

Exact analytical expressions for the final epidemic size of an SIR model on small networks

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Appendix A: Line network, $N = 3$

To verify our analytical expressions for the final epidemic size we ran 2×10^5 stochastic simulations and calculated the frequency of each final size occurring. In Figure A.1 we illustrate what can happen in the stochastic model by plotting the results from three realisations. These results were computed using $\mathcal{R} = 1$ with the initial state SSI . We compute stochastic simulation results for all small networks discussed in this paper, however we omit graphical results for the remaining seven networks.

A.1. Explanation of Figure A.1

Realisation 1: Initially we start the epidemic with two susceptible nodes, one infectious node and zero nodes in the recovered state ($S = 2$, $I = 1$ and $R = 0$). After one time step one susceptible node becomes infected and the initial infectious node recovers ($S = 1$, $I = 1$, $R = 1$). During the next time step the last susceptible node becomes infectious and one infectious node recovers ($S = 0$, $I = 1$ and $R = 2$). Finally the last infectious node recovers which gives a final epidemic size of 3 for this realisation ($S = 0$, $I = 0$ and $R = 3$).

Realisation 2: Initially we start the epidemic with two susceptible nodes, one infectious node and zero nodes in the recovered state ($S = 2$, $I = 1$ and $R = 0$). After one time step one susceptible node becomes infected ($S = 1$, $I = 2$, $R = 0$). During the next time step one infectious node recovers ($S = 1$, $I = 1$, $R = 1$). Finally the last infectious node recovers which gives a final epidemic size of 2 for this realisation ($S = 1$, $I = 0$ and $R = 2$).

Realisation 3: Initially we start the epidemic with two susceptible nodes, one infectious node and zero nodes in the recovered state ($S = 2$, $I = 1$ and $R = 0$). After one time step the infectious node recovers which gives a final epidemic size of 1 for this realisation ($S = 2$, $I = 0$ and $R = 1$).

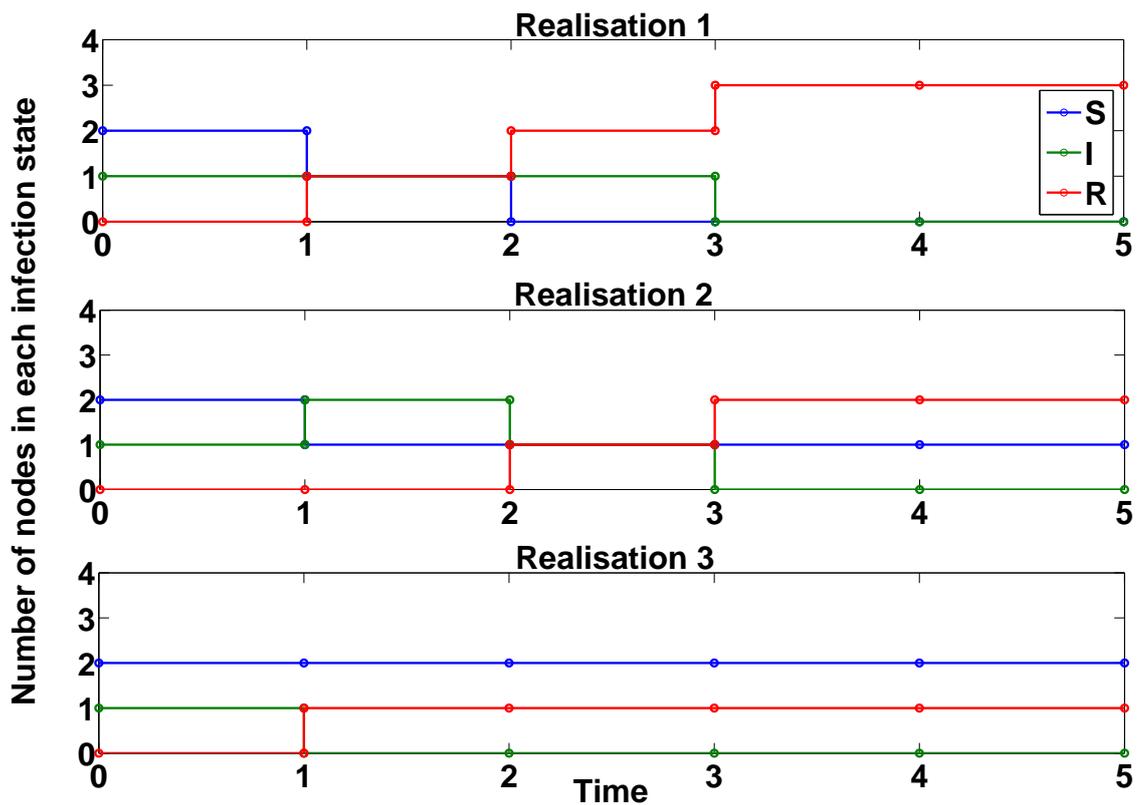


Figure A.1: Illustration of three stochastic realisations. Each graph tracks the number of nodes in each infection state (S , I and R) over time. The resulting final size for each of the realisations are 3, 2 and 1 respectively.

Appendix B: Methods for an SIR model on small networks

B.1. Triangle network

The triangle network is the simplest case of a network of $N = 3$ nodes; it is a complete network and all nodes have degree two. There are 27 possible states which the triangle network can be in for an SIR epidemic process. As we are dealing with a complete network, we can group states together based on those with the same number of nodes in each infection state to reduce the size of the system. In the triangle network, if the epidemic is started with one initial infectious node it does not matter if that is node a , b or c as each node is topologically equivalent. From the transition diagram of the SIR model on the triangle network (Figure B.1) we derive the individual transition probabilities between network states. We obtain the final size probabilities for the SIR epidemic process on the triangle network, given that the epidemic was started with one initial infectious node

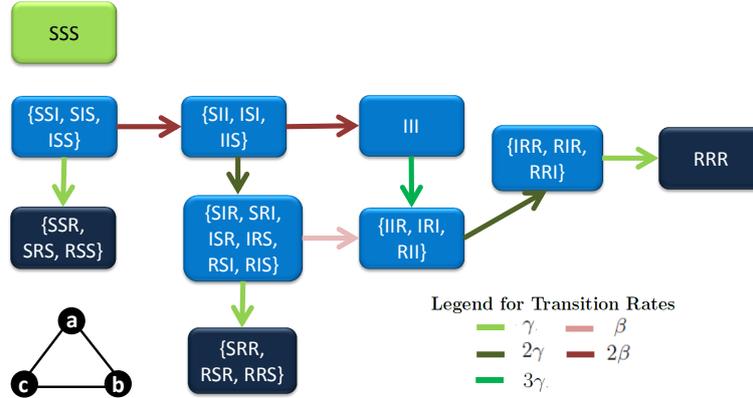


Figure B.1: Triangle (complete) network with $N = 3$ nodes.

shown in Table 1. Note that the probability of the final size equaling one is the same as the infection not taking off, that is the one infectious node recovers before it could infect one of its neighbours.

B.1.1. Catalogue of transition probabilities

From the transition diagram of the *SIR* model on the triangle network we derive the individual transition probabilities between network states. In the following \mathcal{P}_{XYZ} denotes the probability that the network is ever in state XYZ , where X , Y and Z denote the infection state (S , I or R) that nodes a , b and c are in respectively. These probabilities are independent of time and depend only on the infection parameters. From the absorbing state probabilities we find the final epidemic size probabilities, \mathbb{P} .

Possible initial state: $E_{SSI} = 1$

Table 1: Triangle network final size PMFs

Initial State	SSI
$\mathbb{P}(\text{Final Size}=1)$	$\frac{1}{2\mathcal{R}+1}$
$\mathbb{P}(\text{Final Size}=2)$	$\frac{2\mathcal{R}}{(\mathcal{R}+1)^2(2\mathcal{R}+1)}$
$\mathbb{P}(\text{Final Size}=3)$	$\frac{2\mathcal{R}^2(\mathcal{R}+2)}{(\mathcal{R}+1)^2(2\mathcal{R}+1)}$
Expected FS	$\frac{6\mathcal{R}^3+13\mathcal{R}^2+6\mathcal{R}+1}{(\mathcal{R}+1)^2(2\mathcal{R}+1)}$

Probability of passing through transient states:

$$\begin{aligned}\mathcal{P}_{SII} &= \frac{2\mathcal{R}}{2\mathcal{R}+1}E_{SSI} \\ \mathcal{P}_{SIR} &= \frac{2}{2\mathcal{R}+2}\mathcal{P}_{SII} \\ \mathcal{P}_{III} &= \frac{2\mathcal{R}}{2\mathcal{R}+2}\mathcal{P}_{SII} \\ \mathcal{P}_{IIR} &= \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{SIR} + \mathcal{P}_{III} \\ \mathcal{P}_{IRR} &= \mathcal{P}_{IIR}\end{aligned}$$

Probability of terminating in absorbing states:

$$\begin{aligned}\mathcal{P}_{SSR} &= \frac{1}{2\mathcal{R}+1}E_{SSI} \\ \mathcal{P}_{SRR} &= \frac{1}{\mathcal{R}+1}\mathcal{P}_{SIR} \\ \mathcal{P}_{RRR} &= \mathcal{P}_{IIR}\end{aligned}$$

To find the equations for the final size probabilities we evaluated the following:

$$\begin{aligned}\mathbb{P}(\text{Final Size}=1) &= \mathcal{P}_{SSR} \\ \mathbb{P}(\text{Final Size}=2) &= \mathcal{P}_{SRR} \\ \mathbb{P}(\text{Final Size}=3) &= \mathcal{P}_{RRR}\end{aligned}$$

Simplifying the above we derive the analytic expressions for the final size probabilities for the triangle network as shown in Table 1. The probability mass function of the final size distribution for the triangle network is shown in Figure 5 of the main text.

B.1.2. Progression of infection over time

In the following we use P_{XYZ} to denote the probability that the triangle network is in the state XYZ at time t , where X , Y and Z denote the infection state (S , I or R) that nodes a , b and c are in respectively. Thus, the equation for the time derivative \dot{P}_{XYZ} shows how the network can enter and leave the state XYZ . The rate the network enters and leaves each state can be found from the transition diagram. These equations allow us to simulate the time course of the epidemic and to check our final size calculations. Equations describing the probability that the network is in a given state at time t for an SIR model on the triangle network are:

Initial states:

$$\dot{P}_{SSS} = 0 \quad (\text{B.1})$$

$$\dot{P}_{SSI} = -(2\mathcal{R} + 1)P_{SSI} \quad (\text{B.2})$$

Transient states:

$$\dot{P}_{SII} = 2\mathcal{R}P_{SSI} - 2(\mathcal{R} + 1)P_{SII} \quad (\text{B.3})$$

$$\dot{P}_{SIR} = 2P_{SII} - (\mathcal{R} + 1)P_{SIR} \quad (\text{B.4})$$

$$\dot{P}_{IIR} = \mathcal{R}P_{SIR} + 3P_{III} - 2P_{IIR} \quad (\text{B.5})$$

$$\dot{P}_{III} = 2\mathcal{R}P_{SII} - 3P_{III} \quad (\text{B.6})$$

$$\dot{P}_{IRR} = 2P_{IIR} - P_{IRR} \quad (\text{B.7})$$

Absorbing states:

$$\dot{P}_{SSR} = P_{SSI} \quad (\text{B.8})$$

$$\dot{P}_{SRR} = P_{SIR} \quad (\text{B.9})$$

$$\dot{P}_{RRR} = P_{IRR} \quad (\text{B.10})$$

We have included the equation for the initial state SSS for completeness, even though it is disjoint from the transition diagram as no infection is present. For the following networks of four nodes we use the same methods and notation as detailed for the triangle and line networks of three nodes.

B.2. Complete Network

We now examine networks of four nodes, again starting with the simplest case which is the complete network. For an SIR model on a network of $N = 4$ nodes there are 81 possible states that the network can be in. As all nodes in this network are topologically equivalent, we reduce the size of the system by grouping states together based on the number of nodes in each infection state. This results in 15 different states shown in the transition diagram in Figure B.3. We consider one initial condition for the SIR epidemic on a complete network where one node (either a , b , c or d) is the initial infectious node. The final size probabilities for the complete network of 4 nodes are shown in Table 2.

