline{X_1 = a_1}\ in\ t_1')\ in\ e_2 & \text{by IH} \\
\to^{l^*} let\ \overline{X_1 = a_1}\ in\ let\ X = t_1'\ in\ e_2 & \text{by (Flat}^*\text{)} \\
\to^{l} let\ \overline{X_1 = a_1}\ in\ e_2[X/t_1'] & \text{by (Bind)} \\
\to^{l^*} let\ \overline{X_1 = a_1}\ in\ let\ \overline{X = a}\ in\ t' & \text{by IH}
\end{array}
$$

Then by the IH's we have $|\overline{a}| = \overline{\perp}$ and $|\overline{a_1}| = \overline{\perp}$. Besides the variables in $\overline{X_1}$ either belong to $BV(e_1)$ or are fresh, hence none of them may appear in t (by Lemma 27 over $let\ X = e_1\ in\ e_2 \twoheadrightarrow t$ or by freshness). So $t'[\overline{X/\perp}] \sqsupseteq t$ implies that $\forall p \in O(t')$ such that $t'|_p = Y$ for some $Y \in \overline{X_1}$ then $t|_p = \perp$. But then $|let\ \overline{X_1 = a_1}\ in\ let\ \overline{X = a}\ in\ t'| \equiv t'[\overline{X/\perp}][\overline{X_1/\perp}] \sqsupseteq t$.

□

A.8 Proofs for Section 5

Lemma 10
If $BV(\mathcal{C}) \cap FV(e_1) = \emptyset$ and $X \notin FV(\mathcal{C})$ then $[\![\mathcal{C}[let\ X = e_1\ in\ e_2]]\!] = [\![let\ X = e_1\ in\ \mathcal{C}[e_2]]\!]$

Proof
One step of the rule (Dist) can be replaced by two steps (CLetIn) + (Bind):

$$\mathcal{C}[let\ X = e_1\ in\ e_2] \to^l let\ U = e_1\ in\ \mathcal{C}[let\ X = U\ in\ e_2] \to^l let\ U = e_1\ in\ \mathcal{C}[e_2[X/U]]$$

followed by a renaming of U by X in the last expression. Then the lemma follows from preservation of hypersemantics by (CLetIn) and (Bind) (Lemma 9 and Proposition 8). □

Proposition 9 ((Hyper)semantic properties of ?)
For any $e_1, e_2 \in LExp_\perp$

i) $[\![e_1\ ?\ e_2]\!] = [\![e_1]\!] \cup [\![e_2]\!]$
ii) $[\![e_1\ ?\ e_2]\!] = [\![e_1]\!] \Cup [\![e_2]\!]$

Proof
i) Direct from definition of ? and the CRWL-proof calculus.
ii)

$$
\begin{array}{ll}
[\![e_1\ ?\ e_2]\!] = \lambda\theta.[\![(e_1\ ?\ e_2)\theta]\!] & \text{by definition of } [\![\ ]\!] \\
= \lambda\theta.[\![e_1\theta\ ?\ e_2\theta]\!] & \\
= \lambda\theta.([\![e_1\theta]\!] \cup [\![e_2\theta]\!]) & \text{by } i) \\
= \lambda\theta.([\![e_1]\!]\theta \cup [\![e_2]\!]\theta) & \text{by definition of } [\![\ ]\!] \\
= [\![e_1]\!] \Cup [\![e_2]\!] & \text{by definition of } \Cup
\end{array}
$$

□

A.9 Proofs for Section 6

Theorem 14 (Soundness of the let-narrowing relation $\leadsto^l$)
For any $e, e' \in LExp$, $e \leadsto^{l^*}_\theta e'$ implies $e\theta \to^{l\,*} e'$.

Proof
First we prove the soundness of narrowing for one step, proceeding by a case distinction over the rule used in $e \leadsto^l{}_\theta\ e'$. The cases of (Elim), (Bind), (Flat) and (LetIn) are trivial, since narrowing and rewriting coincide for these rules.

(Narr) Then we have $f(\overline{t}) \leadsto^l{}_\theta\ r\theta$ for $(f(\overline{p}) \to r) \in \mathcal{P}$ fresh, $\theta \in CSubst$ such that $f(\overline{t})\theta \equiv f(\overline{p})\theta$. But then $(f(\overline{p}) \to r)\theta \equiv f(\overline{p})\theta \to r\theta \equiv f(\overline{t})\theta \to r\theta$, so we can do $e\theta \equiv f(\overline{t})\theta \to^l r\theta \equiv e'$ by (Fapp).

(Contxt) Then we have $\mathcal{C}[e] \leadsto^l{}_\theta\ \mathcal{C}\theta[e']$ because $e \leadsto^l{}_\theta e'$. Let us do a case distinction over the rule applied in $e \leadsto^l{}_\theta\ e'$:

a) $e \leadsto^l{}_\theta\ e' \equiv f(\overline{t}) \leadsto^l{}_\theta\ r\theta$ by (Narr), for $(f(\overline{p}) \to r) \in \mathcal{P}$ fresh, so $f(\overline{t})\theta \to^l r\theta$ by (Fapp). Then $(\mathcal{C}[e])\theta \equiv (\mathcal{C}[e])\theta|_{\backslash var(\overline{p})}$, because the variables in $var(\overline{p})$ are fresh as $(f(\overline{p}) \to r)$ is. But then, as $dom(\theta) \cap BV(\mathcal{C}) = \emptyset$ and $vRan(\theta|_{\backslash var(\overline{p})}) \cap BV(\mathcal{C}) = \emptyset$ by the conditions in (Contx), and $dom(\theta) \cap BV(\mathcal{C}) = \emptyset$ implies $dom(\theta|_{\backslash var(\overline{p})}) \cap BV(\mathcal{C}) = \emptyset$, we can apply Lemma 25 getting $(\mathcal{C}[e])\theta|_{\backslash var(\overline{p})} \equiv \mathcal{C}\theta|_{\backslash var(\overline{p})}[e\theta|_{\backslash var(\overline{p})}] \equiv \mathcal{C}\theta|_{\backslash var(\overline{p})}[f(\overline{t})\theta|_{\backslash var(\overline{p})}] \equiv \mathcal{C}\theta[f(\overline{t})\theta]$, because the variables in $var(\overline{p})$ are fresh. Besides $vran(\theta|_{\backslash var(\overline{p})}) \cap BV(\mathcal{C}) = \emptyset$, so we can apply (Contx) combined with an inner (Fapp) to do $(\mathcal{C}[e])\theta \equiv \mathcal{C}\theta[f(\overline{t})\theta] \to^l \mathcal{C}\theta[r\theta] \equiv \mathcal{C}\theta[e']$.

b) In case a different rule was applied in $e \leadsto^l{}_\theta\ e'$ then $\theta = \epsilon$. By the proof of the other cases we have $e\theta \equiv e \to^l e'$, so $(\mathcal{C}[e])\theta \equiv \mathcal{C}[e] \to^l \mathcal{C}[e'] \equiv \mathcal{C}\theta[e']$ (remember $\theta = \epsilon$).

