

# Dynamic relationship between the mutual interference and gestation delay of a hybrid tritrophic food chain model: Supplementary material

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## Appendix A

### A.1.

Obviously,  $(x(t), y(t), z(t))$  remains non-negative. Therefore, we need to show that  $x(t) \leq 1, y(t) \leq M_2$  and  $z(t) \leq M_3$  only. From the first equation of the model system (2.3), we obtain  $dx/dt \leq x(1 - x)$ . Therefore, by using standard comparison rule, we have  $\limsup_{t \rightarrow +\infty} x(t) \leq 1$ . Now, we define a

function  $\sigma(t) = x(t - \tau_1) + \frac{y(t)}{\omega_6}$ , the time derivative of which is

$$\begin{aligned} \frac{d\sigma(t)}{dt} &= \frac{dx(t - \tau_1)}{dt} + \frac{1}{\omega_6} \frac{dy(t)}{dt} \\ &= x(t - \tau_1)(1 - x(t - \tau_1)) - \frac{x(t - \tau_1)y(t - \tau_1)}{x(t - \tau_1) + \omega_4} \\ &\quad - \frac{\omega_5}{\omega_6} y(t) + \frac{x(t - \tau_1)y(t - \tau_1)}{x(t - \tau_1) + \omega_4} - \frac{y(t)z(t)}{\omega_6(y(t) + (\omega_8 + \omega_9 y(t))z(t) + \omega_{10})}, \\ \frac{d\sigma(t)}{dt} &\leq x(t - \tau_1)(1 - x(t - \tau_1)) - \frac{\omega_5}{\omega_6} y(t) - \omega_5 x(t - \tau_1) + \omega_5 x(t - \tau_1), \\ \frac{d\sigma(t)}{dt} + \omega_5 \sigma(t) &\leq \frac{1}{4} + \omega_5, \text{ since, } \max x(1 - x) = \frac{1}{4}. \end{aligned}$$

Therefore, we can say that for  $t \geq \tilde{T} \geq 0$ ,

$$\sigma(t) \leq \frac{1}{4\omega_5} + 1 - \left( \frac{1}{4\omega_5} + 1 - \sigma(\tilde{T}) \right) e^{-\omega_5(t - \tilde{T})},$$

then, if  $(\tilde{T}) = 0$ ,  $\sigma(t) \leq \frac{1}{4\omega_5} + 1 - \left( \frac{1}{4\omega_5} + 1 - \sigma(0) \right) e^{-\omega_5(t)}$ ,

$$x(t - \tau_1) + \frac{y(t)}{\omega_6} \leq \frac{1}{4\omega_5} + 1,$$

$\frac{y(t)}{\omega_6} \leq \frac{1}{4\omega_5}$ . Therefore,  $y(t) \leq \frac{\omega_6}{4\omega_5}$ . Again, from the third equation of the system, we have, for  $t > \tau_2$ ,  $dz/dt \leq -\omega_{11}z$ . Thus,  $z(t - \tau_2) \geq z(t)e^{\omega_{11}\tau_2}$ , for  $t > \tau_2$ . Therefore, for  $t > \tau_2$ , we have  $\limsup_{t \rightarrow +\infty} z(t) \leq \frac{M_2\omega_{12}}{\omega_{11}(m_2\omega_9 + \omega_8)}$ .

### A.2. Proof of Theorem 4.1.

The method used to prove this theorem is to construct a suitable Lyapunov function and derive sufficient conditions which guarantee that the positive interior equilibrium  $E^*(x^*, y^*, z^*)$  of the model system (2.3) is globally asymptotically stable. For mathematical convenience, we make the following transformations of the variables:

$$x(t) = x^* e^{X(t)}, y(t) = y^* e^{Y(t)}, z(t) = z^* e^{Z(t)}. \quad (A2.1)$$

These coordinate changes transform the positive equilibrium  $E^*$  into the trivial equilibrium  $X(t) = Y(t) = Z(t) = 0$  for all  $t > 0$ . Due to the above transformations, the model system (2.3) is reduced as follows:

$$\frac{dX}{dt} = \frac{x^* y^*}{(x + \omega_4)(x^* + \omega_4)} (e^{X(t)} - 1) - \frac{y^*}{(x + \omega_4)} (e^{Y(t)} - 1), \quad (A2.2)$$

$$\begin{aligned} \frac{dY}{dt} = & -\frac{\omega_6 x^* y^* (e^{Y(t)} - 1)}{y(x^* + \omega_7)} + \frac{\omega_6 y^* x(t - \tau_1) (e^{Y(t - \tau_1)} - 1)}{y(x(t - \tau_1) + \omega_7)} \\ & + \frac{\omega_6 \omega_7 x^* y^* (e^{X(t - \tau_1)} - 1)}{y(x^* + \omega_7)(x(t - \tau_1) + \omega_7)} \\ & + \frac{y^* z^* (1 + \omega_9 z)}{(y^* + (\omega_8 + \omega_9 y^*) z^* + \omega_{10})(y + (\omega_8 + \omega_9 y) z + \omega_{10})} (e^{Y(t)} - 1) \\ & - \frac{z^* (y^* + \omega_{10})}{(y^* + (\omega_8 + \omega_9 y^*) z^* + \omega_{10})(y + (\omega_8 + \omega_9 y) z + \omega_{10})} (e^{Z(t)} - 1), \end{aligned} \quad (A2.3)$$

$$\begin{aligned} \frac{dZ}{dt} = & -\frac{\omega_{12} y^* z^*}{z(y^* + (\omega_8 + \omega_9 y^*) z^* + \omega_{10})} (e^{Z(t)} - 1) \\ & + \frac{\omega_{12} y(t - \tau_2) z^* (y^* + \omega_{10}) (e^{Z(t - \tau_2)} - 1)}{z(y^* + (\omega_8 + \omega_9 y^*) z^* + \omega_{10})(y(t - \tau_2) + (\omega_8 + \omega_9 y(t - \tau_2)) z(t - \tau_2) + \omega_{10})} \\ & + \frac{\omega_{12} y^* z^* (\omega_8 z(t - \tau_2) + \omega_{10}) (e^{Y(t - \tau_2)} - 1)}{z(y^* + (\omega_8 + \omega_9 y^*) z^* + \omega_{10})(y(t - \tau_2) + (\omega_8 + \omega_9 y(t - \tau_2)) z(t - \tau_2) + \omega_{10})}. \end{aligned} \quad (A2.4)$$