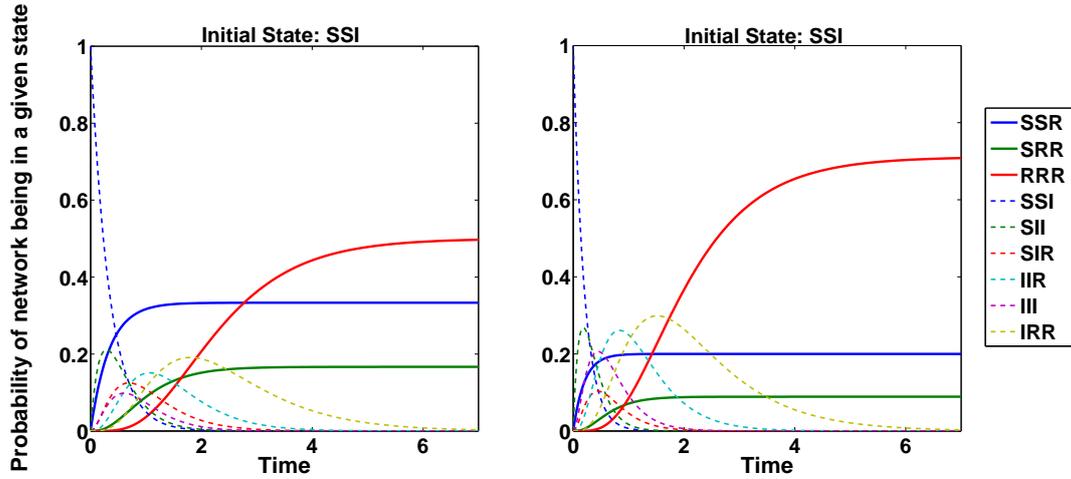


Figure B.2: Numerical results for solving the system of differential equations (B.1) - (B.10) which describes the progression of infection over time for an *SIR* model on a triangle network with $N = 3$ nodes. Left and right columns contain graphical results for $\mathcal{R} = 1$ and $\mathcal{R} = 2$ respectively for the specified initial conditions. The numerical results are in agreement with the analytical expressions for the same set of initial conditions.

B.2.1. Catalogue of transition probabilities

In the following \mathcal{P}_{WXYZ} denotes the probability that the network is ever in state $WXYZ$, where W, X, Y and Z denote the infection state (S, I or R) that nodes a, b, c and d are in respectively.

Possible initial state: $E_{SSSI} = 1$

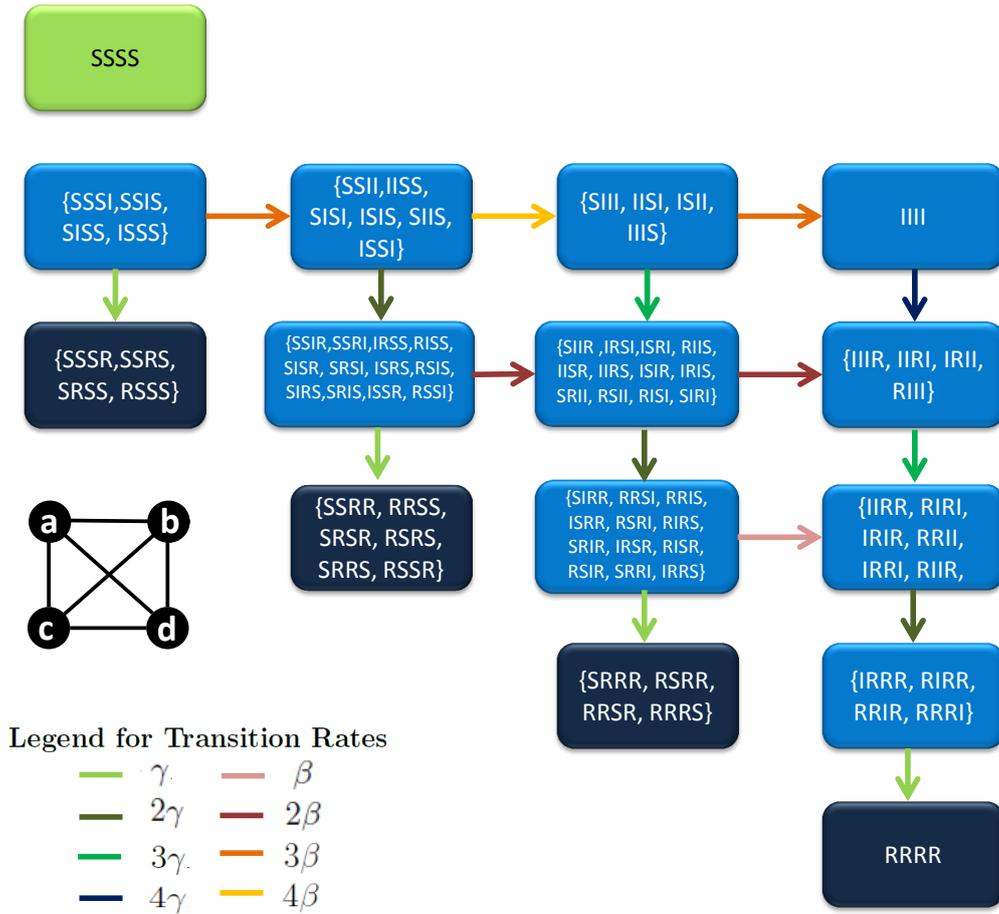


Figure B.3: Complete network with $N = 4$ nodes.

Probability of passing through transient states:

$$\begin{aligned}
 \mathcal{P}_{SSII} &= \frac{3\mathcal{R}}{3\mathcal{R} + 1} E_{SSSI} \\
 \mathcal{P}_{SSIR} &= \frac{2}{4\mathcal{R} + 2} \mathcal{P}_{SSII} \\
 \mathcal{P}_{SIII} &= \frac{4\mathcal{R}}{4\mathcal{R} + 2} \mathcal{P}_{SSII} \\
 \mathcal{P}_{SIIR} &= \frac{3}{3\mathcal{R} + 3} \mathcal{P}_{SIII} + \frac{2\mathcal{R}}{2\mathcal{R} + 1} \mathcal{P}_{SSIR} \\
 \mathcal{P}_{SIRR} &= \frac{2}{2\mathcal{R} + 2} \mathcal{P}_{SIIR} \\
 \mathcal{P}_{IIII} &= \frac{3\mathcal{R}}{3\mathcal{R} + 3} \mathcal{P}_{SIII} \\
 \mathcal{P}_{IIIR} &= \mathcal{P}_{IIII} + \frac{2\mathcal{R}}{2\mathcal{R} + 2} \mathcal{P}_{SIIR} \\
 \mathcal{P}_{IIRR} &= \mathcal{P}_{IIIR} + \frac{\mathcal{R}}{\mathcal{R} + 1} \mathcal{P}_{SIRR} \\
 \mathcal{P}_{IRRR} &= \mathcal{P}_{IIRR}
 \end{aligned}$$

Table 2: Complete network final size PMFs

Initial State	$SSSI$
$\mathbb{P}(\text{Final Size}=1)$	$\frac{1}{3\mathcal{R}+1}$
$\mathbb{P}(\text{Final Size}=2)$	$\frac{3\mathcal{R}}{(2\mathcal{R}+1)^2(3\mathcal{R}+1)}$
$\mathbb{P}(\text{Final Size}=3)$	$\frac{6\mathcal{R}^2(3\mathcal{R}+2)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2(3\mathcal{R}+1)}$
$\mathbb{P}(\text{Final Size}=4)$	$\frac{6\mathcal{R}^3(2\mathcal{R}^3+8\mathcal{R}^2+12\mathcal{R}+5)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2(3\mathcal{R}+1)}$
Expected FS	$\frac{48\mathcal{R}^6+196\mathcal{R}^5+310\mathcal{R}^4+217\mathcal{R}^3+73\mathcal{R}^2+13\mathcal{R}+1}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2(3\mathcal{R}+1)}$

Probability of terminating in absorbing states:

$$\begin{aligned}\mathcal{P}_{SSSR} &= \frac{1}{3\mathcal{R}+1} \mathbb{E}_{SSSI} \\ \mathcal{P}_{SSRR} &= \frac{1}{2\mathcal{R}+1} \mathcal{P}_{SSIR} \\ \mathcal{P}_{SRRR} &= \frac{1}{\mathcal{R}+1} \mathcal{P}_{SIRR} \\ \mathcal{P}_{RRRR} &= \mathcal{P}_{IRRR}\end{aligned}$$

To find the equations for the final size probabilities we evaluated the following:

$$\begin{aligned}\mathbb{P}(\text{Final Size}=1) &= \mathcal{P}_{SSSR} \\ \mathbb{P}(\text{Final Size}=2) &= \mathcal{P}_{SSRR} \\ \mathbb{P}(\text{Final Size}=3) &= \mathcal{P}_{SRRR} \\ \mathbb{P}(\text{Final Size}=4) &= \mathcal{P}_{RRRR}\end{aligned}$$

Simplifying the above we obtained the final size equations for the complete network with four nodes as shown in Table 2.

B.2.2. Progression of infection over time

In the following we use P_{WXYZ} to denote the probability that the complete network is in the state $WXYZ$ at time t , where W , X , Y and Z denote the infection state (S , I or R) that nodes a , b , c and d are in respectively. Thus, the equation for the time derivative \dot{P}_{WXYZ} shows how the network can enter and leave the state $WXYZ$.

Equations describing the probability that the network is in a given state at time t for an SIR model on the complete network of $N = 4$ are:

Initial states:

$$\dot{P}_{SSSS} = 0 \quad (\text{B.11})$$

$$\dot{P}_{SSSI} = -(3\mathcal{R} + 1)P_{SSSI} \quad (\text{B.12})$$

$$(\text{B.13})$$

Transient states:

$$\dot{P}_{SSII} = 3\mathcal{R}P_{SSSI} - (4\mathcal{R} + 2)P_{SSII} \quad (\text{B.14})$$

$$\dot{P}_{SSIR} = 2P_{SSII} - (2\mathcal{R} + 1)P_{SSIR} \quad (\text{B.15})$$

$$\dot{P}_{SIII} = 4\mathcal{R}P_{SSII} - 3(\mathcal{R} + 1)P_{SIII} \quad (\text{B.16})$$

$$\dot{P}_{SIIR} = 3P_{SIII} + 2\mathcal{R}P_{SSIR} - 2(\mathcal{R} + 1)P_{SIIR} \quad (\text{B.17})$$

$$\dot{P}_{SIRR} = 2P_{SIIR} - (\mathcal{R} + 1)P_{SIRR} \quad (\text{B.18})$$

$$\dot{P}_{IIII} = 3\mathcal{R}P_{SIII} - 4P_{IIII} \quad (\text{B.19})$$

$$\dot{P}_{IIIR} = 2\mathcal{R}P_{SIIR} + 4P_{IIII} - 3P_{IIIR} \quad (\text{B.20})$$

$$\dot{P}_{IIRR} = \mathcal{R}P_{SIRR} + 3P_{IIIR} - 2P_{IIRR} \quad (\text{B.21})$$

$$\dot{P}_{IRRR} = 2P_{IIRR} - P_{IRRR} \quad (\text{B.22})$$

Absorbing states:

$$\dot{P}_{SSSR} = P_{SSSI} \quad (\text{B.23})$$

$$\dot{P}_{SSRR} = P_{SSIR} \quad (\text{B.24})$$

$$\dot{P}_{SRRR} = P_{SIRR} \quad (\text{B.25})$$

$$\dot{P}_{RRRR} = P_{IRRR} \quad (\text{B.26})$$

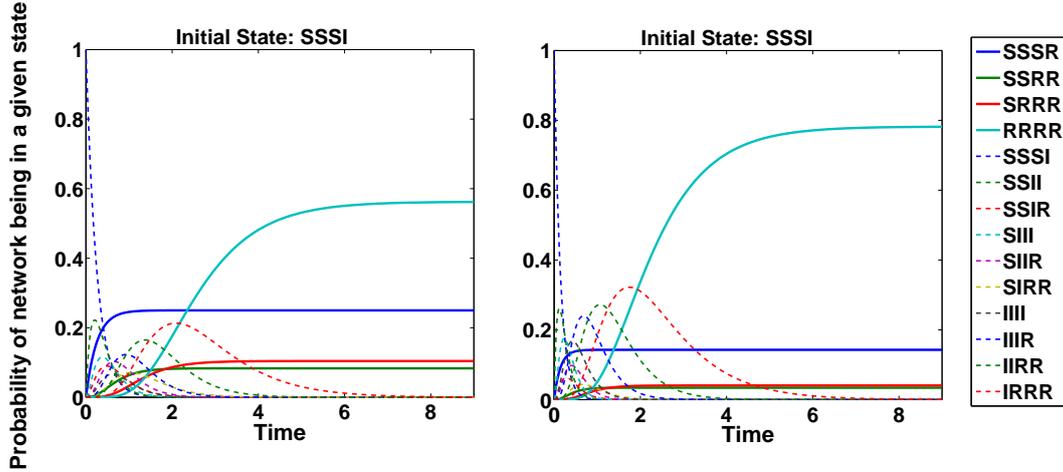


Figure B.4: Numerical results for solving the system of differential equations (B.11) - (B.26) which describes the progression of infection over time for an SIR model on a complete network with $N = 4$ nodes. Left and right columns contain graphical results for $\mathcal{R} = 1$ and $\mathcal{R} = 2$ respectively for the specified initial conditions. The numerical results are in agreement with the analytical expressions for the same set of initial conditions.