Now we prove the lemma for any number of steps $\to^l$, proceeding by induction over the length n of $e \leadsto^{l^n}_\theta\ e'$. The case $e \leadsto^{l^0}_\epsilon\ e \equiv e'$ is straightforward because $e \to^{l^0} e \equiv e'$. For $n > 0$ we have the derivation $e \leadsto^l_\sigma e'' \leadsto^{l^{n-1}}_\gamma\ e'$ with $\theta = \gamma \circ \sigma$. By the proof for one step $e\sigma \to^l e''$, and by the closeness under $CSubst$ of let-rewriting (Lemma 2) $e\sigma\gamma \to^l e''\gamma$. By IH $e''\gamma \to^{l^*} e'$, so we can link $e\theta \equiv e\sigma\gamma \to^l e''\gamma \to^{l^*} e'$.
□

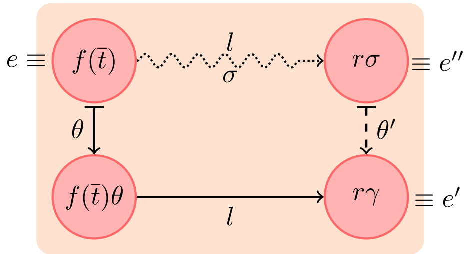
Lemma 11 (Lifting lemma for the let-rewriting relation $\to^l$)
Let $e, e' \in LExp$ such that $e\theta \to^{l^*} e'$ for some $\theta \in CSubst$, and let $\mathcal{W}, \mathcal{B} \subseteq \mathcal{V}$ with $dom(\theta) \cup FV(e) \subseteq \mathcal{W}$, $BV(e) \subseteq \mathcal{B}$ and $(dom(\theta) \cup vran(\theta)) \cap \mathcal{B} = \emptyset$, and for each (Fapp) step of $e\theta \to^{l^*} e'$ using a rule $R \in \mathcal{P}$ and a substitution $\gamma \in CSubst$ then $vran(\gamma|_{vExtra(R)}) \cap \mathcal{B} = \emptyset$. Then there exist a derivation $e \leadsto^{l^*}_\sigma e''$ and $\theta' \in CSubst$ such that:

(i) $e''\theta' = e'$ (ii) $\sigma\theta' = \theta[\mathcal{W}]$ (iii) $(dom(\theta') \cup vran(\theta')) \cap \mathcal{B} = \emptyset$

Besides, the let-narrowing derivation can be chosen to use mgu's at each (Narr) step.

Proof
Let us do a case distinction over the rule applied in $e\theta \rightarrow^{l} e'$:

(Fapp) $e \equiv f(\overline{t})$, so:



With an (Fapp) step $e\theta \equiv f(\overline{t})\theta \rightarrow^{l} r\gamma$ with $(f(\overline{p}) \rightarrow r) \in \mathcal{P}$, $\gamma \in CSubst$, such that $f(\overline{t})\theta \equiv f(\overline{p})\gamma$ and $f(\overline{p}) \rightarrow r$ is a fresh variant. We can assume that $dom(\gamma) \subseteq FV(f(\overline{p}) \rightarrow r)$ without loss of generality. But then $dom(\theta) \cap dom(\gamma) = \emptyset$, and so $\theta \uplus \gamma$ is correctly defined, and it is a unifier of $f(\overline{t})$ and $f(\overline{p})$. So, there must exist $\sigma = mgu(f(\overline{t}), f(\overline{p}))$, which we can use to perform a (Narr) step, because $\sigma \in CSubst$ and $f(\overline{t})\sigma \equiv f(\overline{p})\sigma$.

$$e \equiv f(\overline{t}) \rightsquigarrow^{l}{}_{\sigma}\, r\sigma \equiv e''$$

As this unifier is an mgu then $dom(\sigma) \subseteq FV(f(\overline{t})) \cup FV(f(\overline{p}))$, $vran(\sigma) \subseteq FV(f(\overline{t})) \cup FV(f(\overline{p}))$ and $\sigma \lesssim (\theta \uplus \gamma)$, so there must exist $\theta'_1 \in CSubst$ such that $\sigma\theta'_1 = \theta \uplus \gamma$. Besides we can define $\theta'_0 = \theta|_{\setminus(dom(\theta'_1) \cup FV(f(\overline{t})))}$ and then we can take $\theta' = \theta'_0 \uplus \theta'_1$ which is correctly defined as obviously $dom(\theta'_0) \cap dom(\theta'_1) = \emptyset$. Besides $dom(\theta'_0) \cap (FV(f(\overline{t})) \cup FV(f(\overline{p})) = \emptyset$, as if $Y \in FV(f(\overline{t}))$ then $Y \notin dom(\theta'_0)$ by definition; and if $Y \in FV(f(\overline{p}))$ then $Y \notin dom(\theta)$ as $\overline{p}$ belong to the fresh variant, and so $Y \notin dom(\theta'_0)$. Then the conditions in Lemma 11 hold:

- Condition i) $e''\theta' \equiv e'$: As $e''\theta' \equiv r\sigma\theta' \equiv r\sigma\theta'_1$ because given $Y \in FV(r\sigma)$, if $Y \in FV(r)$ then it belongs to the fresh variant and so $Y \notin dom(\theta) \supseteq dom(\theta'_0)$; and if $Y \in vran(\sigma) \subseteq FV(f(\overline{t})) \cup FV(f(\overline{p}))$ then $Y \notin dom(\theta'_0)$ because $dom(\theta'_0) \cap (FV(f(\overline{t})) \cup FV(f(\overline{p}))) = \emptyset$. But $r\sigma\theta'_1 \equiv r(\theta \uplus \gamma) \equiv r\gamma \equiv e'$, because $\sigma\theta'_1 = \theta \uplus \gamma$ and r is part of the fresh variant.
- Condition ii) $\sigma\theta' = \theta[\mathcal{W}]$: Given $Y \in \mathcal{W}$, if $Y \in FV(f(\overline{t}))$ then $Y \notin dom(\gamma)$ and so $Y\theta \equiv Y(\theta \uplus \gamma) \equiv Y\sigma\theta'_1$, as $\sigma\theta'_1 = \theta \uplus \gamma$. But $Y\sigma\theta'_1 \equiv Y\sigma\theta'$ because given $Z \in var(Y\sigma)$, if $Z \equiv Y$ then as $Y \in FV(f(\overline{t}))$ then $Z \equiv Y \notin dom(\theta'_0)$ by definition of θ'_0; if $Z \in vran(\sigma)$ then $Z \notin dom(\theta'_0)$, as we saw before.
  On the other hand, $(\mathcal{W} \setminus FV(f(\overline{t}))) \cap (FV(f(\overline{t})) \cup FV(f(\overline{p}))) = (\mathcal{W} \setminus FV(f(\overline{t})) \cap FV(f(\overline{t}))) \cup (\mathcal{W} \setminus FV(f(\overline{t})) \cap FV(f(\overline{p}))) = \emptyset \cup \emptyset = \emptyset$, because $FV(f(\overline{p}))$ are part of the fresh variant. So, if $Y \in \mathcal{W} \setminus FV(f(\overline{t}))$, then $Y \notin dom(\sigma) \subseteq FV(f(\overline{t})) \cup FV(f(\overline{p}))$. Now if $Y \in dom(\theta'_0)$ then $Y\theta \equiv Y\theta'_0$ (by definition of θ'_0), $Y\theta'_0 \equiv Y\theta'$ (as $Y \in dom(\theta'_0)$), $Y\theta' \equiv Y\sigma\theta'$ (as $Y \notin dom(\sigma)$). If $Y \in dom(\theta'_1)$, $Y\theta \equiv Y(\theta \uplus \gamma)$ (as $Y \in \mathcal{W} \setminus FV(f(\overline{t}))$ implies it does not appear in the fresh instance), $Y(\theta \uplus \gamma) \equiv Y\sigma\theta'_1$ (as $\sigma\theta'_1 = \theta \uplus \gamma$), $Y\sigma\theta'_1 \equiv Y\theta'_1$ (as $Y \notin dom(\sigma)$), $Y\theta'_1 \equiv Y\theta'$ (as $Y \in dom(\theta'_1)$) and $Y\theta' \equiv Y\sigma\theta'$ (as $Y \notin dom(\sigma)$).