Let  $V_1 = |X(t)|$ . Computing the upper right derivative of  $V_1(t)$  along the solutions of (2.3), we get

$$\begin{aligned} D^+V_1(t) &\leq \frac{x^*y^*}{(x+\omega_4)(x^*+\omega_4)}|e^{X(t)}-1| - \frac{y^*}{(x+\omega_4)}|e^{Y(t)}-1| \\ &\leq \frac{x^*y^*}{m_1(x^*+\omega_4)}|e^{X(t)}-1| - \frac{y^*}{M_1}|e^{Y(t)}-1| \end{aligned} \quad (A2.5)$$

Now, Eq. (A2.3) can be rewritten as

$$\begin{aligned} \frac{dY}{dt} &= -\frac{\omega_6x^*y^*}{y(x^*+\omega_7)}(e^{Y(t)}-1) + \frac{\omega_6\omega_7x^*y^*(e^{X(t-\tau_1)}-1)}{y(x^*+\omega_7)(x(t-\tau_1)+\omega_7)} \\ &+ \frac{y^*z^*(1+\omega_9z)}{(y^*+(\omega_8+\omega_9y^*)z^*+\omega_{10})(y+(\omega_8+\omega_9y)z+\omega_{10})}(e^{Y(t)}-1) \\ &- \frac{z^*(y^*+\omega_{10})}{(y^*+(\omega_8+\omega_9y^*)z^*+\omega_{10})(y+(\omega_8+\omega_9y)z+\omega_{10})}(e^{Z(t)}-1) \\ &+ \frac{\omega_6y^*x(t-\tau_1)(e^{Y(t)}-1)}{y(x(t-\tau_1)+\omega_7)} - \frac{\omega_6y^*x(t-\tau_1)}{y(x(t-\tau_1)+\omega_7)} \int_{t-\tau_1}^t e^{Y(s)} \left[ -\frac{\omega_6x^*y^*(e^{Y(s)}-1)}{y(x^*+\omega_7)} \right. \\ &+ \frac{\omega_6\omega_7x^*y^*(e^{X(s-\tau_1)}-1)}{y(x^*+\omega_7)(x(t-\tau_1)+\omega_7)} + \frac{y^*z^*(1+\omega_9z)(e^{Y(s)}-1)}{(y^*+(\omega_8+\omega_9y^*)z^*+\omega_{10})(y+(\omega_8+\omega_9y)z+\omega_{10})} \\ &- \frac{z^*(y^*+\omega_{10})}{(y^*+(\omega_8+\omega_9y^*)z^*+\omega_{10})(y+(\omega_8+\omega_9y)z+\omega_{10})}(e^{Z(s)}-1) \\ &\left. + \frac{\omega_6y^*x(s-\tau_1)(e^{Y(s-\tau_1)}-1)}{y(x(s-\tau_1)+\omega_7)} \right] \end{aligned} \quad (A2.6)$$

In the above equation we have used the following relation

$$e^{Y(t-\tau_1)} = e^{Y(t)} - \int_{t-\tau_1}^t e^{Y(s)} \frac{dY}{ds} ds.$$

Let  $V_2(t) = |y(t)|$ . Computing the upper right derivative of  $V_2(t)$  along the solution of (2.3), it follows from Eq. (A2.6)

$$\begin{aligned} D^+V_2 &\leq -\frac{\omega_6x^*y^*}{M_2(x^*+\omega_7)}|e^{Y(t)}-1| + \frac{\omega_6x^*y^*}{m_2(x^*+\omega_7)}|e^{X(t-\tau_1)}-1| \\ &+ \frac{y^*z^*|e^{Y(t)}-1|}{m_2(y^*+(\omega_8+\omega_9y^*)z^*+\omega_{10})} - \frac{z^*(y^*+\omega_{10})|e^{Z(t)}-1|}{m_2(y^*+(\omega_8+\omega_9y^*)z^*+\omega_{10})} \end{aligned}$$

$$\begin{aligned}
& + \frac{\omega_6 y^*}{m_2} |e^{Y(t)} - 1| + \frac{\omega_6 y^*}{m_2} \times \int_{t-\tau_1}^t e^{Y(s)} \left[ \frac{\omega_6 x^* y^*}{m_2(x^* + \omega_7)} |e^{Y(s)} - 1| \right. \\
& - \frac{\omega_6 x^* y^*}{m_2(x^* + \omega_7)} |e^{X(s-\tau_1)} - 1| - \frac{y^* z^*}{m_2(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} |e^{Y(s)} - 1| \\
& \left. + \frac{z^*(y^* + \omega_{10})}{m_2(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} |e^{Z(s)} - 1| - \frac{\omega_6 y^*}{m_2} |e^{Y(s-\tau_1)} - 1| \right] ds.
\end{aligned} \tag{A2.7}$$

We find that there exists a  $T > 0$ , such that  $y^* e^{Y(t)} < M_2$  for all  $t > T$  and for  $t > T + \tau_1$  we have

$$\begin{aligned}
D^+ V_2 & \leq - \frac{\omega_6 x^* y^*}{M_2(x^* + \omega_7)} |e^{Y(t)} - 1| + \frac{\omega_6 x^* y^*}{m_2(x^* + \omega_7)} |e^{X(t-\tau_1)} - 1| \\
& + \frac{y^* z^* |e^{Y(t)} - 1|}{m_2(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} - \frac{z^*(y^* + \omega_{10}) |e^{Z(t)} - 1|}{m_2(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} \\
& + \frac{\omega_6 y^*}{m_2} |e^{Y(t)} - 1| + \frac{\omega_6 M_2}{m_2} \times \int_{t-\tau_1}^t \left[ \frac{\omega_6 x^* y^*}{m_2(x^* + \omega_7)} |e^{Y(s)} - 1| \right. \\
& - \frac{\omega_6 x^* y^*}{m_2(x^* + \omega_7)} |e^{X(s-\tau_1)} - 1| - \frac{y^* z^*}{m_2(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} |e^{Y(s)} - 1| \\
& \left. + \frac{z^*(y^* + \omega_{10})}{m_2(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} |e^{Z(s)} - 1| - \frac{\omega_6 y^*}{m_2} |e^{Y(s-\tau_1)} - 1| \right] ds.
\end{aligned} \tag{A2.8}$$