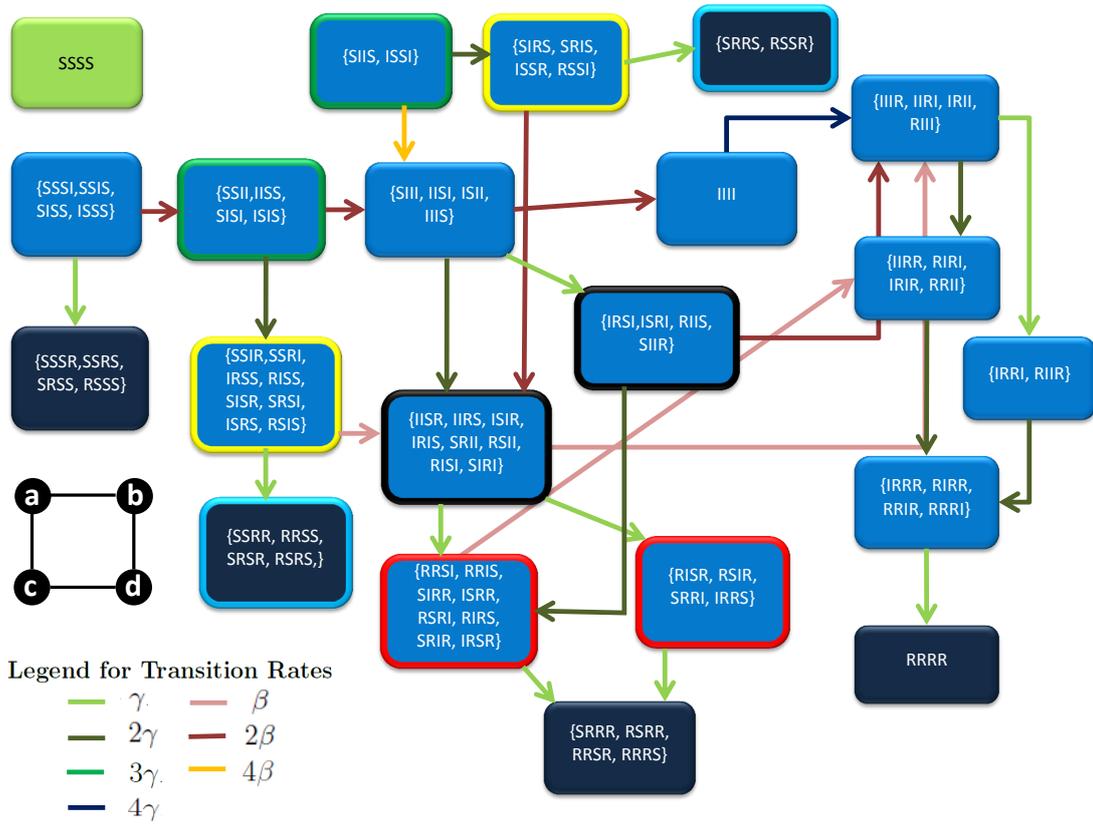
B.3. Square Network

The square network is a regular network in which each node has degree 2. As all nodes are topologically equivalent if we start the epidemic with one initial infectious node it can be either node a , b , c or d . The first initial state, $SSSI$, represents starting the epidemic with one infectious node. The second initial state, $SIIS$, represents starting the epidemic with two infectious nodes which are not neighbours; that is nodes a and d or nodes b and c are the initial infectious nodes (see Figure B.5). We decided to look into the case of starting with two initial infectious nodes (second initial state) within a network as it is not unlikely that two individuals within a population acquire an infection independently of each other and from which an epidemic may occur. We illustrate the probability mass functions of the final size for the square network found with two initial conditions in Table 3.

B.3.1. Catalogue of transition probabilities

Possible initial state indicator variables:

$$E_{SSSI} = \begin{cases} 1, & \text{if initial state is } SSSI. \\ 0, & \text{otherwise.} \end{cases}$$

Figure B.5: Square network with $N = 4$ nodes.

$$E_{SIIS} = \begin{cases} 1, & \text{if initial state is } SIIS. \\ 0, & \text{otherwise.} \end{cases}$$

Probability of passing through transient states:

$$\mathcal{P}_{SSII} = \frac{2\mathcal{R}}{2\mathcal{R} + 1} E_{SSSI}$$

$$\mathcal{P}_{SSIR} = \frac{2}{2\mathcal{R} + 2} \mathcal{P}_{SSII}$$

$$\mathcal{P}_{SIRS} = \frac{2}{4\mathcal{R} + 2} E_{SIIS}$$

$$\begin{aligned}
\mathcal{P}_{SIII} &= \frac{4\mathcal{R}}{4\mathcal{R}+2}E_{SIII} + \frac{2\mathcal{R}}{2\mathcal{R}+2}\mathcal{P}_{SSII} \\
\mathcal{P}_{IISR} &= \frac{2\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{SIRS} + \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{SSIR} + \frac{2}{2\mathcal{R}+2}\mathcal{P}_{SIII} \\
\mathcal{P}_{RRSI} &= \frac{1}{\mathcal{R}+2}\mathcal{P}_{IISR} + \frac{2}{2\mathcal{R}+2}\mathcal{P}_{IRSI} \\
\mathcal{P}_{IIII} &= \frac{2\mathcal{R}}{2\mathcal{R}+3}\mathcal{P}_{SIII} \\
\mathcal{P}_{IRSI} &= \frac{1}{2\mathcal{R}+3}\mathcal{P}_{SIII} \\
\mathcal{P}_{RISR} &= \frac{1}{\mathcal{R}+2}\mathcal{P}_{IISR} \\
\mathcal{P}_{IIIR} &= \mathcal{P}_{IIII} + \frac{2\mathcal{R}}{2\mathcal{R}+2}\mathcal{P}_{IRSI} + \frac{\mathcal{R}}{\mathcal{R}+2}\mathcal{P}_{IISR} \\
\mathcal{P}_{IIRR} &= \frac{2}{3}\mathcal{P}_{IIIR} + \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{RRSI} \\
\mathcal{P}_{IRRI} &= \frac{1}{3}\mathcal{P}_{IIIR} \\
\mathcal{P}_{IRRR} &= \mathcal{P}_{IRRI} + \mathcal{P}_{IIRR}
\end{aligned}$$

Probability of terminating in absorbing states:

$$\begin{aligned}
\mathcal{P}_{SSSR} &= \frac{1}{2\mathcal{R}+1}E_{SSSI} \\
\mathcal{P}_{SSRR} &= \frac{1}{\mathcal{R}+1}\mathcal{P}_{SSIR} \\
\mathcal{P}_{SRRS} &= \frac{1}{2\mathcal{R}+1}\mathcal{P}_{SIRS} \\
\mathcal{P}_{SRRR} &= \mathcal{P}_{RRSI} + \mathcal{P}_{RISR} \\
\mathcal{P}_{RRRR} &= \mathcal{P}_{IRRR}
\end{aligned}$$

To find the equations for the final size probabilities we evaluated the following:

$$\begin{aligned}
\mathbb{P}(\text{Final Size} = 1) &= \mathcal{P}_{SSSR} \\
\mathbb{P}(\text{Final Size} = 2) &= \mathcal{P}_{SSRR} + \mathcal{P}_{SRRS} \\
\mathbb{P}(\text{Final Size} = 3) &= \mathcal{P}_{SRRR} \\
\mathbb{P}(\text{Final Size} = 4) &= \mathcal{P}_{RRRR}
\end{aligned}$$

Simplifying the above we obtained the final size equations for the square network with four nodes as shown in Table 3.

Table 3: Square Network Final Size PMFs

Initial State	<i>SSSI</i>	<i>SIIS</i>
$\mathbb{P}(\text{Final Size}=1)$	$\frac{1}{2\mathcal{R}+1}$	0
$\mathbb{P}(\text{Final Size}=2)$	$\frac{2\mathcal{R}}{(2\mathcal{R}+1)(\mathcal{R}+1)^2}$	$\frac{1}{(2\mathcal{R}+1)^2}$
$\mathbb{P}(\text{Final Size}=3)$	$\frac{4\mathcal{R}^2}{(\mathcal{R}+1)^3(2\mathcal{R}+1)}$	$\frac{2\mathcal{R}(3\mathcal{R}^2+5\mathcal{R}+2)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2}$
$\mathbb{P}(\text{Final Size}=4)$	$1 - \frac{\mathcal{R}^3+9\mathcal{R}^2+5\mathcal{R}+1}{(\mathcal{R}+1)^3(2\mathcal{R}+1)}$	$1 - \frac{7\mathcal{R}^2+6\mathcal{R}+1}{(\mathcal{R}+1)^2(2\mathcal{R}+1)^2}$
Expected FS	$4 - \frac{3\mathcal{R}^3+17\mathcal{R}^2+13\mathcal{R}+3}{(\mathcal{R}+1)^3(2\mathcal{R}+1)}$	$4 - \frac{2}{(\mathcal{R}+1)^2}$

B.3.2. Progression of infection over time

Equations describing the probability that the network is in a given state at time t for an *SIR* model on the square network are:

Initial states:

$$\dot{P}_{SSSS} = 0 \quad (\text{B.27})$$

$$\dot{P}_{SSSI} = -(2\mathcal{R}+1)P_{SSSI} \quad (\text{B.28})$$

$$\dot{P}_{SIIS} = -(4\mathcal{R}+2)P_{SIIS} \quad (\text{B.29})$$

Transient states:

$$\dot{P}_{SSII} = 2\mathcal{R}P_{SSSI} - 2(\mathcal{R}+1)P_{SSII} \quad (\text{B.30})$$

$$\dot{P}_{SSIR} = 2P_{SSII} - (\mathcal{R}+1)P_{SSIR} \quad (\text{B.31})$$

$$\dot{P}_{SIII} = 2\mathcal{R}P_{SSII} + 4\mathcal{R}P_{SIIS} - (2\mathcal{R}+3)P_{SIII} \quad (\text{B.32})$$

$$\dot{P}_{SIRI} = 2P_{SIII} + \mathcal{R}P_{SSIR} + 2\mathcal{R}P_{SIRS} - (\mathcal{R}+2)P_{SIRI} \quad (\text{B.33})$$

$$\dot{P}_{SIRR} = P_{SIRI} + 2P_{SIIR} - (\mathcal{R}+1)P_{SIRR} \quad (\text{B.34})$$

$$\dot{P}_{SRRI} = P_{SIRI} - P_{SRRI} \quad (\text{B.35})$$

$$\dot{P}_{SIIR} = P_{SIII} - 2(\mathcal{R}+1)P_{SIIR} \quad (\text{B.36})$$

$$\dot{P}_{IIII} = 2\mathcal{R}P_{SIII} - 4P_{IIII} \quad (\text{B.37})$$

$$\dot{P}_{IIIR} = 4P_{IIII} + 2\mathcal{R}P_{SIIR} + \mathcal{R}P_{SIRI} - 3P_{IIIR} \quad (\text{B.38})$$

$$\dot{P}_{IIRR} = 2P_{IIIR} + \mathcal{R}P_{SIRR} - 2P_{IIRR} \quad (\text{B.39})$$

$$\dot{P}_{IRRI} = P_{IIRR} - 2P_{IRRI} \quad (\text{B.40})$$

$$\dot{P}_{IRRR} = 2(P_{IIRR} + P_{IRRI}) - P_{IRRR} \quad (\text{B.41})$$

$$\dot{P}_{SIRS} = 2P_{SIIS} - (2\mathcal{R}+1)P_{SIRS} \quad (\text{B.42})$$

$$(\text{B.43})$$

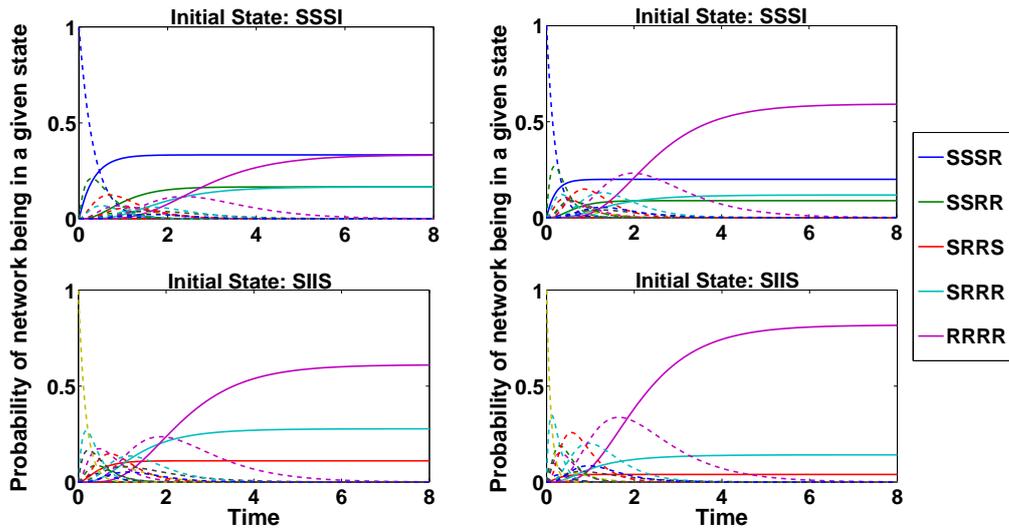


Figure B.6: Numerical results for solving the system of differential equations (B.27) - (B.48) which describes the progression of infection over time for an SIR model on a square network with $N = 4$ nodes. Left and right columns contain graphical results for $\mathcal{R} = 1$ and $\mathcal{R} = 2$ respectively for the specified initial conditions. Dashed lines represent transient states and solid lines are the absorbing states. The numerical results are in agreement with the analytical expressions for the same set of initial conditions.