And if $Y \notin (dom(\theta'_0) \cup dom(\theta'_1))$ then $Y \notin dom(\theta')$, and as $Y \notin dom(\sigma)$ and $Y\theta \equiv Y(\theta \uplus \gamma)$, then $Y\theta \equiv Y(\theta \uplus \gamma) \equiv Y\sigma\theta'_1 \equiv Y \equiv Y\sigma\theta'$.

- Condition iii.1) $dom(\theta') \cap \mathcal{B} = \emptyset$. Remember $\theta' = \theta'_0 \uplus \theta'_1$:

— $dom(\theta'_0) \cap \mathcal{B} = \emptyset$: Given $Y \in dom(\theta'_0)$ then $Y \in dom(\theta)$ by definition of θ'_0, and so $Y \notin \mathcal{B}$, because $dom(\theta) \cap \mathcal{B} = \emptyset$ by hypothesis.

— $dom(\theta'_1) \cap \mathcal{B} = \emptyset$: As σ is an mgu and $\sigma \lesssim \theta \uplus \gamma$, then $dom(\sigma) \subseteq dom(\theta \uplus \gamma)$. Given $Z \in \mathcal{B}$ then $Z \notin dom(\theta)$, as $dom(\theta) \cap \mathcal{B} = \emptyset$ by hypothesis, and $Z \notin dom(\gamma) \subseteq FV(f(\overline{p}) \to r)$ which are fresh, so $Z \notin dom(\sigma)$. But then, as $\sigma\theta'_1 = \theta \uplus \gamma$, $Z \equiv Z(\theta \uplus \gamma) \equiv Z\sigma\theta'_1 \equiv Z\theta'_1$, so $Z \notin dom(\theta'_1)$.

- Condition iii.2) $vran(\theta') \cap \mathcal{B} = \emptyset$. Remember $\theta' = \theta'_0 \uplus \theta'_1$:

— $vran(\theta'_0) \cap \mathcal{B} = \emptyset$: Given $Y \in dom(\theta'_0)$ then $Y\theta'_0 \equiv Y\theta$ by definition of θ'_0. As $vran(\theta) \cap \mathcal{B} = \emptyset$ by hypothesis then it must happen $var(Y\theta) \cap \mathcal{B} = \emptyset$, so $var(Y\theta'_0) \cap \mathcal{B} = \emptyset$.

— $vran(\theta'_1) \cap \mathcal{B} = \emptyset$: As $\sigma\theta'_1 = \theta \uplus \gamma$ then we can assume $dom(\theta'_1) \subseteq vran(\sigma) \cup (dom(\theta \uplus \gamma) \setminus dom(\sigma))$.

  - Let $X \in dom(\theta'_1) \cap vran(\sigma)$ be such that $X\theta'_1 \equiv r[Z]$ with $Z \in \mathcal{B}$. We will see that this $Z \in \mathcal{B}$ can appear in $X\theta'_1$ without leading to contradiction. The intuition is, as $vran(\theta) \cap \mathcal{B} = \emptyset$ and $vran(\gamma|_{vExtra(R)}) \cap \mathcal{B} = \emptyset$, then every $Z \in \mathcal{B}$ must come from an appearance in e of the same variable, transmitted to e' by the matching substitution γ, and so transmitted to e'' by σ.

    As $X \in vran(\sigma)$ then there must exist $Y \in dom(\sigma)$ such that $Y \longmapsto^{\sigma} r_1[X]_p \longmapsto^{\theta'_1} r_2[s[Z]]_p$. But as $\sigma\theta'_1 = \theta \uplus \gamma$ then $Y \longmapsto^{\theta \uplus \gamma} r_2[s[Z]]_p$. Then, $Z \in vran(\theta \uplus \gamma)$, but $Z \in \mathcal{B}, vran(\theta) \cap \mathcal{B} = \emptyset, vran(\gamma|_{vExtra(R)}) \cap \mathcal{B} = \emptyset, dom(\gamma) \subseteq FV(f(\overline{p}) \to s)$, so it must happen $Z \in vran(\gamma|_{FV(\overline{p})})$, and as a consequence $Y \in FV(\overline{p})$. Let $o \in O(f(\overline{p}))$ (set of positions in $f(\overline{p})$) be such that $f(\overline{p})|_o \equiv Y$, then:

    - $((f(\bar{t}))\sigma)|_o \equiv ((f(\overline{p}))\sigma)|_o \equiv ((f(\overline{p}))|_o)\sigma \equiv Y\sigma \equiv r_1[X]_p$.
    - As $f(\bar{t}) \notin dom(\gamma)$, which are the fresh variables of the variant of the program rule, $((f(\bar{t}))\theta)|_o \equiv ((f(\bar{t}))(\theta \uplus \gamma))|_o \equiv ((f(\overline{p}))(\theta \uplus \gamma))|_o \equiv ((f(\overline{p}))|_o)(\theta \uplus \gamma) \equiv Y(\theta \uplus \gamma) \equiv r_2[s[Z]]_p$

    So, as $X \in dom(\theta'_1)$ then $X \notin \mathcal{B}$ and $Z \in \mathcal{B}$ has been introduced by θ, but this is impossible as $vran(\theta) \cap \mathcal{B} = \emptyset$.

  - Let $Y \in dom(\theta) \setminus dom(\sigma)$ be. Then $Y\theta \equiv Y(\theta \uplus \gamma)$ (as $Y \in dom(\theta)$, $Y(\theta \uplus \gamma) \equiv Y\sigma\theta'_1$ (as $\sigma\theta'_1 = \theta \uplus \gamma$), $Y\sigma\theta'_1 \equiv Y\theta'_1$ (as $Y \notin dom(\sigma)$. But then no variable in $\mathcal{B}$ can appear in $Y\theta'_1 \equiv Y\theta$ as $(dom(\theta) \cup vran(\theta)) \cap \mathcal{B} = \emptyset$.

  - Let $Y \in dom(\gamma) \setminus dom(\sigma)$ be. Then $Y\gamma \equiv Y(\theta \uplus \gamma) \equiv Y\sigma\theta'_1 \equiv Y\theta'_1$, reasoning like in the previous case. As $dom(\gamma) \subseteq FV(f(\overline{p}) \to s)$ it can happen:

    - $Y \notin FV(f(\overline{p}))$: Then no variable in $\mathcal{B}$ can appear in $Y\gamma$ because $vran(\gamma|_{vExtra(R)}) \cap \mathcal{B} = \emptyset$ by the hypothesis.

- $Y \in FV(f(\overline{p}))$: Let $Z \in \mathcal{B}$ appearing in $Y\gamma$, then Z appears in $f(\overline{t})$, so it must happen $Y \in dom(\sigma)$ because otherwise σ could not be a unifier of $f(\overline{t})$ and $f(\overline{p})$. But this is a contradiction so this case is impossible.