Again due to the structure of (A2.8) we consider the following functional

$$\begin{aligned}
V_{22}(t) & = V_2(t) + \frac{\omega_6 M_2}{m_2} \times \int_{t-\tau_1}^t \int_v^t \left[ \frac{\omega_6 x^* y^* |e^{Y(s)} - 1|}{m_2(x^* + \omega_7)} - \frac{\omega_6 x^* y^* |e^{X(s-\tau_1)} - 1|}{m_2(x^* + \omega_7)} \right. \\
& - \frac{y^* z^* |e^{Y(s)} - 1|}{m_2(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} + \frac{z^*(y^* + \omega_{10}) |e^{Z(s)} - 1|}{m_2(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} \\
& \left. - \frac{\omega_6 y^*}{m_2} |e^{Y(s-\tau_1)} - 1| \right] ds dv - \frac{\omega_6^2 M_2 x^* y^* \tau_1}{m_2^2(x^* + \omega_7)} \int_{t-\tau_1}^t |e^{X(s)} - 1| ds \\
& - \frac{\omega_6^2 M_2 y^* \tau_1}{m_2^2} \int_{t-\tau_1}^t |e^{Y(s)} - 1| ds + \frac{\omega_6 x^* y^*}{m_2(x^* + \omega_7)} \int_{t-\tau_1}^t |e^{X(s)} - 1| ds,
\end{aligned}$$

whose upper right derivative along the solutions of the system (2.3) is given

by

$$\begin{aligned}
D^+V_{22}(t) &= D^+V_2 + \frac{\omega_6 M_2 \tau_1}{m_2} \left[ \frac{\omega_6 x^* y^*}{m_2(x^* + \omega_7)} |e^{Y(t)} - 1| - \frac{\omega_6 x^* y^*}{m_2(x^* + \omega_7)} |e^{X(t-\tau_1)} - 1| \right. \\
&\quad - \frac{y^* z^* |e^{Y(t)} - 1|}{m_2(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} + \frac{z^*(y^* + \omega_{10}) |e^{Z(t)} - 1|}{m_2(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} \\
&\quad \left. - \frac{\omega_6 y^*}{m_2} |e^{Y(t-\tau_1)} - 1| \right] - \frac{\omega_6 M_2}{m_2} \times \int_{t-\tau_1}^t \left[ \frac{\omega_6 x^* y^* |e^{Y(s)} - 1|}{m_2(x^* + \omega_7)} - \frac{\omega_6 x^* y^* |e^{X(s-\tau_1)} - 1|}{m_2(x^* + \omega_7)} \right. \\
&\quad - \frac{y^* z^* |e^{Y(s)} - 1|}{m_2(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} + \frac{z^*(y^* + \omega_{10}) |e^{Z(s)} - 1|}{m_2(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} \\
&\quad \left. - \frac{\omega_6 y^*}{m_2} |e^{Y(s-\tau_1)} - 1| \right] ds - \frac{\omega_6^2 M_2 x^* y^* \tau_1}{m_2^2(x^* + \omega_7)} [|e^{X(t)} - 1| - |e^{X(t-\tau_1)} - 1|] \\
&\quad - \frac{\omega_6^2 M_2 y^* \tau_1}{m_2^2} [|e^{Y(t)} - 1| - |e^{Y(t-\tau_1)} - 1|] + \frac{\omega_6 x^* y^*}{m_2(x^* + \omega_7)} [|e^{X(t)} - 1| - |e^{X(t-\tau_1)} - 1|] \\
&\leq \frac{\omega_6 x^* y^*}{m_2(x^* + \omega_7)} \left[ 1 - \frac{\omega_6 M_2 \tau_1}{m_2} \right] |e^{X(t)} - 1| \\
&\quad + \left[ \frac{\omega_6 x^* y^*}{(x^* + \omega_7)} \left( -\frac{1}{M_2} + \frac{\omega_6 M_2 \tau_1}{m_2^2} \right) + \frac{\omega_6 y^*}{m_2} \left( 1 - \frac{\omega_6 M_2 \tau_1}{m_2} \right) \right. \\
&\quad \left. + \frac{y^* z^*}{m_2 \Omega} \left( 1 - \omega_6 M_2 \tau_1 \right) \right] |e^{Y(t)} - 1| + \frac{z^*(y^* + \omega_{10})}{m_2 \Omega} \left[ -1 + \omega_6 M_2 \tau_1 \right] |e^{Z(t)} - 1|.
\end{aligned} \tag{A2.9}$$

Now, Eq. (A2.4) can be rewritten as

$$\begin{aligned}
\frac{dZ}{dt} &= -\frac{\omega_{12} y^* z^*}{z(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} (e^{Z(t)} - 1) \\
&+ \frac{\omega_{12} y^* z^* (\omega_8 z(t - \tau_2) + \omega_{10}) (e^{Y(t-\tau_2)} - 1)}{z(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10}) (y(t - \tau_2) + (\omega_8 + \omega_9 y(t - \tau_2))z(t - \tau_2) + \omega_{10})} \\
&+ \frac{\omega_{12} y(t - \tau_2) z^* (y^* + \omega_{10}) (e^{Z(t)} - 1)}{z(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10}) (y(t - \tau_2) + (\omega_8 + \omega_9 y(t - \tau_2))z(t - \tau_2) + \omega_{10})} \\
&- \frac{\omega_{12} y(t - \tau_2) z^* (y^* + \omega_{10})}{z(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10}) (y(t - \tau_2) + (\omega_8 + \omega_9 y(t - \tau_2))z(t - \tau_2) + \omega_{10})} \\
&\times \int_{t-\tau_2}^t e^{Z(s)} \left[ -\frac{\omega_{12} y^* z^*}{z(y^* + (\omega_8 + \omega_9 y^*)z^* + \omega_{10})} (e^{Z(s)} - 1) \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{\omega_{12}y^*z^*(\omega_8z(s-\tau_2) + \omega_{10})(e^{Y(s-\tau_2)} - 1)}{z(y^* + (\omega_8 + \omega_9y^*)z^* + \omega_{10})(y(s-\tau_2) + (\omega_8 + \omega_9y(s-\tau_2))z(s-\tau_2) + \omega_{10})} \\
& + \frac{\omega_{12}y(s-\tau_2)z^*(y^* + \omega_{10})(e^{Z(s-\tau_2)} - 1)}{z(y^* + (\omega_8 + \omega_9y^*)z^* + \omega_{10})(y(s-\tau_2) + (\omega_8 + \omega_9y(s-\tau_2))z(s-\tau_2) + \omega_{10})} \Big] ds.
\end{aligned}
\tag{A2.10}$$