Absorbing states:

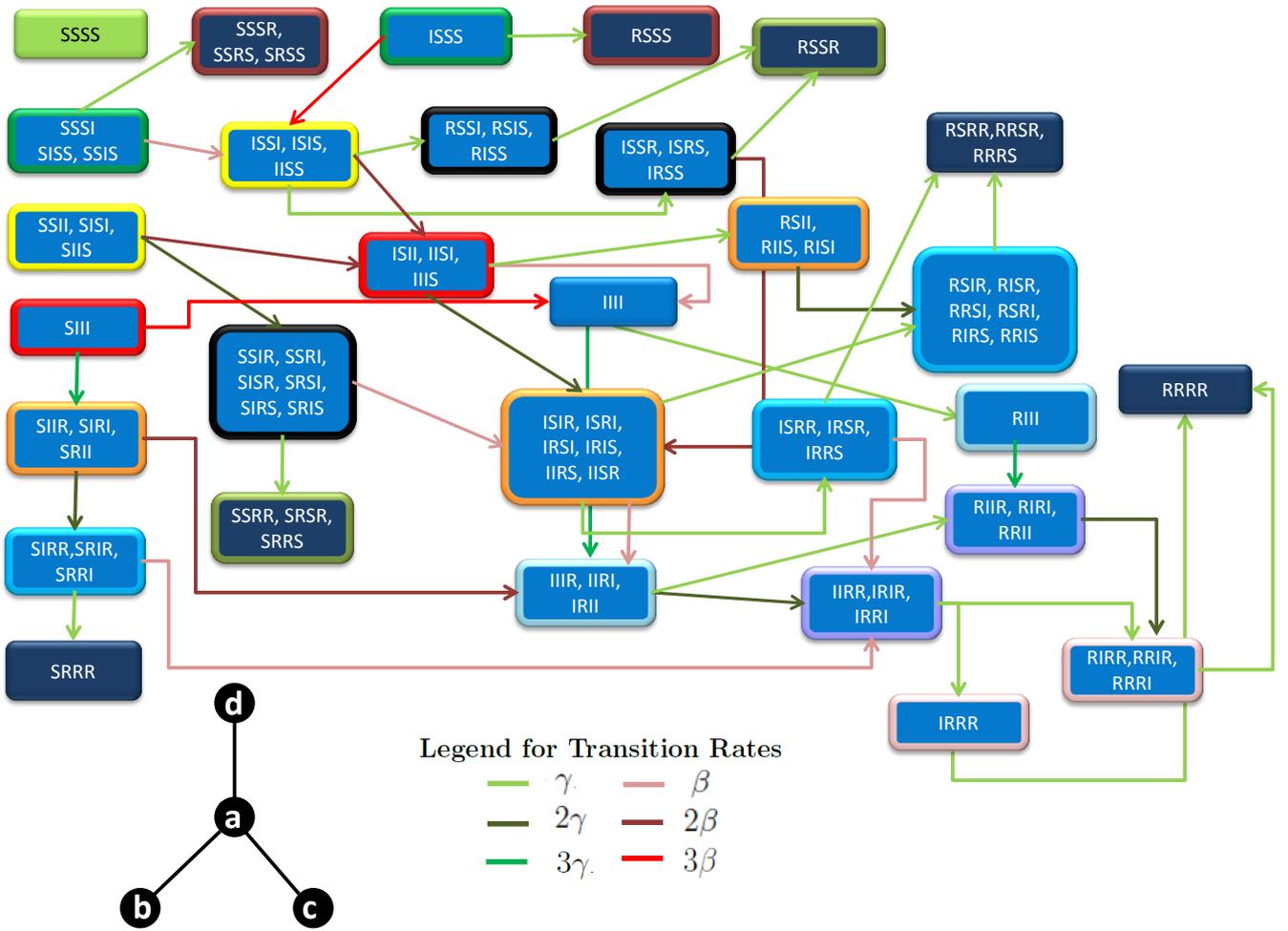
$$\dot{P}_{SSSR} = P_{SSSI} \quad (\text{B.44})$$

$$\dot{P}_{SSRR} = P_{SSIR} \quad (\text{B.45})$$

$$\dot{P}_{SRRS} = P_{SIRS} \quad (\text{B.46})$$

$$\dot{P}_{SRRR} = P_{SIRR} + P_{SRRI} \quad (\text{B.47})$$

$$\dot{P}_{RRRR} = P_{IRRR} \quad (\text{B.48})$$

Figure B.7: Star network with $N = 4$ nodes.

B.4. Star Network

For a star network of N nodes there are two different types of nodes, the centre node with degree $N - 1$ and the $N - 1$ outer nodes each of which have degree one. Here we have a centre node with degree 3 and three outer nodes with degree one. Therefore, the final size probabilities will again depend on which type of node is initially infected. We denote $ISSS$ as the initial state in which the centre node is infectious; $SSSI$, $SSII$ and $SIII$ denote the initial state in which 1, 2 and 3 of the outer nodes are infectious respectively. The probability mass functions of the final size for the star network found with four initial conditions are shown in Table 4.

B.4.1. Catalogue of transition probabilities

Possible initial state indicator variables:

$$E_{SSSI} = \begin{cases} 1, & \text{if initial state is } SSSI. \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{ISSS} = \begin{cases} 1, & \text{if initial state is } ISSS. \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{SSII} = \begin{cases} 1, & \text{if initial state is } SSII. \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{SIII} = \begin{cases} 1, & \text{if initial state is } SIII. \\ 0, & \text{otherwise.} \end{cases}$$

Probability of passing through transient states:

$$\begin{aligned} \mathcal{P}_{SIIR} &= \frac{3}{3\mathcal{R} + 3} E_{SIII} \\ \mathcal{P}_{SIRR} &= \frac{2}{2\mathcal{R} + 2} \mathcal{P}_{SIIR} \\ \mathcal{P}_{ISSI} &= \frac{\mathcal{R}}{\mathcal{R} + 1} E_{SSSI} + \frac{3\mathcal{R}}{3\mathcal{R} + 1} E_{ISSS} \\ \mathcal{P}_{SSIR} &= \frac{2}{2\mathcal{R} + 2} E_{SSII} \\ \mathcal{P}_{RSSI} &= \frac{1}{2\mathcal{R} + 2} \mathcal{P}_{ISSI} \\ \mathcal{P}_{ISII} &= \frac{2\mathcal{R}}{2\mathcal{R} + 2} (\mathcal{P}_{ISSI} + E_{SSII}) \\ \mathcal{P}_{ISSR} &= \frac{1}{2\mathcal{R} + 2} \mathcal{P}_{ISSI} \\ \mathcal{P}_{IIII} &= \frac{\mathcal{R}}{\mathcal{R} + 3} \mathcal{P}_{ISII} + \frac{3\mathcal{R}}{3\mathcal{R} + 3} E_{SIII} \\ \mathcal{P}_{ISIR} &= \frac{2}{\mathcal{R} + 3} \mathcal{P}_{ISII} + \frac{2\mathcal{R}}{2\mathcal{R} + 1} \mathcal{P}_{ISSR} + \frac{\mathcal{R}}{\mathcal{R} + 1} \mathcal{P}_{SSIR} \\ \mathcal{P}_{RSII} &= \frac{1}{\mathcal{R} + 3} \mathcal{P}_{ISII} \\ \mathcal{P}_{ISRR} &= \frac{1}{\mathcal{R} + 2} \mathcal{P}_{ISIR} \end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{RSIR} &= \mathcal{P}_{RSII} + \frac{1}{\mathcal{R}+2} \mathcal{P}_{ISIR} \\
\mathcal{P}_{IIIR} &= \frac{3}{4} \mathcal{P}_{IIII} + \frac{\mathcal{R}}{\mathcal{R}+2} \mathcal{P}_{ISIR} + \frac{2\mathcal{R}}{2\mathcal{R}+2} \mathcal{P}_{SIIR} \\
\mathcal{P}_{RIII} &= \frac{1}{4} \mathcal{P}_{IIII} \\
\mathcal{P}_{RIIR} &= \mathcal{P}_{RIII} + \frac{1}{3} \mathcal{P}_{IIIR} \\
\mathcal{P}_{IIRR} &= \frac{\mathcal{R}}{\mathcal{R}+1} (\mathcal{P}_{ISRR} + \mathcal{P}_{SIRR}) + \frac{2}{3} \mathcal{P}_{IIIR} \\
\mathcal{P}_{IRRR} &= \frac{1}{2} \mathcal{P}_{IIRR} \\
\mathcal{P}_{RIRR} &= \frac{1}{2} \mathcal{P}_{IIRR} + \mathcal{P}_{RIIR}
\end{aligned}$$

Probability of terminating in absorbing states:

$$\begin{aligned}
\mathcal{P}_{SSSR} &= \frac{1}{\mathcal{R}+1} \mathbb{E}_{SSSI} \\
\mathcal{P}_{RSSS} &= \frac{1}{3\mathcal{R}+1} \mathbb{E}_{ISSS} \\
\mathcal{P}_{SSRR} &= \frac{1}{\mathcal{R}+1} \mathcal{P}_{SSIR} \\
\mathcal{P}_{RSSR} &= \mathcal{P}_{RSSI} + \frac{1}{2\mathcal{R}+1} \mathcal{P}_{ISSR} \\
\mathcal{P}_{SRRR} &= \frac{1}{\mathcal{R}+1} \mathcal{P}_{SIRR} \\
\mathcal{P}_{RSRR} &= \mathcal{P}_{RSIR} + \frac{1}{\mathcal{R}+1} \mathcal{P}_{ISRR} \\
\mathcal{P}_{RRRR} &= \mathcal{P}_{IRRR} + \mathcal{P}_{RIRR}
\end{aligned}$$

To find the equations for the final size probabilities we evaluated the following:

$$\begin{aligned}
\mathbb{P}(\text{Final Size} = 1) &= \mathcal{P}_{SSSR} + \mathcal{P}_{RSSS} \\
\mathbb{P}(\text{Final Size} = 2) &= \mathcal{P}_{SSRR} + \mathcal{P}_{RSSR} \\
\mathbb{P}(\text{Final Size} = 3) &= \mathcal{P}_{SRRR} + \mathcal{P}_{RSRR} \\
\mathbb{P}(\text{Final Size} = 4) &= \mathcal{P}_{RRRR}
\end{aligned}$$

Simplifying the above we obtained the final size equations for the star network with four nodes as shown in Table 4.