(LetIn) In this case $e\theta \equiv h(e_1\theta, \ldots, e\theta, \ldots, e_n\theta)$ and $e \equiv h(e_1, \ldots, e, \ldots, e_n)$. Then the let-rewriting step is

$$e\theta \equiv h(e_1\theta, \ldots, e\theta, \ldots, e_n\theta) \to^l let\ X = e\theta\ in\ h(e_1\theta, \ldots, X, \ldots, e_n\theta) \equiv e'$$

with $h \in \Sigma$, $e\theta \equiv f(\overline{e'})$ —$f \in FS$— or $e\theta \equiv let\ Y = e'_1\ in\ e'_2$, and X is a fresh variable. Notice that $e\theta$ is a let-rooted expression or a $f(\overline{e'})$ iff e is a let-rooted expression or a function application, as $\theta \in CTerm$. Then we can apply a let-narrowing step:

$$e \equiv h(e_1, \ldots, e, \ldots, e_n) \rightsquigarrow^l{}_\sigma\ let\ X = e\ in\ h(e_1, \ldots, X, \ldots, e_n) \equiv e''$$

with $\sigma \equiv \epsilon$ and $\theta' \equiv \theta$. Then the conditions in Lemma 11 hold:

i) $e''\theta' \equiv (let\ X = e\ in\ h(e_1, \ldots, X, \ldots, e_n))\theta \equiv$
$let\ X = e\theta\ in\ h(e_1\theta, \ldots, X\theta, \ldots, e_n\theta) \equiv$
$let\ X = e\theta\ in\ h(e_1\theta, \ldots, X, \ldots, e_n\theta) \equiv e'$, since X is fresh an it cannot appear in $dom(\theta')$.

ii) $\sigma\theta' \equiv \epsilon\theta \equiv \theta = \theta[\mathcal{W}]$.

iii) $(dom(\theta') \cup vran(\theta')) \cap \mathcal{B} = (dom(\theta) \cup vran(\theta)) \cap \mathcal{B} = \emptyset$ by hypothesis.

(Bind) In this case $e\theta \equiv let\ X = t\theta\ in\ e_2\theta$ and $e \equiv let\ X = t\ in\ e_2$. Then the let-rewriting step is $let\ X = t\theta\ in\ e_2\theta \to^l e_2\theta[X/t\theta]$ with $t\theta \in CTerm$. As $\theta \in CTerm$, if $t\theta \in CTerm$ then $t \in CTerm$, so we can apply a let-narrowing step:

$$e \equiv let\ X = t\ in\ e_2 \rightsquigarrow^l{}_\sigma\ e_2[X/t] \equiv e''$$

with $\sigma \equiv \epsilon$ and $\theta' \equiv \theta$. Then the conditions in Lemma 11 hold:

i) $e''\theta' \equiv e_2[X/t]\theta$. By the variable convention we can assume that $X \notin dom(\theta) \cup vran(\theta)$, so by Lemma 1 $e_2[X/t]\theta \equiv e_2\theta[X/t\theta] \equiv e'$.

ii) and iii) As before.

(Elim) We have $e\theta \equiv let\ X = e_1\theta\ in\ e_2\theta$, so $e \equiv let\ X = e_1\ in\ e_2$. Then the let-rewriting step is $e\theta \equiv let\ X = e_1\theta\ in\ e_2\theta \to^l e_2\theta$ with $X \notin FV(e_2\theta)$. By the variable convention $(dom(\theta) \cup vran(\theta)) \cap BV(e) = \emptyset$, so as $X \in BV(e)$ then $X \notin dom(\theta) \cup vran(\theta)$. Then $X \notin FV(e_2\theta)$ implies $X \notin FV(e_2)$ and we can apply a let-narrowing step:

$$e \equiv let\ X = e_1\ in\ e_2 \rightsquigarrow^l{}_\sigma\ e_2 \equiv e''$$

with $\sigma \equiv \epsilon$ and $\theta' \equiv \theta$. Then the conditions in Lemma 11 hold trivially.

(Flat) In this case $e\theta \equiv let\ X = (let\ Y = e_1\theta\ in\ e_2\theta)\ in\ e_3\theta$ and $e \equiv let\ X = (let\ Y = e_1\ in\ e_2)\ in\ e_3$. The let-rewriting step is $e\theta \equiv let\ X = (let\ Y = e_1\theta\ in\ e_2\theta)\ in\ e_3\theta \to^l let\ Y = e_1\theta\ in\ let\ X = e_2\theta\ in\ e_3\theta \equiv e'$ with $Y \notin FV(e_3\theta)$. By a similar reasoning as in the (Elim) case we conclude that $Y \notin dom(\theta) \cup$

$vran(\theta)$, so $Y \notin FV(e_3)$. Then we can apply a let-narrowing step:

$$e \equiv let\ X = (let\ Y = e_1\ in\ e_2)\ in\ e_3 \rightsquigarrow^l{}_\sigma\ let\ Y = e_1\ in\ let\ X =\ e_2\ in\ e_3 \equiv e''$$

with $\sigma \equiv \epsilon$ and $\theta' \equiv \theta$. Then the conditions in Lemma 11 hold trivially.

(Contx) Then we have $e \equiv \mathcal{C}[s]$. By the variable convention $(dom(\theta) \cup vran(\theta)) \cap BV(e) = \emptyset$, so by lemma 25 $e\theta \equiv (\mathcal{C}[s])\theta \equiv \mathcal{C}\theta[s\theta]$, and the step was

$$e\theta \equiv \mathcal{C}\theta[s\theta] \to^l \mathcal{C}\theta[s'] \equiv e',\text{ because } s\theta \to^l s'$$

Then we know that the lemma holds for $s\theta \to^l s'$, by the proof of the other cases, so taking $\mathcal{W}' = \mathcal{W} \cup FV(s)$ and $\mathcal{B}' = \mathcal{B}$ (as $BV(s) \subseteq BV(\mathcal{C}[s])$) we can do $s \rightsquigarrow^l{}_{\sigma_2} s''$ for some θ'_2 under the conditions stipulated. Now we can put this step into (Contx) to do:

$$e \equiv \mathcal{C}[s] \rightsquigarrow^l{}_{\sigma_2} \mathcal{C}\sigma_2[s''] \equiv e''\text{ taking } \sigma = \sigma_2 \text{ and } \theta' = \theta'_2$$

because if $s \rightsquigarrow^l{}_{\sigma_2} s''$ was a (Narr) step which lifts a (Fapp) step that uses the fresh variant $(f(\overline{p}) \to r) \in \mathcal{P}$ and adjusts with $\gamma \in CSubst$, then:

- $dom(\sigma_2) \cap BV(\mathcal{C}) = \emptyset$: As $\sigma_2 = mgu(s, f(\overline{p}))$ then $dom(\sigma_2) \subseteq FV(s) \cup FV(f(\overline{p}))$. As $\sigma_2 \lesssim \theta \uplus \gamma$ and it is an mgu then $dom(\sigma_2) \subseteq dom(\theta \uplus \gamma)$. If $X \in FV(s) \cap dom(\sigma_2)$ then $X \notin dom(\gamma) \subseteq FV(f(\overline{p}) \to r)$, so it must happen $X \in dom(\theta)$; but then $X \notin BV(\mathcal{C})$ because $dom(\theta) \cap BV(\mathcal{C}) = \emptyset$ by the variable convention.
  Otherwise it could happen $X \in FV(f(\overline{p})) \cap dom(\sigma_2)$, then X appears in the fresh variant and so it cannot appear in $\mathcal{C}$.
- $vran(\sigma_2|_{\setminus var(\overline{p})}) \cap BV(\mathcal{C}) = \emptyset$: As $dom(\sigma_2) \subseteq FV(s) \cup FV(f(\overline{p}))$ then we have $vran(\sigma_2|_{\setminus var(\overline{p})}) = vran(\sigma_2|_{FV(s)})$. But as $\sigma_2 = mgu(s, f(\overline{p}))$ then $vran(\sigma|_{FV(s)}) \subseteq FV(f(\overline{p}))$, which are part of the fresh variant, so every variable in $vran(\sigma_2|_{\setminus var(\overline{p})})$ is fresh and so cannot appear in $\mathcal{C}$.