Again, let  $V_3 = |z(t)|$ . Computing the upper right derivative of  $V_3(t)$  along the solutions of (2.3), we get

$$\begin{aligned}
D^+V_3(t) & \leq -\frac{\omega_{12}y^*z^*}{M_3\Omega}|e^{Z(t)} - 1| + \frac{\omega_{12}y^*z^*}{m_3\Omega}|e^{Y(t-\tau_2)} - 1| + \frac{\omega_{12}z^*(y^* + \omega_{10})}{m_3\Omega}|e^{Z(t)} - 1| \\
& + \frac{\omega_{12}z^*(y^* + \omega_{10})}{m_3\Omega} \times \int_{t-\tau_2}^t e^{Z(s)} \left[ \frac{\omega_{12}y^*z^*}{m_3\Omega}|e^{Z(s)} - 1| \right. \\
& \left. - \frac{\omega_{12}y^*z^*}{m_3\Omega}|e^{Y(s-\tau_2)} - 1| - \frac{\omega_{12}z^*(y^* + \omega_{10})}{m_3\Omega}|e^{Z(s-\tau_2)} - 1| \right] ds.
\end{aligned}$$

We find that there exist a  $T > 0$  such that  $z^*e^{Z(t)} < M_3$  for all  $t > T$  and for  $t > T + \tau_2$ , we have

$$\begin{aligned}
D^+V_3(t) & \leq -\frac{\omega_{12}y^*z^*}{M_3\Omega}|e^{Z(t)} - 1| + \frac{\omega_{12}y^*z^*}{m_3\Omega}|e^{Y(t-\tau_2)} - 1| + \frac{\omega_{12}z^*(y^* + \omega_{10})}{m_3\Omega}|e^{Z(t)} - 1| \\
& + \frac{\omega_{12}M_3(y^* + \omega_{10})}{m_3\Omega} \times \int_{t-\tau_2}^t \left[ \frac{\omega_{12}y^*z^*}{m_3\Omega}|e^{Z(s)} - 1| \right. \\
& \left. - \frac{\omega_{12}y^*z^*}{m_3\Omega}|e^{Y(s-\tau_2)} - 1| - \frac{\omega_{12}z^*(y^* + \omega_{10})}{m_3\Omega}|e^{Z(s-\tau_2)} - 1| \right] ds.
\end{aligned}$$

Again, due to structure of the above equation, we consider the following functional

$$\begin{aligned}
V_{33}(t) & = V_3(t) + \frac{\omega_{12}M_3(y^* + \omega_{10})}{m_3\Omega} \int_{t-\tau_2}^t \int_v^t \left[ \frac{\omega_{12}y^*z^*}{m_3\Omega}|e^{Z(s)} - 1| - \frac{\omega_{12}y^*z^*}{m_3\Omega}|e^{Y(s-\tau_2)} - 1| \right. \\
& \left. - \frac{\omega_{12}z^*(y^* + \omega_{10})}{m_3\Omega}|e^{Z(s-\tau_2)} - 1| \right] ds dv - \frac{\omega_{12}^2y^*z^*M_3(y^* + \omega_{10})\tau_2}{m_3^2\Omega^2} \int_{(t-\tau_2)}^t |e^{Y(s)} - 1| ds \\
& - \frac{\omega_{12}^2z^*M_3(y^* + \omega_{10})^2\tau_2}{m_3^2\Omega^2} \int_{(t-\tau_2)}^t |e^{Z(s)} - 1| ds + \frac{\omega_{12}y^*z^*}{m_3\Omega} \int_{(t-\tau_2)}^t |e^{Y(s)} - 1| ds.
\end{aligned}$$

The upper right derivative along the solutions of the system (2.3) is given by

$$\begin{aligned}
D^+V_{33}(t) &= D^+V_3(t) + \frac{\omega_{12}M_3(y^* + \omega_{10})\tau_2}{m_3\Omega} \left[ \frac{\omega_{12}y^*z^*}{m_3\Omega} |e^{Z(t)} - 1| - \frac{\omega_{12}y^*z^*}{m_3\Omega} |e^{Y(t-\tau_2)} - 1| \right. \\
&\quad \left. - \frac{\omega_{12}z^*(y^* + \omega_{10})}{m_3\Omega} |e^{Z(t-\tau_2)} - 1| \right] - \frac{\omega_{12}M_3(y^* + \omega_{10})}{m_3\Omega} \int_{t-\tau_2}^t \left[ \frac{\omega_{12}y^*z^*}{m_3\Omega} |e^{Z(s)} - 1| \right. \\
&\quad \left. - \frac{\omega_{12}y^*z^*}{m_3\Omega} |e^{Y(s-\tau_2)} - 1| - \frac{\omega_{12}z^*(y^* + \omega_{10})}{m_3\Omega} |e^{Z(s-\tau_2)} - 1| \right] ds \\
&\quad - \frac{\omega_{12}^2y^*z^*M_3(y^* + \omega_{10})\tau_2}{m_3^2\Omega^2} [|e^{Y(t)} - 1| - |e^{Y(t-\tau_2)} - 1|] - \frac{\omega_{12}^2z^*M_3(y^* + \omega_{10})^2\tau_2}{m_3^2\Omega^2} \\
&\quad \times [|e^{Z(t)} - 1| - |e^{Z(t-\tau_2)} - 1|] + \frac{\omega_{12}y^*z^*}{m_3\Omega} [|e^{Y(t)} - 1| - |e^{Y(t-\tau_2)} - 1|] \\
&\leq \frac{\omega_{12}y^*z^*}{m_3\Omega} \left[ 1 - \frac{\omega_{12}M_3(y^* + \omega_{10})\tau_2}{m_3\Omega} \right] |e^{Y(t)} - 1| \\
&\quad + \left[ -\frac{\omega_{12}y^*z^*}{M_3\Omega} + \frac{\omega_{12}z^*(y^* + \omega_{10})}{m_3\Omega} \left( 1 + \frac{\omega_{12}M_3y^*\tau_2}{m_3\Omega} - \frac{\omega_{12}M_3(y^* + \omega_{10})\tau_2}{m_3\Omega} \right) \right] \\
&\quad \times |e^{Z(t)} - 1|. \tag{A2.11}
\end{aligned}$$