Table 4: Star Network final size PMFs

Initial State	$SSSI$	$ISSS$	$SSII$	$SIII$
$\mathbb{P}(\text{Final Size}=1)$	$\frac{1}{\mathcal{R}+1}$	$\frac{1}{3\mathcal{R}+1}$	0	0
$\mathbb{P}(\text{Final Size}=2)$	$\frac{\mathcal{R}}{(\mathcal{R}+1)(2\mathcal{R}+1)}$	$\frac{3\mathcal{R}}{(2\mathcal{R}+1)(3\mathcal{R}+1)}$	$\frac{1}{(\mathcal{R}+1)^2}$	0
$\mathbb{P}(\text{Final Size}=3)$	$\frac{2\mathcal{R}^2}{(\mathcal{R}+1)^2(2\mathcal{R}+1)}$	$\frac{6\mathcal{R}^2}{(\mathcal{R}+1)(2\mathcal{R}+1)(3\mathcal{R}+1)}$	$\frac{\mathcal{R}}{(\mathcal{R}+1)^2}$	$\frac{1}{(\mathcal{R}+1)^3}$
$\mathbb{P}(\text{Final Size}=4)$	$\frac{6\mathcal{R}^4 - 5\mathcal{R}^2 - 4\mathcal{R} - 1}{(\mathcal{R}+1)^2(2\mathcal{R}+1)(3\mathcal{R}+1)}$	$\frac{6\mathcal{R}^3}{(\mathcal{R}+1)(2\mathcal{R}+1)(3\mathcal{R}+1)}$	$\frac{\mathcal{R}^2(\mathcal{R}+2)}{(\mathcal{R}+1)^3}$	$1 - \frac{1}{(\mathcal{R}+1)^3}$
Expected FS	$4 - \frac{5\mathcal{R}+3}{(\mathcal{R}+1)^2}$	$4 - \frac{3}{\mathcal{R}+1}$	$4 - \frac{\mathcal{R}^2+4\mathcal{R}+2}{(\mathcal{R}+1)^3}$	$4 - \frac{1}{(\mathcal{R}+1)^3}$

B.4.2. Progression of infection over time

Equations describing the probability that the network is in a given state at time t for an SIR model on the star network are:

Initial states:

$$\dot{P}_{SSSS} = 0 \quad (\text{B.49})$$

$$\dot{P}_{SSSI} = -(\mathcal{R}+1)P_{SSSI} \quad (\text{B.50})$$

$$\dot{P}_{ISSS} = -(3\mathcal{R}+1)P_{ISSS} \quad (\text{B.51})$$

$$\dot{P}_{SSII} = -2(\mathcal{R}+1)P_{SSII} \quad (\text{B.52})$$

$$\dot{P}_{SIII} = -3(\mathcal{R}+1)P_{SIII} \quad (\text{B.53})$$

Transient states:

$$\dot{P}_{ISSI} = 3\mathcal{R}P_{ISSS} + \mathcal{R}P_{SSSI} - 2(\mathcal{R}+1)P_{ISSI} \quad (\text{B.54})$$

$$\dot{P}_{ISII} = 2\mathcal{R}(P_{ISSI} + P_{SSII}) - (\mathcal{R}+3)P_{ISII} \quad (\text{B.55})$$

$$\dot{P}_{SSIR} = 2P_{SSII} - (\mathcal{R}+1)P_{SSIR} \quad (\text{B.56})$$

$$\dot{P}_{RSSI} = P_{ISSI} - P_{RSSI} \quad (\text{B.57})$$

$$\dot{P}_{ISSR} = P_{ISSI} - (2\mathcal{R}+1)P_{ISSR} \quad (\text{B.58})$$

$$\dot{P}_{SIIR} = 3P_{SIII} - 2(\mathcal{R}+1)P_{SIIR} \quad (\text{B.59})$$

$$\dot{P}_{RSII} = P_{ISII} - 2P_{RSII} \quad (\text{B.60})$$

$$\dot{P}_{ISIR} = 2P_{ISII} + \mathcal{R}(P_{SSIR} + 2P_{ISSR}) - (\mathcal{R}+2)P_{ISIR} \quad (\text{B.61})$$

$$\dot{P}_{IIII} = \mathcal{R}(3P_{SIII} + P_{ISII}) - 4P_{IIII} \quad (\text{B.62})$$

$$\dot{P}_{SIRR} = 2P_{SIIR} - (\mathcal{R}+1)P_{SIRR} \quad (\text{B.63})$$

$$\dot{P}_{RSIR} = (2P_{RSII} + P_{ISIR}) - 1P_{RSIR} \quad (\text{B.64})$$

$$\dot{P}_{ISRR} = P_{ISIR} - (\mathcal{R}+1)P_{ISRR} \quad (\text{B.65})$$

$$\dot{P}_{IIIR} = \mathcal{R}(2P_{SIIR} + P_{ISIR}) + 3P_{IIII} - 3P_{IIIR} \quad (\text{B.66})$$

$$\dot{P}_{RIII} = P_{IIII} - 3P_{RIII} \quad (\text{B.67})$$

$$\dot{P}_{RIIR} = (3P_{RIII} + P_{IIIR}) - 2P_{RIIR} \quad (\text{B.68})$$

$$\dot{P}_{IIRR} = \mathcal{R}(P_{ISRR} + P_{SIRR}) + 2P_{IIIR} - 2P_{IIRR} \quad (\text{B.69})$$

$$\dot{P}_{IRRR} = P_{IIRR} - P_{IRRR} \quad (\text{B.70})$$

$$\dot{P}_{RIIR} = (P_{IIRR} + 2P_{RIIR}) - P_{RIIR} \quad (\text{B.71})$$

$$(\text{B.72})$$

Absorbing states:

$$\dot{P}_{SSSR} = P_{SSSI} \quad (\text{B.73})$$

$$\dot{P}_{RSSS} = P_{ISSS} \quad (\text{B.74})$$

$$\dot{P}_{SSRR} = P_{SSIR} \quad (\text{B.75})$$

$$\dot{P}_{RSSR} = (P_{RSSI} + P_{ISSR}) \quad (\text{B.76})$$

$$\dot{P}_{SRRR} = P_{SIRR} \quad (\text{B.77})$$

$$\dot{P}_{RRSR} = (P_{ISRR} + P_{RSIR}) \quad (\text{B.78})$$

$$\dot{P}_{RRRR} = (P_{IRRR} + P_{RIRR}) \quad (\text{B.79})$$

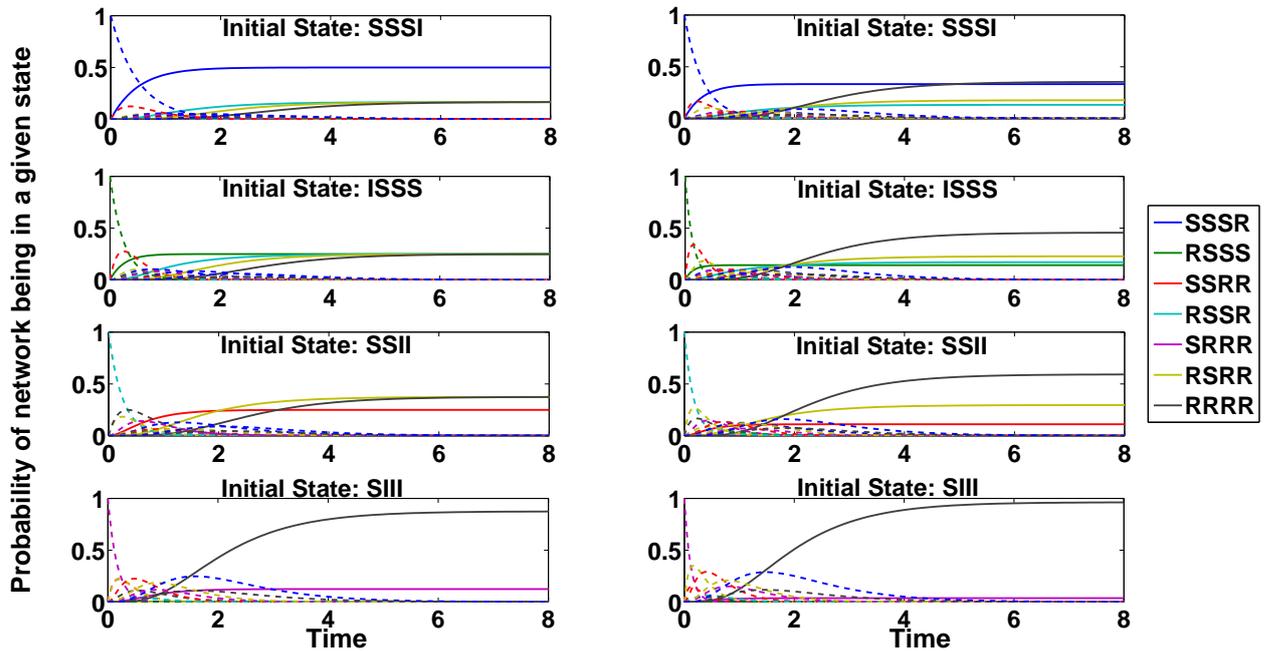


Figure B.8: Numerical results for solving the system of differential equations (B.49) - (B.79) which describes the progression of infection over time for an *SIR* model on a star network with $N = 4$ nodes. Left and right columns contain graphical results for $\mathcal{R} = 1$ and $\mathcal{R} = 2$ respectively for the specified initial conditions. Dashed lines represent transient states and solid lines are the absorbing states. The numerical results are in agreement with the analytical expressions for the same set of initial conditions.

B.5. Toast Network

The toast network is simply a square network with one diagonal edge through it. There are two different types of nodes in the toast network, two nodes of degree 2 (nodes b and

c) and two nodes of degree 3 (nodes a and d). The probability mass functions of the final size for the toast network found with three initial conditions are shown in Table 5. We denote $SSIS$ as the initial state in which node b or c is infectious; $SSSI$ as the initial state in which node a or d is infectious and $SIIS$ as the initial state in which nodes b and c are infectious.

B.5.1. Catalogue of transition probabilities

Possible initial state indicator variables:

$$E_{SISS} = \begin{cases} 1, & \text{if initial state is } SISS. \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{ISSS} = \begin{cases} 1, & \text{if initial state is } ISSS. \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{SIIS} = \begin{cases} 1, & \text{if initial state is } SIIS. \\ 0, & \text{otherwise.} \end{cases}$$

Probability of passing through transient states:

$$\mathcal{P}_{IISS} = \frac{2\mathcal{R}}{2\mathcal{R}+1}E_{SISS} + \frac{2\mathcal{R}}{3\mathcal{R}+1}\mathcal{P}_{SSSI}$$

$$\mathcal{P}_{ISSI} = \frac{\mathcal{R}}{3\mathcal{R}+1}\mathcal{P}_{SSSI}$$

$$\mathcal{P}_{RSSI} = \frac{2}{4\mathcal{R}+2}\mathcal{P}_{ISSI}$$

$$\mathcal{P}_{SRIS} = \frac{2}{4\mathcal{R}+2}E_{SIIS}$$

$$\mathcal{P}_{IIIS} = \frac{4\mathcal{R}}{4\mathcal{R}+2}E_{SIIS} + \frac{\mathcal{R}}{3\mathcal{R}+2}\mathcal{P}_{IISS}$$

$$\mathcal{P}_{ISII} = \frac{2\mathcal{R}}{3\mathcal{R}+2}\mathcal{P}_{IISS} + \frac{4\mathcal{R}}{4\mathcal{R}+2}\mathcal{P}_{ISSI}$$

$$\mathcal{P}_{SSIR} = \frac{1}{3\mathcal{R}+2}\mathcal{P}_{IISS}$$

$$\mathcal{P}_{IRSS} = \frac{1}{3\mathcal{R}+2}\mathcal{P}_{IISS}$$

$$\mathcal{P}_{SIIR} = \frac{1}{3\mathcal{R}+3}\mathcal{P}_{IIIS}$$

$$\mathcal{P}_{ISRI} = \frac{1}{2\mathcal{R}+3}\mathcal{P}_{ISII} + \frac{\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{IRSS}$$

$$\begin{aligned}
\mathcal{P}_{ISIR} &= \frac{2}{2\mathcal{R}+3}\mathcal{P}_{ISII} + \frac{2\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{RSSI} + \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{SSIR} \\
\mathcal{P}_{IRIS} &= \frac{2}{3\mathcal{R}+3}\mathcal{P}_{IIIS} + \frac{2\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{SRIS} + \frac{\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{IRSS} \\
\mathcal{P}_{IIII} &= \frac{3\mathcal{R}}{3\mathcal{R}+3}\mathcal{P}_{IIIS} + \frac{2\mathcal{R}}{2\mathcal{R}+3}\mathcal{P}_{ISII} \\
\mathcal{P}_{SIRR} &= \frac{2}{2\mathcal{R}+2}\mathcal{P}_{SIIR} + \frac{1}{2\mathcal{R}+2}\mathcal{P}_{IRIS} \\
\mathcal{P}_{ISRR} &= \frac{2}{2\mathcal{R}+2}\mathcal{P}_{ISRI} + \frac{1}{\mathcal{R}+2}\mathcal{P}_{ISIR} \\
\mathcal{P}_{SRRI} &= \frac{1}{2\mathcal{R}+2}\mathcal{P}_{IRIS} \\
\mathcal{P}_{RSIR} &= \frac{1}{\mathcal{R}+2}\mathcal{P}_{ISIR} \\
\mathcal{P}_{IIIR} &= \frac{2\mathcal{R}}{2\mathcal{R}+2}\mathcal{P}_{SIIR} + \frac{1}{2}\mathcal{P}_{IIII} + \frac{\mathcal{R}}{\mathcal{R}+2}\mathcal{P}_{ISIR} \\
\mathcal{P}_{IRII} &= \frac{1}{2}\mathcal{P}_{IIII} + \frac{2\mathcal{R}}{2\mathcal{R}+2}\mathcal{P}_{ISRI} + \frac{2\mathcal{R}}{2\mathcal{R}+2}\mathcal{P}_{IRIS} \\
\mathcal{P}_{IRIR} &= \frac{\mathcal{R}}{\mathcal{R}+1}(\mathcal{P}_{SIRR} + \mathcal{P}_{ISRR}) + \frac{2}{3}(\mathcal{P}_{IRII} + \mathcal{P}_{IIIR}) \\
\mathcal{P}_{IRRI} &= \frac{1}{3}\mathcal{P}_{IRII} + \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{SRRI} \\
\mathcal{P}_{RIIR} &= \frac{1}{3}\mathcal{P}_{IIIR} \\
\mathcal{P}_{IRRR} &= \frac{1}{2}\mathcal{P}_{IRIR} + \mathcal{P}_{IRRI} \\
\mathcal{P}_{RIRR} &= \frac{1}{2}\mathcal{P}_{IRIR} + \mathcal{P}_{RIIR}
\end{aligned}$$