Then the conditions in Lemma 11 hold:

ii) $\sigma\theta' = \theta[\mathcal{W}]$: Because $\mathcal{W} \subseteq \mathcal{W}'$, and $\sigma_2\theta'_2 = \theta[\mathcal{W}']$, by the proof of the other cases.

i) $e''\theta' \equiv e'$: As $BV(\mathcal{C}\sigma_2) = BV(\mathcal{C})$, by the variable convention, $BV(\mathcal{C}) \subseteq BV(e) \subseteq BV(\mathcal{B})$, by the hypothesis, and $(dom(\theta'_2) \cup vran(\theta'_2)) \cap \mathcal{B} = \emptyset$, by the proof of the other cases, then $(dom(\theta'_2) \cup vran(\theta'_2)) \cap BV(\mathcal{C}\sigma_2) = \emptyset$. But then:

$$e''\theta' \equiv (\mathcal{C}\sigma_2[s''])\theta'_2 \equiv \underbrace{\mathcal{C}\sigma_2\theta'_2}_{\mathcal{C}\theta}[\underbrace{s''\theta'_2}_{s'}] \equiv e'$$

Because we have $s''\theta'_2 \equiv s'$, by the proof of the other cases, and because $FV(\mathcal{C}) \subseteq FV(e) \subseteq \mathcal{W}$ and $\sigma_2\theta'_2 = \theta[\mathcal{W}]$, as we saw in the previous case (remember $\sigma = \sigma_2$ and $\theta' = \theta'_2$).

iii) $(dom(\theta') \cup vran(\theta')) \cap \mathcal{B} = \emptyset$: Because $\theta' = \theta'_2$ and the proof of the other cases.

The proof for any number of steps proceeds by induction over the number n of steps of the derivation $e\theta \to^{l\,n} e'$. The base case where $n = 0$ is straightforward, as then we have $e\theta \to^{l\,0} e\theta \equiv e'$ so we can do $e \rightsquigarrow^{l\,0}_{\epsilon} e \equiv e''$, so $\sigma = \epsilon$ and taking $\theta' = \theta$ the lemma holds. In the inductive step we have $e\theta \to^{l} e_1 \to^{l^*} e'$, and we will try to build the following diagram:

$$
\begin{tikzcd}
e \arrow[r, squiggly, "l", "\sigma_1"'] \arrow[d, mapsto, "\theta"'] & e''_1 \arrow[r, squiggly, "l^*", "\sigma_2"'] \arrow[d, mapsto, dashed, "\theta_1"'] & e''_2 \equiv e'' \arrow[d, mapsto, dashed, "\theta'_2 = \theta'"] \\
e\theta \arrow[r, "l"'] & e_1 \arrow[r, "*", "l"'] & e'
\end{tikzcd}
$$

By the previous proof for one step we have $e \rightsquigarrow^{l}{}_{\sigma_1} e''_1$ and $\theta'_1 \in CSubst$ under the conditions stipulated. In order to this with the IH we define the sets $\mathcal{B}_1 = \mathcal{B} \cup BV(e_1)$ and $\mathcal{W}_1 = (\mathcal{W} \setminus dom(\sigma_1)) \cup vran(\sigma_1) \cup vE$, where vE is the set of extra variables in the fresh variant $f(\overline{p}) \to s$ used in $e \rightsquigarrow^{l}{}_{\sigma_1} e''_1$, if it was a (Narr) step; or empty otherwise. We also define $\theta_1 = \theta'_1|_{\mathcal{W}_1}$. Then:

- $FV(e''_1) \cup dom(\theta_1) \subseteq \mathcal{W}_1$: We have $dom(\theta_1) \subseteq \mathcal{W}_1$ by definition of θ_1. On the other hand we have $FV(e''_1) \subseteq \mathcal{W}_1$ because given $X \in FV(e''_1)$ we have two possibilities:
  a) $X \in FV(e))$: then $X \notin dom(\sigma_1)$ since otherwise it disappears in the step $e \rightsquigarrow^{l}{}_{\sigma_1} e''$. As $dom(\theta) \cup FV(e) \subseteq \mathcal{W}$ then $X \in \mathcal{W} \setminus dom(\sigma_1)$, so $X \in \mathcal{W}_1$.
  b) $X \notin FV(e))$: then there are two possibilities:
     i) X has been inserted by σ_1, so $X \in vran(\sigma_1)$ and $X \in \mathcal{W}_1$.
     ii) X has been inserted as an extra variable in a (Narr) step. Since the narrowing substitution is a mgu then σ_1 cannot affect X, so $X \in \mathcal{W}_1$ because $X \in vE$.
- $e''_1\theta_1 \equiv e_1$: Because as we have seen, $FV(e''_1) \subseteq \mathcal{W}_1$, and so $e''_1\theta_1 \equiv e''_1\theta'_1|_{\mathcal{W}_1} \equiv e''_1\theta'_1 \equiv e_1$, by the proof for one step.
- $BV(e''_1) \subseteq \mathcal{B}_1$: As $\theta'_1 \in CSubst$, $e''_1\theta'_1 \equiv e_1$ and no $CSubst$ can introduce any binding then $BV(e_1) = BV(e''_1)$. But $\mathcal{B}_1 = \mathcal{B} \cup BV(e_1)$, so $BV(e''_1) = BV(e_1) \subseteq \mathcal{B}_1$.
- $(dom(\theta_1) \cup vran(\theta_1)) \cap \mathcal{B}_1 = \emptyset$: As $\theta'_1 \in CSubst$, $e''_1\theta'_1 \equiv e_1$ and no $CSubst$ can introduce any binding then $BV(e_1) = BV(e''_1)$. Then it can happen:
  a) $BV(e''_1) \subseteq BV(e)$: Then $\mathcal{B} = \mathcal{B}_1$, as $BV(e_1) = BV(e''_1) \subseteq BV(e) \subseteq \mathcal{B}$ by hypothesis. Then, as $(dom(\theta'_1) \cup vran(\theta'_1)) \cap \mathcal{B} = \emptyset$ by the proof for one step, then $(dom(\theta'_1) \cup vran(\theta'_1)) \cap \mathcal{B}_1 = \emptyset$, and so $(dom(\theta_1) \cup vran(\theta_1)) \cap \mathcal{B}_1 = \emptyset$, because $\theta_1 = \theta'_1|_{\mathcal{W}_1}$ and so its domain and variable range is smaller than the domain of θ'_1.
  b) $BV(e''_1) \supset BV(e)$: Then $e \rightsquigarrow^{l}{}_{\sigma_1} e''_1$ must have been a (LetIn) step and so $\sigma = \epsilon$ and $\theta'_1 = \theta$. As the new bounded variable Z is fresh wrt. θ then it is also fresh for $\theta'_1 = \theta$, and so $\mathcal{B}_1 = \mathcal{B} \cup \{Z\}$ has no intersection with $dom(\theta'_1) \cup vran(\theta'_1)$ nor with $dom(\theta_1) \cup vran(\theta_1)$, which is smaller.
- $\sigma_1\theta_1 = \theta[\mathcal{W}]$: It is enough to see that $\sigma_1\theta_1 = \sigma_1\theta'_1[\mathcal{W}]$, because we have $\sigma_1\theta'_1 = \theta[\mathcal{W}]$ by the proof for one step, and this is true because given $X \in \mathcal{W}$:

a) If $X \in dom(\sigma_1)$ then $FV(X\sigma_1) \subseteq vran(\sigma_1) \subseteq \mathcal{W}_1$, so as $\theta_1 = \theta'_1|_{\mathcal{W}_1}$ then $X\sigma_1\theta_1 \equiv X\sigma_1\theta'_1|_{\mathcal{W}_1} \equiv X\sigma_1\theta'_1$.

b) If $X \in \mathcal{W} \setminus dom(\sigma_1)$ then $X \in \mathcal{W}_1$ by definition, and so $X\sigma_1\theta_1 \equiv X\theta_1$ (as $X \notin dom(\sigma_1)$), $X\theta_1 \equiv X\theta'_1|_{\mathcal{W}_1} \equiv X\theta'_1$ (as $X \in \mathcal{W}_1$), and $X\theta'_1 \equiv X\sigma\theta'_1$ (as $X \notin dom(\sigma_1)$).