We now define a Lyapunov functional  $V(t)$  as

$$V(t) = V_1(t) + V_{22}(t) + V_{33}(t) > |x(t)| + |y(t)| + |z(t)|. \tag{A2.12}$$

Computing the upper right derivative of  $V(t)$  along the solutions of the system (2.3), and by using Eqs. (A2.5), (A2.9) and (A2.11), we get

$$\begin{aligned}
D^+V(t) &= D^+V_1(t) + D^+V_{22}(t) + D^+V_{33}(t) \\
&\leq \left[ \frac{x^*y^*}{m_1(x^* + \omega_4)} + \frac{\omega_6x^*y^*}{m_2(x^* + \omega_7)} \left( 1 - \frac{\omega_6M_2\tau_1}{m_2} \right) \right] |e^{X(t)} - 1| \\
&\quad + \left[ -\frac{y^*}{M_1} + \frac{\omega_6x^*y^*}{(x^* + \omega_7)} \left( -\frac{1}{M_2} + \frac{\omega_6M_2\tau_1}{m_2^2} \right) + \frac{\omega_6y^*}{m_2} \left( 1 - \frac{\omega_6M_2\tau_1}{m_2} \right) \right] \\
&\quad + \frac{y^*z^*}{m_2\Omega} \left( 1 - \omega_6M_2\tau_1 \right) + \frac{\omega_{12}y^*z^*}{m_3\Omega} \left( 1 - \frac{\omega_{12}M_3(y^* + \omega_{10})\tau_2}{m_3\Omega} \right) |e^{Y(t)} - 1| \\
&\quad + \left[ \frac{z^*(y^* + \omega_{10})}{m_2\Omega} \left( -1 + \omega_6M_2\tau_1 \right) - \frac{\omega_{12}y^*z^*}{M_3\Omega} + \frac{\omega_{12}z^*(y^* + \omega_{10})}{m_3\Omega} \right]
\end{aligned}$$

$$\begin{aligned}
& \times \left( 1 + \frac{\omega_{12}M_3y^*\tau_2}{m_3\Omega} - \frac{\omega_{12}M_3(y^* + \omega_{10})\tau_2}{m_3\Omega} \right) \Big] |e^{Z(t)} - 1| \\
& \leq -\gamma_1 x^* |e^{X(t)} - 1| - \gamma_2 y^* |e^{Y(t)} - 1| - \gamma_3 z^* |e^{Z(t)} - 1|,
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &= \left\{ -\frac{y^*}{m_1(x^* + \omega_4)} + \frac{\omega_6 y^*}{m_2(x^* + \omega_7)} \left( -1 + \frac{\omega_6 M_2 \tau_1}{m_2} \right) \right\} > 0, \\
\gamma_2 &= \left\{ \frac{1}{M_1} + \frac{\omega_6 x^*}{(x^* + \omega_7)} \left( \frac{1}{M_2} - \frac{\omega_6 M_2 \tau_1}{m_2^2} \right) - \frac{\omega_6}{m_2} \left( 1 - \frac{\omega_6 M_2 \tau_1}{m_2} \right) \right. \\
&\quad \left. - \frac{z^*}{m_2 \Omega} \left( 1 - \omega_6 M_2 \tau_1 \right) - \frac{\omega_{12} z^*}{m_3 \Omega} \left( 1 - \frac{\omega_{12} M_3 (y^* + \omega_{10}) \tau_2}{m_3 \Omega} \right) \right\} > 0, \\
\gamma_3 &= \left\{ \frac{y^* + \omega_{10}}{m_2 \Omega} \left( 1 - \omega_6 M_2 \tau_1 \right) + \frac{\omega_{12} y^*}{M_3 \Omega} - \frac{\omega_{12} (y^* + \omega_{10})}{m_3 \Omega} \right. \\
&\quad \left. \times \left( 1 + \frac{\omega_{12} M_3 y^* \tau_2}{m_3 \Omega} - \frac{\omega_{12} M_3 (y^* + \omega_{10}) \tau_2}{m_3 \Omega} \right) \right\} > 0.
\end{aligned}$$

Since the model system (2.3) is positive invariant, therefore, for all  $t > T^*$ , we have

$$X^* e^{x(t)} = X(t) > \underline{X}, Y^* e^{y(t)} = Y(t) > \underline{Y}, Z^* e^{z(t)} = Z(t) > \underline{Z}.$$

Using the mean value theorem, we have

$$\begin{aligned}
X^* |e^{x(t)} - 1| &= X^* e^{\theta_1(t)} |x(t)| > m_1 |x(t)|, \\
Y^* |e^{y(t)} - 1| &= Y^* e^{\theta_2(t)} |y(t)| > m_2 |y(t)|, \\
Z^* |e^{z(t)} - 1| &= Z^* e^{\theta_3(t)} |z(t)| > m_3 |z(t)|,
\end{aligned}$$

where  $X^* e^{\theta_1(t)}$  lies between  $X^*$  and  $X(t)$ ,  $Y^* e^{\theta_2(t)}$  lies between  $Y^*$  and  $Y(t)$ ,  $Z^* e^{\theta_3(t)}$  lies between  $Z^*$  and  $Z(t)$ . Therefore,

$$D^+V(t) \leq -\gamma_1 \underline{X} |x(t)| - \gamma_2 \underline{Y} |y(t)| - \gamma_3 \underline{Z} |z(t)| \leq -\eta (|x(t)| + |y(t)| + |z(t)|), \quad (\text{A2.13})$$

where  $\eta = \min \{ \gamma_1 \underline{X}, \gamma_2 \underline{Y}, \gamma_3 \underline{Z} \}$ .