Table 5: Toast network final size PMFs

Initial State	<i>SSIS</i>	<i>SSSI</i>	<i>SIIS</i>
$\mathbb{P}(\text{Final Size}=1)$	$\frac{1}{2\mathcal{R}+1}$	$\frac{1}{3\mathcal{R}+1}$	0
$\mathbb{P}(\text{Final Size}=2)$	$\frac{2\mathcal{R}}{(\mathcal{R}+1)(2\mathcal{R}+1)^2}$	$\frac{\mathcal{R}(5\mathcal{R}+3)}{(\mathcal{R}+1)(2\mathcal{R}+1)^2(3\mathcal{R}+1)}$	$\frac{1}{(2\mathcal{R}+1)^2}$
$\mathbb{P}(\text{Final Size}=3)$	$\frac{2\mathcal{R}^2(2\mathcal{R}+3)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2}$	$\frac{2\mathcal{R}^2(7\mathcal{R}^2+13\mathcal{R}+5)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2(3\mathcal{R}+1)}$	$\frac{2\mathcal{R}(3\mathcal{R}+2)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2}$
$\mathbb{P}(\text{Final Size}=4)$	$\frac{2\mathcal{R}^3(2\mathcal{R}+3)(\mathcal{R}+2)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2}$	$\frac{2\mathcal{R}^3(6\mathcal{R}^3+24\mathcal{R}^2+28\mathcal{R}+9)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2(3\mathcal{R}+1)}$	$1 - \frac{\mathcal{R}^3+9\mathcal{R}^2+7\mathcal{R}+1}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2}$
Expected FS	$4 - \frac{6\mathcal{R}^4+29\mathcal{R}^3+4\mathcal{R}^2+19\mathcal{R}+3}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2}$	$4 - \frac{6\mathcal{R}^4+33\mathcal{R}^3+47\mathcal{R}^2+21\mathcal{R}+3}{(\mathcal{R}+1)^3(2\mathcal{R}+1)(3\mathcal{R}+1)}$	$4 - \frac{2\mathcal{R}^3+12\mathcal{R}^2+10\mathcal{R}+2}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2}$

Probability of terminating in absorbing states:

$$\begin{aligned}
\mathcal{P}_{SRSS} &= \frac{1}{2\mathcal{R}+1} \mathbb{E}_{SISS} \\
\mathcal{P}_{SSSR} &= \frac{1}{3\mathcal{R}+1} \mathcal{P}_{SSSI} \\
\mathcal{P}_{RSSR} &= \frac{1}{2\mathcal{R}+1} \mathcal{P}_{RSSI} \\
\mathcal{P}_{SRRS} &= \frac{1}{2\mathcal{R}+1} \mathcal{P}_{SRIS} \\
\mathcal{P}_{RRSS} &= \frac{1}{\mathcal{R}+1} \mathcal{P}_{SSIR} + \frac{1}{2\mathcal{R}+1} \mathcal{P}_{IRSS} \\
\mathcal{P}_{RSRR} &= \mathcal{P}_{RSIR} + \frac{1}{\mathcal{R}+1} \mathcal{P}_{ISRR} \\
\mathcal{P}_{SRRR} &= \frac{1}{\mathcal{R}+1} (\mathcal{P}_{SRRI} + \mathcal{P}_{SIRR}) \\
\mathcal{P}_{RRRR} &= \mathcal{P}_{IRRR} + \mathcal{P}_{RIRR}
\end{aligned}$$

To find the equations for the final size probabilities we evaluated the following:

$$\begin{aligned}
\mathbb{P}(\text{Final Size} = 1) &= \mathcal{P}_{SRSS} + \mathcal{P}_{SSSR} \\
\mathbb{P}(\text{Final Size} = 2) &= \mathcal{P}_{RSSR} + \mathcal{P}_{SRRS} + \mathcal{P}_{RRSS} \\
\mathbb{P}(\text{Final Size} = 3) &= \mathcal{P}_{RSRR} + \mathcal{P}_{SRRR} \\
\mathbb{P}(\text{Final Size} = 4) &= \mathcal{P}_{RRRR}
\end{aligned}$$

Simplifying the above we obtained the final size equations for the toast network with four nodes as shown in Table 5.

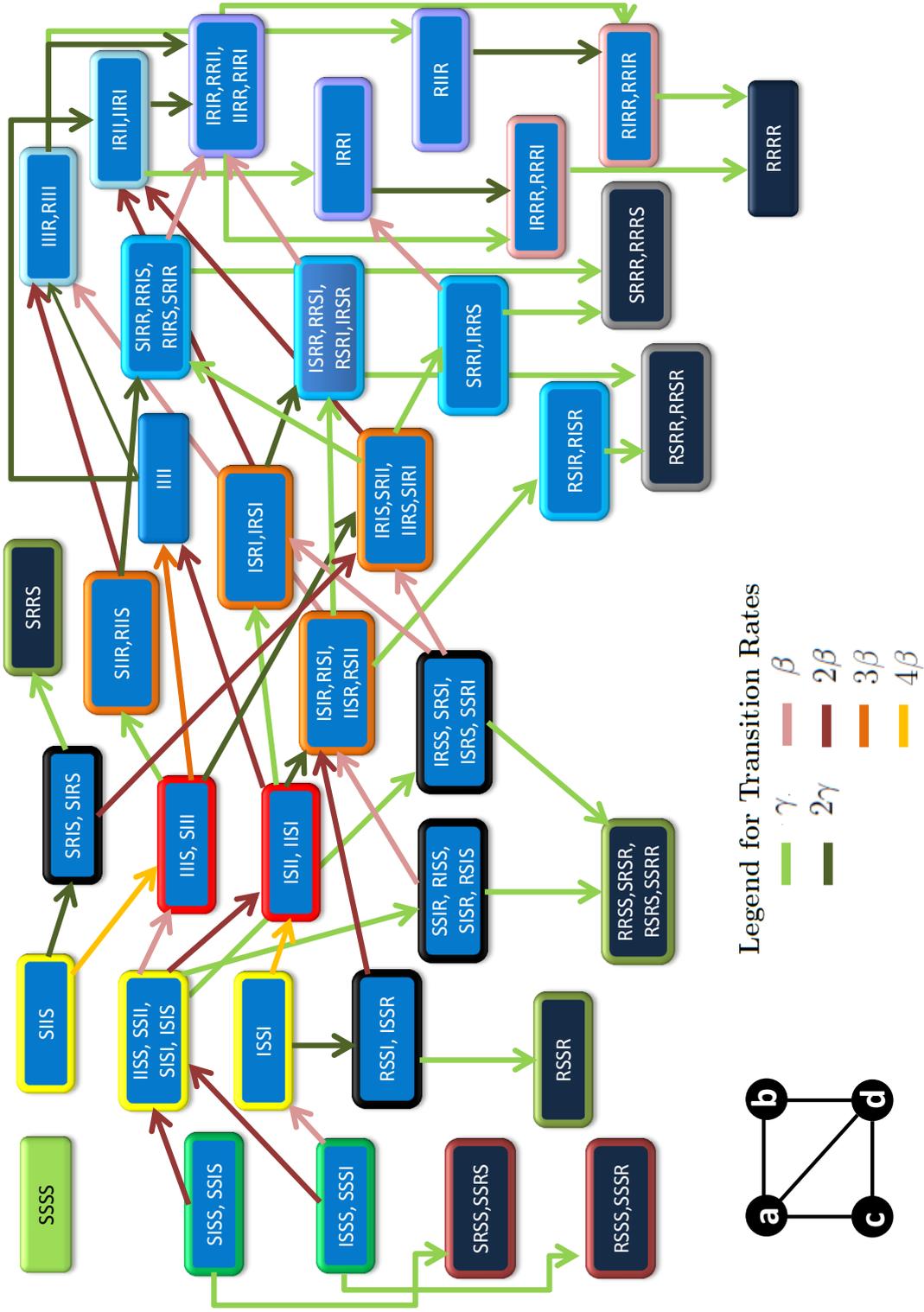


Figure B.9: Toast network with $N = 4$ nodes.

B.5.2. Progression of infection over time

Equations describing the probability that the network is in a given state at time t for an SIR model on the toast network are:

Initial states:

$$\dot{P}_{SSSS} = 0 \quad (\text{B.80})$$

$$\dot{P}_{SSIS} = -(2\mathcal{R} + 1)P_{SSIS} \quad (\text{B.81})$$

$$\dot{P}_{SSSI} = -(3\mathcal{R} + 1)P_{SSSI} \quad (\text{B.82})$$

$$\dot{P}_{SIIS} = -(2 + 4\mathcal{R})P_{SIIS} \quad (\text{B.83})$$

Transient states:

$$\dot{P}_{SSII} = -(3\mathcal{R} + 2)P_{SSII} + 2\mathcal{R}P_{SSIS} + 2\mathcal{R}P_{SSSI} \quad (\text{B.84})$$

$$\dot{P}_{ISSI} = \mathcal{R}P_{SSSI} - (4\mathcal{R} + 2)P_{ISSI} \quad (\text{B.85})$$

$$\dot{P}_{RSSI} = 2P_{ISSI} - (2\mathcal{R} + 1)P_{RSSI} \quad (\text{B.86})$$

$$\dot{P}_{SRIS} = 2P_{SIIS} - P_{SRIS} \quad (\text{B.87})$$

$$\dot{P}_{SIII} = 4\mathcal{R}P_{SIIS} + \mathcal{R}P_{SSII} - (3\mathcal{R} + 3)P_{SIII} \quad (\text{B.88})$$

$$\dot{P}_{ISII} = 2\mathcal{R}P_{SSII} + 4\mathcal{R}P_{ISSI} - (2\mathcal{R} + 3)P_{ISII} \quad (\text{B.89})$$

$$\dot{P}_{SSIR} = P_{SSII} - (\mathcal{R} + 1)P_{SSIR} \quad (\text{B.90})$$

$$\dot{P}_{SRSI} = P_{SSII} - (2\mathcal{R} + 1)P_{SRSI} \quad (\text{B.91})$$

$$\dot{P}_{RISI} = 2P_{ISII} + 2\mathcal{R}P_{RSSI} + \mathcal{R}P_{SSIR} - (\mathcal{R} + 2)P_{RISI} \quad (\text{B.92})$$

$$\dot{P}_{SIIR} = P_{SIII} - 2(\mathcal{R} + 1)P_{SIIR} \quad (\text{B.93})$$

$$\dot{P}_{IRSI} = P_{ISII} + \mathcal{R}P_{SRSI} - 2(\mathcal{R} + 1)P_{IRSI} \quad (\text{B.94})$$

$$\dot{P}_{IRIS} = 2P_{SIII} + \mathcal{R}P_{SRSI} - 2(\mathcal{R} + 1)P_{IRIS} \quad (\text{B.95})$$

$$\dot{P}_{IIII} = 3\mathcal{R}P_{SIII} + 2\mathcal{R}P_{ISII} - 4P_{IIII} \quad (\text{B.96})$$

$$\dot{P}_{SIRR} = 2P_{SIIR} + 2P_{IRIS} - (\mathcal{R} + 1)P_{SIRR} \quad (\text{B.97})$$

$$\dot{P}_{ISRR} = P_{RISI} + 2P_{IRSI} - (\mathcal{R} + 1)P_{ISRR} \quad (\text{B.98})$$

$$\dot{P}_{SRRI} = P_{IRIS} - (\mathcal{R} + 1)P_{SRRI} \quad (\text{B.99})$$

$$\dot{P}_{RSIR} = P_{RISI} - P_{RSIR} \quad (\text{B.100})$$

$$\dot{P}_{IIIR} = 2\mathcal{R}P_{SIIR} + \mathcal{R}P_{RISI} + 2P_{IIII} - 3P_{IIIR} \quad (\text{B.101})$$

$$\dot{P}_{IRII} = 2\mathcal{R}(P_{IRSI} + P_{IRIS}) + 2P_{IIII} - 3P_{IRII} \quad (\text{B.102})$$