So we have $e''_1\theta_1 \equiv e_1$ and $e_1 \to^{l^*} e'$, but then we can apply the induction hypothesis to $e''_1\theta_1 \to^{l^*} e'$ using $\mathcal{W}_1$ and $\mathcal{B}_1$, which fulfill the hypothesis of the lemma, as we have seen. Then we get $e''_1 \rightsquigarrow^{l^*}_{\sigma_2} e''_2$ and $\theta'_2 \in CSubst$ under the conditions stipulated. But then we have:

$$e \rightsquigarrow^{l}_{\sigma_1} e''_1 \rightsquigarrow^{l^*}_{\sigma_2} e''_2 \text{ taking } e'' \equiv e''_2,\ \sigma = \sigma_1\sigma_2 \text{ and } \theta' = \theta'_2$$

for which we can prove the conditions in Lemma 11:

i) $e''\theta' \equiv e'$: As $e''\theta' \equiv e''_2\theta'_2 \equiv e'$ by IH.

ii) $\sigma\theta' = \theta[\mathcal{W}]$: That is, $\sigma_1\sigma_2\theta'_2 = \theta[\mathcal{W}]$. As we have $\sigma_1\theta_1 = \theta[\mathcal{W}]$, as we saw before, all that is left is proving $\sigma_1\sigma_2\theta'_2 = \sigma_1\theta_1[\mathcal{W}]$, which happens because given $X \in \mathcal{W}$:

   a) If $X \in dom(\sigma_1)$ then $FV(X\sigma_1) \subseteq vran(\sigma_1) \subseteq \mathcal{W}_1$, so as $\sigma_2\theta'_2 = \theta_1[\mathcal{W}_1]$ by IH, then $(X\sigma_1)\sigma_2\theta'_2 \equiv (X\sigma_1)\theta_1$.

   b) If $X \in \mathcal{W} \setminus dom(\sigma_1)$ then $X \in \mathcal{W}_1$ by definition, and so, as $\sigma_2\theta'_2 = \theta_1[\mathcal{W}_1]$ by IH, then $X\sigma_1\sigma_2\theta'_2 \equiv X\sigma_2\theta'_2$ (as $X \notin dom(\sigma_1)$), $X\sigma_2\theta'_2 \equiv X\theta_1$ (as $X \in \mathcal{W}_1$), $X\theta_1 \equiv X\sigma_1\theta_1$ (as $X \notin dom(\sigma_1)$).

iii) $(dom(\theta') \cup vran(\theta')) \cap \mathcal{B} = \emptyset$: That is $(dom(\theta'_2) \cup vran(\theta'_2)) \cap \mathcal{B} = \emptyset$, which happens as $(dom(\theta'_2) \cup vran(\theta'_2)) \cap \mathcal{B}_1 = \emptyset$ by IH and $\mathcal{B} \subseteq \mathcal{B}_1$.

□

A.10 Proofs for Section 7

The let-binding elimination transformation $\widehat{\ }$ satisfies the following interesting properties, which illustrate that its definition is sound.

Lemma 33
For all $e, e' \in LExp$, $\mathcal{C} \in Cntxt$, $X \in \mathcal{V}$ we have:

i) $|\widehat{e}| \equiv |e|$.
ii) If $e \in Exp$ then $\widehat{e} \equiv e$.
iii) $FV(\widehat{e}) \subseteq FV(e)$
iv) $\widehat{e[X/e']} = \widehat{e}[X/\widehat{e'}]$.

Proof
i–iii) Easily by induction on the structure of e.
iv) A trivial induction on the structure of e, using Lemma 1 for the case when e has the shape $e \equiv let\ X = e_1\ in\ e_2$.

□

Lemma 12 (Copy lemma)
For all $e, e_1, e_2 \in Exp$, $X \in \mathcal{V}$:

i) $e_1 \to e_2$ implies $e[X/e_1] \to^* e[X/e_2]$.
ii) $e_1 \to^* e_2$ implies $e[X/e_1] \to^* e[X/e_2]$.

Proof
To prove *i)* we proceed by induction on the structure of e. Concerning the base cases:

- If $e \equiv X$ then $e[X/e_1] \equiv e_1 \to e_2 \equiv e[X/e_2]$, by hypothesis.
- If $e \equiv Y \in \mathcal{V} \setminus \{X\}$ then $e[X/e_1] \equiv Y \to^0 Y \equiv e[X/e_2]$.
- Otherwise $e \equiv h$ for some $h \in \Sigma$, so $e[X/e_1] \equiv h \to^0 h \equiv e[X/e_2]$

Regarding the inductive step, then $e \equiv h(e'_1, \ldots, e'_n)$ and so

$$\begin{array}{ll} e[X/e_1] \equiv h(e'_1[X/e_1], \ldots, e'_n[X/e_1]) & \\ \to^* h(e'_1[X/e_2], \ldots, e'_n[X/e_2]) & \text{by IH, } n \text{ times} \\ \equiv e[X/e_2] & \end{array}$$

The proof for *ii)* follows the same structure. $\square$

Lemma 13 (One-Step Soundness of let-rewriting wrt. term rewriting)
For all $e, e' \in LExp$ we have that $e \to^l e'$ implies $\widehat{e} \to^* \widehat{e'}$.

Proof
We proceed by a case distinction over the rule of let-rewriting used in the step $e \to^l e'$.

(Fapp) Then we have:

$$e \equiv f(\overline{p})\theta \to^l r\theta \equiv e' \text{ for some } (f(\overline{p}) \to r) \in \mathcal{P}, \theta \in CSubst$$

But then $f(\overline{p})\theta, r\theta \in Exp$, therefore $\widehat{f(\overline{p})\theta} \equiv f(\overline{p})\theta$ and $\widehat{r\theta} \equiv r\theta$, by Lemma 33 *ii)*, and so we can link $\widehat{e} \equiv \widehat{f(\overline{p})\theta} \equiv f(\overline{p})\theta \to r\theta \equiv \widehat{r\theta} \equiv \widehat{e'}$, by a term rewriting step.

(LetIn) Then we have:

$$e \equiv h(e_1, \ldots, e_k, \ldots, e_n) \to^l let\ X = e_k\ in\ h(e_1, \ldots, X, \ldots, e_n) \equiv e'$$

where X is a fresh variable (among other conditions). But then

$$\begin{array}{ll} \widehat{e'} \equiv \widehat{h(e_1, \ldots, X, \ldots, e_n)}[X/\widehat{e_k}] \equiv h(\widehat{e_1}, \ldots, X, \ldots, \widehat{e_n})[X/\widehat{e_k}] & \\ \equiv h(\widehat{e_1}, \ldots, \widehat{e_k}, \ldots, \widehat{e_n}) & \text{as } X \text{ is fresh} \\ \equiv \widehat{h(e_1, \ldots, e_k, \ldots, e_n)} \equiv \widehat{e} & \end{array}$$

Therefore $\widehat{e} \to^0 \widehat{e} \equiv \widehat{e'}$.

(Bind) Then we have:

$$e \equiv let\ X = t\ in\ e_1 \to^l e_1[X/t] \equiv e' \text{ with } t \in CTerm$$

But then $\widehat{e} \equiv \widehat{e_1}[X/\widehat{t}] \equiv \widehat{e_1[X/t]} \equiv \widehat{e'}$, by Lemma 33 *iv)*, hence $\widehat{e} \to^0 \widehat{e} \equiv \widehat{e'}$.