We note that  $V(t) > |x(t)| + |y(t)| + |z(t)|$ . Hence, by using Eq. (A2.13) and Lyapunov direct method, we can conclude that the zero solution of the reduced system (A2.2)-(A2.4) is globally asymptotically stable. Therefore,



the positive equilibrium of the original model system (2.3) is globally asymptotically stable.

### A.3.

$$\begin{aligned}
e_1 &= \frac{2y^*\omega_4}{(x^*+\omega_4)^3}, e_2 = -\frac{\omega_4}{(x^*+\omega_4)^2}, E^* = 0, e_4 = -\frac{6y^*\omega_4}{(x^*+\omega_4)^4}, e_5 = \frac{2\omega_4}{(x^*+\omega_4)^3}, e_6 = 0, \\
e_7 &= 0, p_1 = \frac{2\omega_6y^*\omega_4}{(x^*+\omega_7)^3}, p_2 = -\frac{\omega_4\omega_6}{(x^*+\omega_7)^2}, p_3 = 0, p_4 = -\frac{6y^*\omega_4\omega_6}{(x^*+\omega_7)^4}, p_5 = \frac{2\omega_4\omega_6}{(x^*+\omega_7)^3}, \\
p_6 &= 0, p_7 = 0, q_1 = \frac{2z^*(\omega_8z^*+\omega_{10})(1+\omega_9z^*)}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^3}, q_2 = \frac{-2\omega_8y^*z^*-\omega_{10}y^*-\omega_8\omega_{10}z^*+\omega_9\omega_{10}y^*z^*-\omega_{10}^2}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^3}, \\
q_3 &= \frac{2y^*(y^*+\omega_{10})(\omega_8+\omega_9y^*)}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^3}, q_4 = \frac{-6z^*(\omega_8z^*+\omega_{10})(1+\omega_9z^*)^2}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^4}, \\
q_5 &= \frac{2\{\omega_8z^*(2y^*-\omega_8z^*+2\omega_9y^*z^*)+\omega_9\omega_{10}z^*(2\omega_8z^*-\omega_9y^*z^*+2\omega_{10})+\omega_{10}(y^*+\omega_{10})\}}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^4}, \\
q_6 &= \frac{(-2\omega_8y^*-\omega_8\omega_{10}+\omega_9\omega_{10}y^*)}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^3} - \frac{3(-2\omega_8y^*z^*-\omega_{10}y^*-\omega_8\omega_{10}z^*+\omega_9\omega_{10}y^*z^*-\omega_{10}^2)(\omega_8+\omega_9y^*)}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^4}, \\
q_7 &= \frac{-6y^*(y^*+\omega_{10})(\omega_8+\omega_9y^*)^2}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^4}, r_1 = \frac{2z^*\omega_{12}(\omega_8z^*+\omega_{10})(1+\omega_9z^*)}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^3}, \\
r_2 &= \frac{\omega_{12}(-2\omega_8y^*z^*-\omega_{10}y^*-\omega_8\omega_{10}z^*+\omega_9\omega_{10}y^*z^*-\omega_{10}^2)}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^3}, r_3 = \frac{2y^*\omega_{12}(y^*+\omega_{10})(\omega_8+\omega_9y^*)}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^3}, \\
r_4 &= \frac{-6z^*\omega_{12}(\omega_8z^*+\omega_{10})(1+\omega_9z^*)^2}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^4}, \\
r_5 &= \frac{2\omega_{12}\{\omega_8z^*(2y^*-\omega_8z^*+2\omega_9y^*z^*)+\omega_9\omega_{10}z^*(2\omega_8z^*-\omega_9y^*z^*+2\omega_{10})+\omega_{10}(y^*+\omega_{10})\}}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^4}, \\
r_6 &= \frac{\omega_{12}(-2\omega_8y^*-\omega_8\omega_{10}+\omega_9\omega_{10}y^*)}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^3} - \frac{3\omega_{12}(-2\omega_8y^*z^*-\omega_{10}y^*-\omega_8\omega_{10}z^*+\omega_9\omega_{10}y^*z^*-\omega_{10}^2)(\omega_8+\omega_9y^*)}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^4}, \\
r_7 &= \frac{-6y^*\omega_{12}(y^*+\omega_{10})(\omega_8+\omega_9y^*)^2}{(y^*+\omega_8z^*+\omega_9y^*z^*+\omega_{10})^4}.
\end{aligned}$$

### A.4.

$$a_1 = \frac{i\omega_0 - a_{11}}{a_{12}}, a_2 = \frac{c_{32}e^{-i\omega_0 \frac{\tau_2^*}{\tau_{10}^*}}}{i\omega_0 - a_{33} - c_{33}e^{-i\omega_0 \frac{\tau_2^*}{\tau_{10}^*}}}, a_1^* = -\frac{i\omega_0 + a_{11}}{b_{21}e^{-i\omega_0 \frac{\tau_2^*}{\tau_{10}^*}}}, a_2^* = -\frac{a_1^* a_{23}}{i\omega_0 + a_{33} + c_{33}e^{-i\omega_0 \frac{\tau_2^*}{\tau_{10}^*}}},$$

$M = 1 \setminus N$ , where,

$$\begin{aligned}
N &= 1 + a_1 \bar{a}_1^* + a_2 \bar{a}_2^* + \tilde{\tau}_{10} \left( \bar{a}_1^* b_{21} e^{-i\omega_0 \tilde{\tau}_{10}} \right. \\
&\quad \left. + a_1 (\bar{a}_1^* b_{22} e^{-i\omega_0 \tilde{\tau}_{10}} + \frac{\tau_2^*}{\tilde{\tau}_{10}} \bar{a}_2^* c_{32} e^{-i\omega_0 \frac{\tau_2^*}{\tilde{\tau}_{10}}}) + \frac{\tau_2^*}{\tilde{\tau}_{10}} a_2 \bar{a}_2^* c_{33} e^{-i\omega_0 \frac{\tau_2^*}{\tilde{\tau}_{10}}} \right).
\end{aligned}$$