$$\dot{P}_{RIRI} = 2(P_{IIIR} + P_{IRII}) + \mathcal{R}(P_{SIRR} + P_{ISRR}) - 2P_{RIRI} \quad (\text{B.103})$$

$$\dot{P}_{IRRI} = P_{IRII} + \mathcal{R}P_{SRRI} - 2P_{IRRI} \quad (\text{B.104})$$

$$\dot{P}_{RIIR} = P_{IIIR} - 2P_{RIIR} \quad (\text{B.105})$$

$$\dot{P}_{IRRR} = 2P_{IRRI} + P_{RIRI} - P_{IRRR} \quad (\text{B.106})$$

$$\dot{P}_{RIIR} = 2P_{RIIR} + P_{RIRI} - P_{RIIR} \quad (\text{B.107})$$

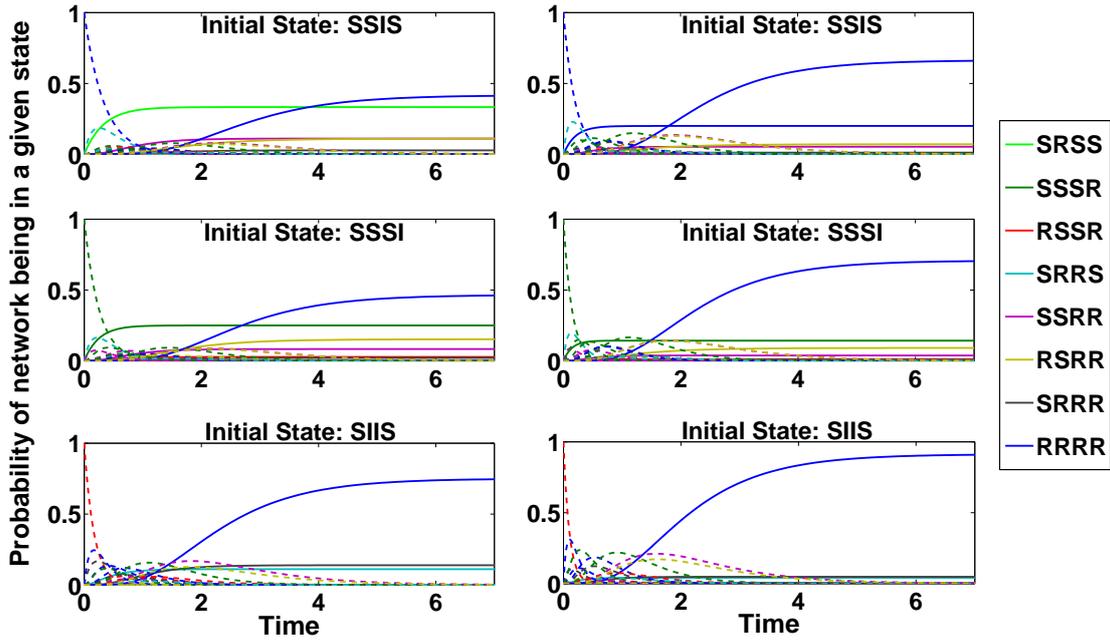


Figure B.10: Numerical results for solving the system of differential equations (B.80) - (B.115) which describes the progression of infection over time for an SIR model on a toast network with $N = 4$ nodes. Left and right columns contain graphical results for $\mathcal{R} = 1$ and $\mathcal{R} = 2$ respectively for the specified initial conditions. Dashed lines represent transient states and solid lines are the absorbing states. The numerical results are in agreement with the analytical expressions for the same set of initial conditions.

Absorbing states:

$$\dot{P}_{SRSS} = P_{SSIS} \quad (\text{B.108})$$

$$\dot{P}_{SSSR} = P_{SSSI} \quad (\text{B.109})$$

$$\dot{P}_{RSSR} = P_{RSSI} \quad (\text{B.110})$$

$$\dot{P}_{SRRS} = P_{SRIS} \quad (\text{B.111})$$

$$\dot{P}_{SSRR} = P_{SSIR} + P_{SRSI} \quad (\text{B.112})$$

$$\dot{P}_{RSRR} = P_{RSIR} + P_{ISRR} \quad (\text{B.113})$$

$$\dot{P}_{SRRR} = P_{SRRI} + P_{SIRR} \quad (\text{B.114})$$

$$\dot{P}_{RRRR} = P_{IRRR} + P_{RIRR} \quad (\text{B.115})$$

B.6. Line Network

Similarly to a line network of $N = 3$ nodes, a line network of $N = 4$ nodes has two different types of nodes, the end nodes of degree 1 (nodes a and d) and the centre nodes of degree 2 (nodes b and c). We denote $SSIS$ as the initial state in which node b or c is infectious; $SSSI$ as the initial state in which node a or d is infectious; $SISI$ as the initial state in which nodes b and d are infectious and $ISSI$ as the initial state in which nodes a and d are infectious. The probability mass functions of the final epidemic size for the line network of four nodes found with four initial conditions are shown in Table 6.

B.6.1. Catalogue of transition probabilities

Possible initial state indicator variables:

$$E_{SISS} = \begin{cases} 1, & \text{if initial state is } SISS. \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{SSSI} = \begin{cases} 1, & \text{if initial state is } SSSI. \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{SISI} = \begin{cases} 1, & \text{if initial state is } SISI. \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{ISSI} = \begin{cases} 1, & \text{if initial state is } ISSI. \\ 0, & \text{otherwise.} \end{cases}$$

Probability of passing through transient states:

$$\mathcal{P}_{SSII} = \frac{\mathcal{R}}{\mathcal{R}+1}E_{SSSI} + \frac{\mathcal{R}}{2\mathcal{R}+1}E_{SISS}$$

$$\mathcal{P}_{SIIS} = \frac{\mathcal{R}}{2\mathcal{R}+1}E_{SISS}$$

$$\mathcal{P}_{SRIS} = \frac{2}{2\mathcal{R}+2}\mathcal{P}_{SIIS}$$

$$\mathcal{P}_{ISSR} = \frac{2}{2\mathcal{R}+2}E_{ISSI}$$

$$\mathcal{P}_{SSIR} = \frac{1}{\mathcal{R}+2}\mathcal{P}_{SSII}$$

$$\mathcal{P}_{SSRI} = \frac{1}{\mathcal{R}+2}\mathcal{P}_{SSII}$$

$$\mathcal{P}_{SIII} = \frac{\mathcal{R}}{\mathcal{R}+2}\mathcal{P}_{SSII} + \frac{2\mathcal{R}}{2\mathcal{R}+2}\mathcal{P}_{SIIS} + \frac{2\mathcal{R}}{3\mathcal{R}+2}E_{SISI}$$

$$\begin{aligned}
\mathcal{P}_{ISII} &= \frac{\mathcal{R}}{3\mathcal{R}+2}E_{SISI} + \frac{2\mathcal{R}}{2\mathcal{R}+2}E_{ISSI} \\
\mathcal{P}_{IIII} &= \frac{\mathcal{R}}{\mathcal{R}+3}\mathcal{P}_{SIII} + \frac{2\mathcal{R}}{2\mathcal{R}+3}\mathcal{P}_{ISII} \\
\mathcal{P}_{SRSI} &= \frac{1}{3\mathcal{R}+2}E_{SISI} \\
\mathcal{P}_{SISR} &= \frac{1}{3\mathcal{R}+2}E_{SISI} \\
\mathcal{P}_{SIIR} &= \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{SSIR} + \frac{\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{SISR} + \frac{1}{\mathcal{R}+3}\mathcal{P}_{SIII} \\
\mathcal{P}_{SIRI} &= \frac{1}{\mathcal{R}+3}\mathcal{P}_{SIII} \\
\mathcal{P}_{SRII} &= \frac{1}{\mathcal{R}+3}\mathcal{P}_{SIII} + \frac{\mathcal{R}}{\mathcal{R}+1}(\mathcal{P}_{SRIS} + \mathcal{P}_{SRSI}) \\
\mathcal{P}_{IISR} &= \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{ISSR} + \frac{\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{SISR} + \frac{1}{2\mathcal{R}+3}\mathcal{P}_{IISI} \\
\mathcal{P}_{ISRI} &= \frac{1}{2\mathcal{R}+3}\mathcal{P}_{IISI} \\
\mathcal{P}_{ISIR} &= \frac{1}{2\mathcal{R}+3}\mathcal{P}_{IISI} \\
\mathcal{P}_{SRIR} &= \frac{1}{2}\mathcal{P}_{SRII} + \frac{1}{\mathcal{R}+2}\mathcal{P}_{SIIR} \\
\mathcal{P}_{SIRR} &= \frac{1}{\mathcal{R}+2}(\mathcal{P}_{SIIR} + \mathcal{P}_{SIRI}) \\
\mathcal{P}_{SRRI} &= \frac{1}{2}\mathcal{P}_{SRII} + \frac{1}{\mathcal{R}+2}\mathcal{P}_{SIRI} \\
\mathcal{P}_{RSIR} &= \frac{1}{\mathcal{R}+2}\mathcal{P}_{IISR} + \frac{1}{2\mathcal{R}+2}\mathcal{P}_{ISIR} \\
\mathcal{P}_{ISRR} &= \frac{1}{\mathcal{R}+2}\mathcal{P}_{ISRI} + \frac{1}{2\mathcal{R}+2}\mathcal{P}_{ISIR} \\
\mathcal{P}_{RSRI} &= \frac{1}{\mathcal{R}+2}(\mathcal{P}_{ISRI} + \mathcal{P}_{IISR}) \\
\mathcal{P}_{IIIR} &= \frac{\mathcal{R}}{\mathcal{R}+2}(\mathcal{P}_{SIIR} + \mathcal{P}_{IISR}) + \frac{2\mathcal{R}}{2\mathcal{R}+2}\mathcal{P}_{ISIR} + \frac{1}{2}\mathcal{P}_{IIII}
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_{IIRI} &= \frac{1}{2}\mathcal{P}_{IIII} + \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{SIRI} + \frac{\mathcal{R}}{\mathcal{R}+2}\mathcal{P}_{ISRI} \\
\mathcal{P}_{IRRI} &= \frac{1}{3}\mathcal{P}_{IIRI} \\
\mathcal{P}_{IIRR} &= \frac{1}{3}(\mathcal{P}_{IIRI} + \mathcal{P}_{IIIR}) + \frac{\mathcal{R}}{\mathcal{R}+1}(\mathcal{P}_{SIRR} + \mathcal{P}_{ISRR}) \\
\mathcal{P}_{IRIR} &= \frac{1}{3}(\mathcal{P}_{IIRI} + \mathcal{P}_{IIIR}) \\
\mathcal{P}_{RIIR} &= \frac{1}{3}\mathcal{P}_{IIRI} + \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{RSIR} \\
\mathcal{P}_{IIRR} &= \frac{1}{2}(\mathcal{P}_{IIRR} + \mathcal{P}_{IRIR}) + \mathcal{P}_{IRRI} \\
\mathcal{P}_{RIIR} &= \frac{1}{2}(\mathcal{P}_{IIRR} + \mathcal{P}_{IRIR}) + \mathcal{P}_{RIIR}
\end{aligned}$$

Probability of terminating in absorbing states:

$$\begin{aligned}
\mathcal{P}_{SSSR} &= \frac{1}{\mathcal{R}+1}\mathcal{E}_{SSSI} \\
\mathcal{P}_{SRSS} &= \frac{1}{2\mathcal{R}+1}\mathcal{E}_{SISS} \\
\mathcal{P}_{SRRS} &= \frac{1}{\mathcal{R}+1}\mathcal{P}_{SRIS} \\
\mathcal{P}_{RSSR} &= \frac{1}{\mathcal{R}+1}\mathcal{P}_{ISSR} \\
\mathcal{P}_{SSRR} &= \frac{1}{\mathcal{R}+1}\mathcal{P}_{SSIR} + \mathcal{P}_{SSRI} \\
\mathcal{P}_{SRSR} &= \frac{1}{\mathcal{R}+1}\mathcal{P}_{SRSI} + \frac{1}{2\mathcal{R}+1}\mathcal{P}_{SISR} \\
\mathcal{P}_{SRRR} &= \mathcal{P}_{SRIR} + \mathcal{P}_{SRRI} + \frac{1}{\mathcal{R}+1}\mathcal{P}_{SIRR} \\
\mathcal{P}_{RSRR} &= \mathcal{P}_{RSRI} + \frac{1}{\mathcal{R}+1}(\mathcal{P}_{ISRR} + \mathcal{P}_{RSIR}) \\
\mathcal{P}_{RRRR} &= \mathcal{P}_{IIRR} + \mathcal{P}_{RIIR}
\end{aligned}$$