(Elim) Then we have:

$$e \equiv let\ X = e_1\ in\ e_2 \to^l e_2 \equiv e' \text{ with } X \notin FV(e_2)$$

But then

$$\begin{array}{ll} \widehat{e} \equiv \widehat{e_2}[X/\widehat{e_1}] & \\ \equiv \widehat{e_2[X/e_1]} & \text{by Lemma 33 } iv) \\ \equiv \widehat{e_2} \equiv \widehat{e'} & \text{as } X \notin FV(e_2) \end{array}$$

Therefore $\widehat{e} \to^0 \widehat{e} \equiv \widehat{e'}$.

(Flat) Then we have:

$$e \equiv let\ X = (let\ Y = e_1\ in\ e_2)\ in\ e_3 \to^l let\ Y = e_1\ in\ (let\ X = e_2\ in\ e_3) \equiv e'$$

where $Y \notin FV(e_3)$. But then

$$\begin{array}{ll} \widehat{e} \equiv \widehat{e_3}[X/\widehat{let\ Y = e_1\ in\ e_2}] \equiv \widehat{e_3}[X/(\widehat{e_2}[Y/\widehat{e_1}])] & \\ \equiv \widehat{e_3}[X/\widehat{e_2}][Y/\widehat{e_1}] & Y \notin FV(\widehat{e_3}) \text{ by Lemma 33 } iii) \\ \equiv (\widehat{let\ X = e_2\ in\ e_3})[Y/\widehat{e_1}] \equiv \widehat{e'} & \end{array}$$

Therefore $\widehat{e} \to^0 \widehat{e} \equiv \widehat{e'}$.

(Contx) Then we have:

$$e \equiv \mathcal{C}[e_1] \to^l \mathcal{C}[e_2] \equiv e'$$

with $e_1 \to^l e_2$ by some of the previous rules, therefore $\widehat{e_1} \to^* \widehat{e_2}$ by the proof of the previous cases. We will prove that $\widehat{e_1} \to^* \widehat{e_2}$ implies $\widehat{\mathcal{C}[e_1]} \to^* \widehat{\mathcal{C}[e_2]}$, thus getting $\widehat{e} \to^* \widehat{e'}$ as a trivial consequence.

We proceed by induction on the structure of $\mathcal{C}$. Regarding the base case then $\mathcal{C} \equiv []$ and so $\widehat{\mathcal{C}[e_1]} \equiv \widehat{e_1} \to^* \widehat{e_2} \equiv \widehat{\mathcal{C}[e_2]}$ by hypothesis. For the inductive step:

- If $\mathcal{C} \equiv let\ X = \mathcal{C}'\ in\ a$ then by IH we get $\widehat{\mathcal{C}'[e_1]} \to^* \widehat{\mathcal{C}'[e_2]}$, and so

  $$\begin{array}{l} \widehat{\mathcal{C}[e_1]} \equiv \widehat{a}[X/\widehat{\mathcal{C}'[e_1]}] \\ \to^* \widehat{a}[X/\widehat{\mathcal{C}'[e_2]}] \text{ by IH and Lemma 12} \\ \equiv \widehat{\mathcal{C}[e_2]} \end{array}$$

  Notice that it is precisely because of this case that we cannot say that $e \to^l e'$ implies $\widehat{e} \to^* \widehat{e'}$ in zero or one steps, because the copies of $\widehat{\mathcal{C}'[e_1]}$ made by the substitution $[X/\widehat{\mathcal{C}'[e_1]}]$ may force the zero or one steps derivation from $\widehat{\mathcal{C}'[e_1]}$ to be repeated several times in derivation $\widehat{a}[X/\widehat{\mathcal{C}'[e_1]}] \to^* \widehat{a}[X/\widehat{\mathcal{C}'[e_2]}]$. This is typical situation when mimicking term graph rewriting derivations by term rewriting.
- If $\mathcal{C} \equiv let\ X = a\ in\ \mathcal{C}'$ then $\widehat{\mathcal{C}[e_1]} \equiv \widehat{\mathcal{C}'[e_1]}[X/\widehat{a}] \to^* \widehat{\mathcal{C}'[e_2]}[X/\widehat{a}] \equiv \widehat{\mathcal{C}[e_2]}$, by IH combined with closedness under substitutions of term rewriting.
- Otherwise $\mathcal{C} \equiv h(a_1, \ldots, \mathcal{C}', \ldots, a_n)$ and then $\widehat{\mathcal{C}[e_1]} \equiv h(\widehat{a_1}, \ldots, \widehat{\mathcal{C}'[e_1]}, \ldots, \widehat{a_n})$ $\to^* h(\widehat{a_1}, \ldots, \widehat{\mathcal{C}'[e_2]}, \ldots, \widehat{a_n}) \equiv \widehat{\mathcal{C}[e_2]}$ by IH.

$\square$

Proposition 10

For all $\sigma \in Subst_\perp$, $\theta \in [\![\sigma]\!]$, we have that $\theta \trianglelefteq \sigma$.

Proof
Given some $X \in \mathcal{V}$, we have two possibilities. If $X \in dom(\theta)$ then taking any $t \in CTerm_\perp$ such that $\mathcal{P} \vdash_{CRWL} \theta(X) \rightarrowtriangle t$, by Lemma 5 we have $t \sqsubseteq \theta(X)$, because $\theta \in [\![\sigma]\!] \subseteq CSubst_\perp$. But $\theta \in [\![\sigma]\!]$ implies $\mathcal{P} \vdash_{CRWL} \sigma(X) \rightarrowtriangle \theta(X)$, therefore $\mathcal{P} \vdash_{CRWL} \sigma(X) \rightarrowtriangle t$ by the polarity from Proposition 3, which holds for CRWL too. Hence $[\![\theta(X)]\!] \subseteq [\![\sigma(X)]\!]$.

On the other hand, if $X \notin dom(\theta)$ then for any $t \in CTerm_\perp$ such that $\mathcal{P} \vdash_{CRWL} \theta(X) \equiv X \rightarrowtriangle t$ we have that $t \equiv \perp$ or $t \equiv X$. If $t \equiv \perp$ then $\mathcal{P} \vdash_{CRWL} \sigma(X) \rightarrowtriangle t$ by rule (B). Otherwise $\theta \in [\![\sigma]\!]$ implies $\mathcal{P} \vdash_{CRWL} \sigma(X) \rightarrowtriangle \theta(X) \equiv X \equiv t$. Hence $[\![\theta(X)]\!] \subseteq [\![\sigma(X)]\!]$. $\square$

Proposition 11
For all $\sigma \in DSusbt_\perp$, $[\![\sigma]\!]$ is a directed set.

Proof
For any preorder $\leq$, any directed set D wrt. it and any elements $e_1, e_2 \in D$ by $e_1 \sqcup_D e_2$ we denote the element $e_3 \in D$ such that $e_1 \leq e_3$ and $e_2 \leq e_3$ that must exist because D is directed.

Now, given any $\sigma \in DSubst_\perp$ we have that $\forall X \in \mathcal{V}, [\![\sigma(X)]\!]$ is a directed set, because if $X \in dom(\sigma)$ then we can apply the definition of $DSubst_\perp$ and otherwise $[\![X]\!] = \{X, \perp\}$, which is directed. Now given $\theta_1, \theta_2 \in [\![\sigma]\!]$ we can define $\theta_3 \in CSubst_\perp$ as $\theta_3(X) = \theta_1(X) \sqcup_{\sigma(X)} \theta_2(X)$, which fulfills:

1. $\theta_i \sqsubseteq \theta_3$ for $i \in \{1, 2\}$, because for any $X \in \mathcal{V}$ we have that $[\![\sigma(X)]\!]$ is directed (as we saw above) and $\theta_i(X) \in [\![\sigma(X)]\!]$ (because $\theta_1, \theta_2 \in [\![\sigma]\!]$), therefore $\theta_i(X) \sqsubseteq \theta_1(X) \sqcup_{\sigma(X)} \theta_2(X) = \theta_3(X)$ by definition.
2. $\theta_3 \in [\![\sigma]\!]$, because $\forall X \in \mathcal{V}, \theta_3(X) = \theta_1(X) \sqcup_{\sigma(X)} \theta_2(X) \in [\![\sigma(X)]\!]$ by definition.