### A.5.

The expressions of  $g(z, \bar{z})$ ,  $g_{20}$ ,  $g_{11}$ ,  $g_{02}$  and  $g_{21}$  are as follows:

$$\begin{aligned}
g(z, \bar{z}) = & \bar{M}\tilde{\tau}_{10} \left[ z^2 \left\{ -1 + e_1 + a_1 e_2 + a_1^2 E^* + \bar{a}_1^* \left( (p_1 + p_2 a_1 + p_3 a_1^2) e^{-2i\omega_0 \tilde{\tau}_{10}} \right. \right. \right. \\
& + q_1 a_1^2 + q_2 a_1 a_2 + q_3 a_2^2 \left. \left. \right) + \bar{a}_2^* e^{-2i\omega_0 \tau_2^*} \left( r_1 a_1^2 + r_2 a_1 a_2 + r_3 a_2^2 \right) \right\} \\
& + z\bar{z} \left\{ -2 + 2e_1 + e_2 \operatorname{Re}\{a_1\} + 2E^*|a_1|^2 + \bar{a}_1^* \left( 2p_1 + p_2 \operatorname{Re}\{a_1\} + 2p_3|a_1|^2 + 2q_1|a_1|^2 \right. \right. \\
& + q_2(a_1 \bar{a}_2 + \bar{a}_1 a_2) + 2q_3|a_2|^2 \left. \left. \right) + \bar{a}_2^* \left( 2r_1|a_1|^2 + r_2(a_1 \bar{a}_2 + \bar{a}_1 a_2) + 2r_3|a_2|^2 \right) \right\} \\
& + \bar{z}^2 \left\{ -1 + e_1 + \bar{a}_1 e_2 + \bar{a}_1^2 E^* + \bar{a}_1^* \left( (p_1 + p_2 \bar{a}_1 + p_3 \bar{a}_1^2) e^{2i\omega_0 \tilde{\tau}_{10}} \right. \right. \\
& + q_1 \bar{a}_1^2 + q_2 \bar{a}_1 \bar{a}_2 + q_3 \bar{a}_2^2 \left. \left. \right) + \bar{a}_2^* e^{2i\omega_0 \tau_2^*} \left( r_1 \bar{a}_1^2 + r_2 \bar{a}_1 \bar{a}_2 + r_3 \bar{a}_2^2 \right) \right\} \\
& + z^2 \bar{z} \left\{ -2W_{11}^1(0) - W_{20}^1(0) + e_1(2W_{11}^1(0) + W_{20}^1(0)) \right. \\
& + e_2(W_{11}^2(0) + \bar{a}_1 \frac{W_{20}^1(0)}{2} + \frac{W_{20}^2(0)}{2} + a_1 W_{11}^1(0)) + E^*(2a_1 W_{11}^2(0) + \bar{a}_1 W_{20}^2(0)) \\
& + \bar{a}_1^* \left( p_1(2e^{-i\omega_0 \tilde{\tau}_{10}} W_{11}^1(-1) + e^{i\omega_0 \tilde{\tau}_{10}} W_{20}^1(-1)) \right. \\
& + p_2(e^{-i\omega_0 \tilde{\tau}_{10}} (W_{11}^2(-1) + a_1 W_{11}^1(-1)) + e^{i\omega_0 \tilde{\tau}_{10}} (\bar{a}_1 \frac{W_{20}^1(-1)}{2} + \frac{W_{20}^2(-1)}{2})) \\
& + p_3(2a_1 e^{-i\omega_0 \tilde{\tau}_{10}} W_{11}^2(-1) + \bar{a}_1 e^{i\omega_0 \tilde{\tau}_{10}} \frac{W_{20}^2(-1)}{2}) + q_1(2a_1 W_{11}^2(0) + \bar{a}_1 W_{20}^2(0)) \\
& + q_2(a_1 W_{11}^3(0) + \bar{a}_2 \frac{W_{20}^2(0)}{2} + \bar{a}_1 \frac{W_{20}^3(0)}{2} + a_2 W_{11}^2(0)) + q_3(2a_2 W_{11}^3(0) + \bar{a}_2 W_{20}^3(0)) \left. \right) \\
& + \bar{a}_2^* \left( r_1(2a_1 e^{-i\omega_0 \tau_2^*} W_{11}^2(-\frac{\tau_2^*}{\tilde{\tau}_{10}}) + \bar{a}_1 e^{i\omega_0 \tau_2^*} W_{20}^2(-\frac{\tau_2^*}{\tilde{\tau}_{10}})) \right. \\
& + r_2(e^{-i\omega_0 \tau_2^*} (a_1 W_{11}^3(-\frac{\tau_2^*}{\tilde{\tau}_{10}}) + a_2 W_{11}^2(-\frac{\tau_2^*}{\tilde{\tau}_{10}})) + e^{i\omega_0 \tau_2^*} (\bar{a}_2 \frac{W_{20}^2(-\frac{\tau_2^*}{\tilde{\tau}_{10}})}{2} \\
& + \bar{a}_1 \frac{W_{20}^3(-\frac{\tau_2^*}{\tilde{\tau}_{10}})}{2})) + r_3(2a_2 e^{-i\omega_0 \tau_2^*} W_{11}^3(-\frac{\tau_2^*}{\tilde{\tau}_{10}}) + \bar{a}_2 e^{i\omega_0 \tau_2^*} W_{20}^3(-\frac{\tau_2^*}{\tilde{\tau}_{10}})) \left. \right) \left. \right\} \Big]
\end{aligned}$$

$$\begin{aligned}
g_{20} &= 2\bar{M}\tilde{\tau}_{10} \left[ -1 + e_1 + a_1 e_2 + a_1^2 E^* + \bar{a}_1^* \left( (p_1 + p_2 a_1 + p_3 a_1^2) e^{-2i\omega_0 \tilde{\tau}_{10}} \right. \right. \\
&\quad \left. \left. + q_1 a_1^2 + q_2 a_1 a_2 + q_3 a_2^2 \right) + \bar{a}_2^* e^{-2i\omega_0 \tau_2^*} \left( r_1 a_1^2 + r_2 a_1 a_2 + r_3 a_2^2 \right) \right], \\
g_{11} &= \bar{M}\tilde{\tau}_{10} \left[ -2 + 2e_1 + e_2 \operatorname{Re}\{a_1\} + 2E^* |a_1|^2 + \bar{a}_1^* \left( 2p_1 + p_2 \operatorname{Re}\{a_1\} + 2p_3 |a_1|^2 + 2q_1 |a_1|^2 \right. \right. \\
&\quad \left. \left. + q_2 (a_1 \bar{a}_2 + \bar{a}_1 a_2) + 2q_3 |a_2|^2 \right) + \bar{a}_2^* \left( 2r_1 |a_1|^2 + r_2 (a_1 \bar{a}_2 + \bar{a}_1 a_2) + 2r_3 |a_2|^2 \right) \right], \\
g_{02} &= 2\bar{M}\tilde{\tau}_{10} \left[ -1 + e_1 + \bar{a}_1 e_2 + \bar{a}_1^2 E^* + \bar{a}_1^* \left( (p_1 + p_2 \bar{a}_1 + p_3 \bar{a}_1^2) e^{2i\omega_0 \tilde{\tau}_{10}} \right. \right. \\
&\quad \left. \left. + q_1 \bar{a}_1^2 + q_2 \bar{a}_1 \bar{a}_2 + q_3 \bar{a}_2^2 \right) + \bar{a}_2^* e^{2i\omega_0 \tau_2^*} \left( r_1 \bar{a}_1^2 + r_2 \bar{a}_1 \bar{a}_2 + r_3 \bar{a}_2^2 \right) \right], \\
g_{21} &= 2\bar{M}\tilde{\tau}_{10} \left[ -2W_{11}^1(0) - W_{20}^1(0) + e_1 (2W_{11}^1(0) + W_{20}^1(0)) \right. \\
&\quad + e_2 (W_{11}^2(0) + \bar{a}_1 \frac{W_{20}^1(0)}{2} + \frac{W_{20}^2(0)}{2} + a_1 W_{11}^1(0)) + E^* (2a_1 W_{11}^2(0) + \bar{a}_1 W_{20}^2(0)) \\
&\quad + \bar{a}_1^* \left( p_1 (2e^{-i\omega_0 \tilde{\tau}_{10}} W_{11}^1(-1) + e^{i\omega_0 \tilde{\tau}_{10}} W_{20}^1(-1)) \right. \\
&\quad + p_2 (e^{-i\omega_0 \tilde{\tau}_{10}} (W_{11}^2(-1) + a_1 W_{11}^1(-1)) + e^{i\omega_0 \tilde{\tau}_{10}} (\bar{a}_1 \frac{W_{20}^1(-1)}{2} + \frac{W_{20}^2(-1)}{2})) \\
&\quad + p_3 (2a_1 e^{-i\omega_0 \tilde{\tau}_{10}} W_{11}^2(-1) + \bar{a}_1 e^{i\omega_0 \tilde{\tau}_{10}} \frac{W_{20}^2(-1)}{2}) + q_1 (2a_1 W_{11}^2(0) + \bar{a}_1 W_{20}^2(0)) \\
&\quad + q_2 (a_1 W_{11}^3(0) + \bar{a}_2 \frac{W_{20}^2(0)}{2} + \bar{a}_1 \frac{W_{20}^3(0)}{2} + a_2 W_{11}^2(0)) + q_3 (2a_2 W_{11}^3(0) + \bar{a}_2 W_{20}^3(0)) \Big) \\
&\quad + \bar{a}_2^* \left( r_1 (2a_1 e^{-i\omega_0 \tau_2^*} W_{11}^2(-\frac{\tau_2^*}{\tilde{\tau}_{10}}) + \bar{a}_1 e^{i\omega_0 \tau_2^*} W_{20}^2(-\frac{\tau_2^*}{\tilde{\tau}_{10}})) \right. \\
&\quad + r_2 (e^{-i\omega_0 \tau_2^*} (a_1 W_{11}^3(-\frac{\tau_2^*}{\tilde{\tau}_{10}}) + a_2 W_{11}^2(-\frac{\tau_2^*}{\tilde{\tau}_{10}})) + e^{i\omega_0 \tau_2^*} (\bar{a}_2 \frac{W_{20}^2(-\frac{\tau_2^*}{\tilde{\tau}_{10}})}{2} + \bar{a}_1 \frac{W_{20}^3(-\frac{\tau_2^*}{\tilde{\tau}_{10}})}{2})) \\
&\quad \left. \left. + r_3 (2a_2 e^{-i\omega_0 \tau_2^*} W_{11}^3(-\frac{\tau_2^*}{\tilde{\tau}_{10}}) + \bar{a}_2 e^{i\omega_0 \tau_2^*} W_{20}^3(-\frac{\tau_2^*}{\tilde{\tau}_{10}})) \right) \right].
\end{aligned}$$

**A.6.**

$$\begin{aligned}
H_1 &= -1 + e_1 + a_1 e_2 + a_1^2 E^*, \\
H_2 &= (p_1 + p_2 a_1 + p_3 a_1^2) e^{-2i\omega_0 \tilde{\tau}_{10}} + q_1 a_1^2 + q_2 a_1 a_2 + q_3 a_2^2, \\
H_3 &= e^{-2i\omega_0 \tau_2^*} (r_1 a_1^2 + r_2 a_1 a_2 + r_3 a_2^2), \\
P_1 &= -2 + 2e_1 + e_2 \operatorname{Re}\{a_1\} + 2E^* |a_1|^2, \\
P_2 &= 2p_1 + p_2 \operatorname{Re}\{a_1\} + 2p_3 |a_1|^2 + 2q_1 |a_1|^2 + q_2 (a_1 \bar{a}_2 + \bar{a}_1 a_2) + 2q_3 |a_2|^2, \\
P_3 &= 2r_1 |a_1|^2 + r_2 (a_1 \bar{a}_2 + \bar{a}_1 a_2) + 2r_3 |a_2|^2.
\end{aligned}$$

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