$\square$

We will use the following lemma about non-triviality of substitution denotations as an auxiliary result for proving Lemma 15.

Lemma 34
For all $\sigma \in Subst_\perp$ we have that $[\![\sigma]\!] \neq \emptyset$ and given $\overline{X} = dom(\sigma)$ then $[\overline{X / \perp}] \in [\![\sigma]\!]$.

Proof
It is enough to prove that if $\overline{X} = dom(\sigma)$ then $[\overline{X/\perp}] \in [\![\sigma]\!]$. First of all $[\overline{X/\perp}] \in CSubst_\perp$ by definition. Now consider some $Y \in \mathcal{V}$.

i) If $Y \in \overline{X}$ then $\sigma(Y) \rightarrowtriangle \perp \equiv Y[\overline{X/\perp}]$, by rule (B).
ii) Otherwise $Y \notin \overline{X} = dom(\sigma)$, hence $\sigma(Y) \equiv Y \rightarrowtriangle Y \equiv Y[\overline{X/\perp}]$, by rule (RR).

$\square$

Lemma 15
For all $\sigma \in DSusbt_\perp$, $e \in Exp_\perp, t \in CTerm_\perp$,

$$\text{if } e\sigma \rightarrowtriangle t \text{ then } \exists \theta \in [\![\sigma]\!] \text{ such that } e\theta \rightarrowtriangle t$$

Proof
We proceed by a case distinction over e:

- If $e \equiv X \in dom(\sigma)$: Then $e\sigma \equiv \sigma(X) \rightarrow t$, so we can define:

$$\theta(Y) = \begin{cases} t & \text{if } Y \equiv X \\ \perp & \text{if } Y \in dom(\sigma) \setminus \{X\} \\ Y & \text{otherwise} \end{cases}$$

  Then $\theta \in [\![\sigma]\!]$ because obviously $\theta \in CSusbt_\perp$, and given $Z \in \mathcal{V}$.

  a) If $Z \equiv X$ then $\sigma(Z) \equiv \sigma(X) \rightarrow t \equiv \theta(Z)$ by hypothesis.
  b) If $Z \in (dom(\sigma) \setminus \{X\})$ then $\sigma(Z) \rightarrow \perp \equiv \theta(Z)$ by rule (B).
  c) Otherwise $Z \notin dom(\sigma)$ and then $\sigma(Z) \equiv Z \rightarrow Z \equiv \theta(Z)$ by rule (RR).

  But then $e\theta \equiv \theta(X) \equiv t \rightarrow t$ by Lemma 5—which also holds for CRWL, because CRWL and CRWL$_{let}$ coincide for c-terms— , as $t \in CTerm_\perp$.
- If $e \equiv X \notin dom(\sigma)$: Then given $\overline{Y} = dom(\sigma)$ we have $[\overline{Y/\perp}] \in [\![\sigma]\!]$ by Lemma 34, so we can take $\theta = \{[\overline{Y/\perp}]\}$ for which $[\![e\sigma]\!] = [\![X\sigma]\!] = [\![X]\!] = [\![X[\overline{Y/\perp}]]\!] = [\![X\theta]\!]$.
- If $e \notin \mathcal{V}$ then we proceed by induction over the structure of $e\sigma \rightarrow t$:

  Base cases

  (B) Then $t \equiv \perp$, so given $\overline{Y} = dom(\sigma)$ we can take $\theta = \{[\overline{Y/\perp}]\}$ for which $e\theta \rightarrow \perp$ by rule (B).

  (RR) Then $e \in \mathcal{V}$ and we are in the previous case.

  (DC) Similar to the case for $e \equiv X \notin dom(\sigma)$.

  Inductive steps

  (DC) Then $e \equiv c(e_1, \ldots, e_n)$, as $e \notin \mathcal{V}$, and we have:

$$\frac{e_1\sigma \rightarrow t_1 \ \ldots \ e_n\sigma \rightarrow t_n}{e\sigma \equiv c(e_1\sigma, \ldots, e_n\sigma) \rightarrow c(t_1, \ldots, t_n) \equiv t}\ DC$$

  Then by IH or the proof of the other cases we have that $\forall i \in \{1, \ldots, n\}$. $\exists \theta_i \in [\![\sigma]\!]$ such that $e_i\theta_i \rightarrow t_i$. But as $\sigma \in DSusbt_\perp$ then $[\![\sigma]\!]$ is directed by Lemma 11, therefore there must exist some $\theta \in [\![\sigma]\!]$ such that $\forall i \in \{1, \ldots, n\}.\theta_i \sqsubseteq \theta$, and so by Proposition 5 —which also holds for CRWL, by Theorem 4— we have $\forall i \in \{1, \ldots, n\}.e_i\theta \rightarrow t_i$, so we can build the following proof:

$$\frac{e_1\theta \rightarrow t_1 \ \ldots e_n\theta \rightarrow t_n}{e\theta \equiv c(e_1\theta, \ldots, e_n\theta) \rightarrow c(t_1, \ldots, t_n) \equiv t}\ DC$$

  (OR) Very similar to the proof of the previous case. We also have $e \equiv f(e_1, \ldots, e_n)$ (as $e \notin \mathcal{V}$) and given a proof for $e\sigma \equiv f(e_1, \ldots, e_n)\sigma \rightarrow t$, so we can apply the IH or the proof of the other cases to every $e_i\sigma \rightarrow p_i\mu$ to get some $\theta_i \in [\![\sigma]\!]$ such that $e_i\theta_i \rightarrow p_i\mu$. Then we can use Lemma 11 and Proposition 5 to use the obtained θ to compute the same values for the arguments of f, thus using the same substitution $\mu \in CSubst_\perp$ for parameter passing in (OR).

□

Theorem 19
Let $\mathcal{P}$ be a CRWL-deterministic program, and $e, e' \in Exp, t \in CTerm$. Then:

a) $e \to^* e'$ implies $e \to^{l^*} e''$ for some $e'' \in LExp$ with $|e''| \sqsupseteq |e'|$.
b) $e \to^* t$ iff $e \to^{l^*} t$ iff $\mathcal{P} \vdash_{CRWL} e \rightarrowtriangle t$.

Proof

a) Assume $e \to^* e'$. By Lemma 16, $[\![e']\!] \subseteq [\![e]\!]$ and by Lemma 5 we have $|e'| \in [\![e']\!]$, then $|e'| \in [\![e]\!]$. Therefore, by Theorem 12 there exists $e'' \in LExp$ such that $e \to^{l^*} e''$ with $|e''| \sqsupseteq |e'|$.
b) The parts $e \to^{l^*} t$ iff $\mathcal{P} \vdash_{CRWL} e \rightarrowtriangle t$, and $e \to^{l^*} t$ implies $e \to^* t$ have been already proved for arbitrary programs in Theorems 12 and 17 respectively. What remains to be proved is that $e \to^* t$ implies $e \to^{l^*} t$ (or the equivalent $\mathcal{P} \vdash_{CRWL} e \rightarrowtriangle t$). Assume $e \to^* t$. Then $[\![t]\!] \subseteq [\![e]\!]$ by Lemma 16. Now, by Lemma 5 $t \in [\![t]\!]$, and therefore $t \in [\![e]\!]$, which exactly means that $\mathcal{P} \vdash_{CRWL} e \rightarrowtriangle t$.

□