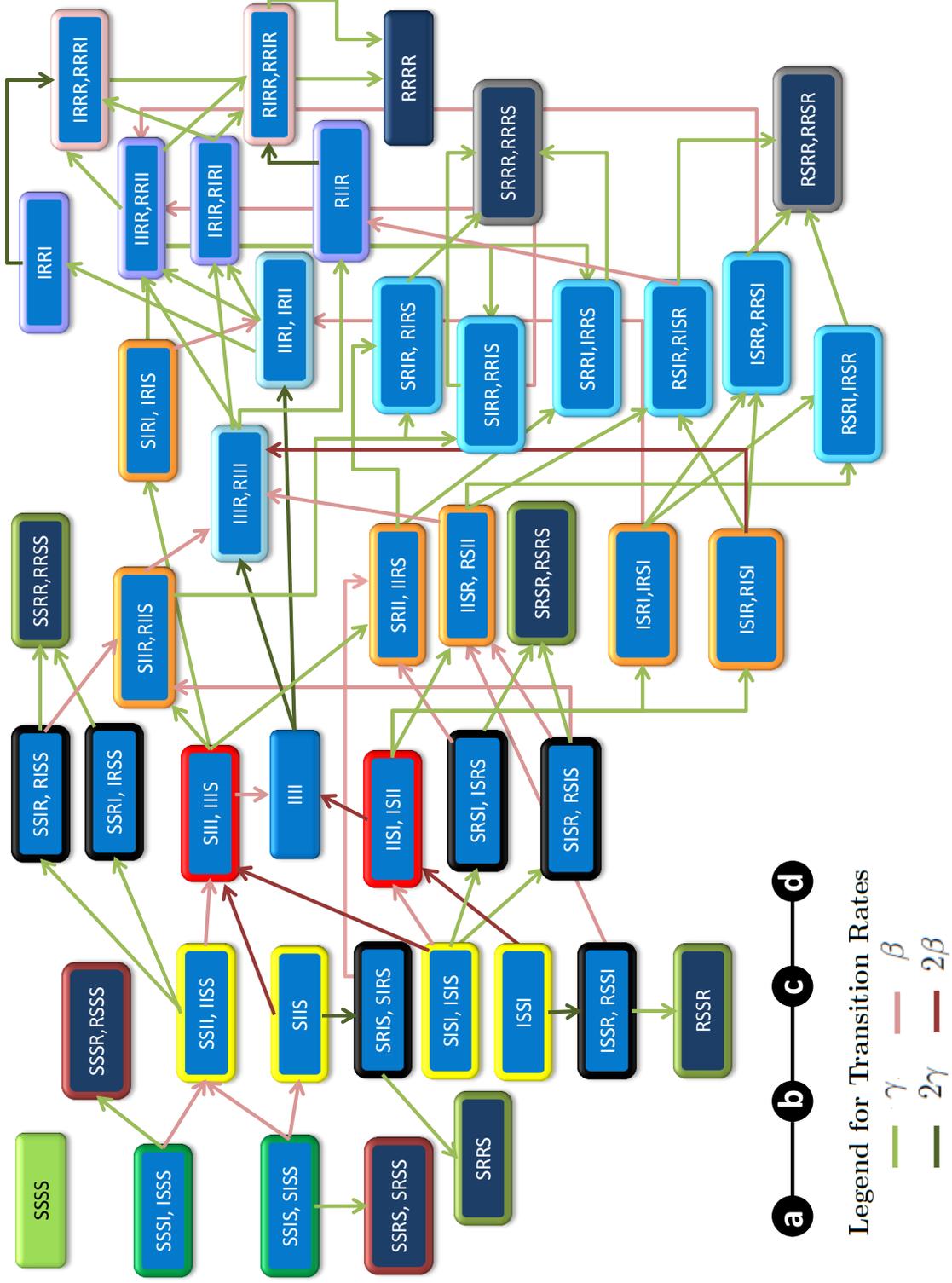


Figure B.11: Line network with $N = 4$ nodes.

Table 6: Line network final size PMFs

Initial State	<i>SSSI</i>	<i>SSIS</i>	<i>ISSI</i>	<i>SISI</i>
$\mathbb{P}(\text{Final Size}=1)$	$\frac{1}{\mathcal{R}+1}$	$\frac{1}{2\mathcal{R}+1}$	0	0
$\mathbb{P}(\text{Final Size}=2)$	$\frac{\mathcal{R}}{(\mathcal{R}+1)^2}$	$\frac{\mathcal{R}(\mathcal{R}+2)}{(\mathcal{R}+1)^2(2\mathcal{R}+1)}$	$\frac{1}{(\mathcal{R}+1)^2}$	$\frac{1}{(\mathcal{R}+1)(2\mathcal{R}+1)}$
$\mathbb{P}(\text{Final Size}=3)$	$\frac{\mathcal{R}^2}{(\mathcal{R}+1)^3}$	$\frac{3\mathcal{R}^2}{(\mathcal{R}+1)^2(2\mathcal{R}+1)}$	$\frac{2\mathcal{R}}{(\mathcal{R}+1)^3}$	$\frac{\mathcal{R}(2\mathcal{R}+3)}{(\mathcal{R}+1)^2(2\mathcal{R}+1)}$
$\mathbb{P}(\text{Final Size}=4)$	$\frac{\mathcal{R}^3}{(\mathcal{R}+1)^3}$	$\frac{2\mathcal{R}^3}{(\mathcal{R}+1)^2(2\mathcal{R}+1)}$	$1 - \frac{3\mathcal{R}+1}{(\mathcal{R}+1)^3}$	$\frac{\mathcal{R}^2(2\mathcal{R}+3)}{(\mathcal{R}+1)^2(2\mathcal{R}+1)}$
Expected FS	$4 - \frac{6\mathcal{R}^2 + 8\mathcal{R} + 3}{(\mathcal{R}+1)^3}$	$\frac{(2\mathcal{R}+1)^2}{(\mathcal{R}+1)^2}$	$4 - \frac{2(2\mathcal{R}+1)}{(\mathcal{R}+1)^3}$	$4 - \frac{\mathcal{R}+2}{(\mathcal{R}+1)^2}$

To find the equations for the final size probabilities we evaluated the following:

$$\begin{aligned} \mathbb{P}(\text{Final Size} = 1) &= \mathcal{P}_{SSSR} + \mathcal{P}_{SRSS} \\ \mathbb{P}(\text{Final Size} = 2) &= \mathcal{P}_{SRRS} + \mathcal{P}_{RSSR} + \mathcal{P}_{SSRR} + \mathcal{P}_{SRSR} \\ \mathbb{P}(\text{Final Size} = 3) &= \mathcal{P}_{SRRR} + \mathcal{P}_{RSRR} \\ \mathbb{P}(\text{Final Size} = 4) &= \mathcal{P}_{RRRR} \end{aligned}$$

Simplifying the above we obtained the final size equations for the line network with four nodes as shown in Table 6.

B.6.2. Progression of infection over time

Equations describing the probability that the network is in a given state at time t for an *SIR* model on the line network of $N = 4$ are:

Initial states:

$$\dot{P}_{SSSS} = 0 \quad (\text{B.116})$$

$$\dot{P}_{SSSI} = -(\mathcal{R}+1)P_{SSSI} \quad (\text{B.117})$$

$$\dot{P}_{SSIS} = -(2\mathcal{R}+1)P_{SSIS} \quad (\text{B.118})$$

$$\dot{P}_{SISI} = -(3\mathcal{R}+2)P_{SISI} \quad (\text{B.119})$$

$$\dot{P}_{ISSI} = -2(\mathcal{R}+1)P_{ISSI} \quad (\text{B.120})$$

Transient states:

$$\dot{P}_{SSII} = \mathcal{R}(P_{SSSI} + P_{SSIS}) - (\mathcal{R}+2)P_{SSII} \quad (\text{B.121})$$

$$\dot{P}_{SIIS} = \mathcal{R}P_{SSIS} - 2(\mathcal{R}+1)P_{SIIS} \quad (\text{B.122})$$

$$\dot{P}_{SRIS} = 2P_{SIIS} - (\mathcal{R}+1)P_{SRIS} \quad (\text{B.123})$$

$$\dot{P}_{ISSR} = 2P_{ISSI} - (\mathcal{R}+1)P_{ISSR} \quad (\text{B.124})$$

$$\dot{P}_{SSIR} = P_{SSII} - (\mathcal{R} + 1)P_{SSIR} \quad (\text{B.125})$$

$$\dot{P}_{SSRI} = P_{SSII} - P_{SSRI} \quad (\text{B.126})$$

$$\dot{P}_{SRSI} = P_{SISI} - (\mathcal{R} + 1)P_{SRSI} \quad (\text{B.127})$$

$$\dot{P}_{SISR} = P_{SISI} - (2\mathcal{R} + 1)P_{SISR} \quad (\text{B.128})$$

$$\dot{P}_{SIII} = \mathcal{R}P_{SSII} + 2\mathcal{R}(P_{SIIS} + P_{SISI}) - (\mathcal{R} + 3)P_{SIII} \quad (\text{B.129})$$

$$\dot{P}_{IISI} = \mathcal{R}P_{SISI} + 2\mathcal{R}P_{ISSI} - (2\mathcal{R} + 3)P_{IISI} \quad (\text{B.130})$$

$$\dot{P}_{IIII} = \mathcal{R}P_{SIII} + 2\mathcal{R}P_{IISI} - 4P_{IIII} \quad (\text{B.131})$$

$$\dot{P}_{SIIR} = \mathcal{R}(P_{SSIR} + P_{SISR}) + P_{SIII} - (\mathcal{R} + 2)P_{SIIR} \quad (\text{B.132})$$

$$\dot{P}_{SIRI} = P_{SIII} - (\mathcal{R} + 2)P_{SIRI} \quad (\text{B.133})$$

$$\dot{P}_{SRII} = P_{SIII} + \mathcal{R}(P_{SRIS} + P_{SRSI}) - 2P_{SRII} \quad (\text{B.134})$$

$$\dot{P}_{IISR} = P_{IISI} + \mathcal{R}(P_{SISR} + P_{ISSR}) - (\mathcal{R} + 2)P_{IISR} \quad (\text{B.135})$$

$$\dot{P}_{ISRI} = P_{IISI} - (\mathcal{R} + 2)P_{ISRI} \quad (\text{B.136})$$

$$\dot{P}_{ISIR} = P_{IISI} - (2\mathcal{R} + 2)P_{ISIR} \quad (\text{B.137})$$

$$\dot{P}_{SRIR} = P_{SIIR} + P_{SRII} - P_{SRIR} \quad (\text{B.138})$$

$$\dot{P}_{SIRR} = P_{SIIR} + P_{SIRI} - (\mathcal{R} + 1)P_{SIRR} \quad (\text{B.139})$$

$$\dot{P}_{SRRI} = P_{SRII} + P_{SIRI} - P_{SRRI} \quad (\text{B.140})$$

$$\dot{P}_{RSIR} = P_{IISR} + P_{ISIR} - (\mathcal{R} + 1)P_{RSIR} \quad (\text{B.141})$$

$$\dot{P}_{ISRR} = P_{ISIR} + P_{ISRI} - (\mathcal{R} + 1)P_{ISRR} \quad (\text{B.142})$$

$$\dot{P}_{RSRI} = P_{ISRI} + P_{IISR} - P_{RSRI} \quad (\text{B.143})$$

$$\dot{P}_{IIIR} = 2P_{IIII} + \mathcal{R}(P_{SIIR} + P_{IISR} + 2P_{ISIR}) - 3P_{IIIR} \quad (\text{B.144})$$

$$\dot{P}_{IIRI} = 2P_{IIII} + \mathcal{R}(P_{SIRI} + P_{ISRI}) - 3P_{IIRI} \quad (\text{B.145})$$

$$\dot{P}_{IRRI} = P_{IIRI} - 2P_{IRRI} \quad (\text{B.146})$$

$$\dot{P}_{IIRR} = P_{IIIR} + P_{IIRI} + \mathcal{R}(P_{SIRR} + P_{ISRR}) - 2P_{IIRR} \quad (\text{B.147})$$

$$\dot{P}_{IRIR} = P_{IIIR} + P_{IIRI} - 2P_{IRIR} \quad (\text{B.148})$$

$$\dot{P}_{RIIR} = P_{IIIR} + \mathcal{R}P_{RSIR} - 2P_{RIIR} \quad (\text{B.149})$$

$$\dot{P}_{IRRR} = 2P_{IRRI} + P_{IIRR} + P_{IRIR} - P_{IRRR} \quad (\text{B.150})$$

$$\dot{P}_{RIRR} = 2P_{RIIR} + P_{IIRR} + P_{IRIR} - P_{RIRR} \quad (\text{B.151})$$

Absorbing states:

$$\dot{P}_{SSSR} = P_{SSSI} \quad (\text{B.152})$$

$$\dot{P}_{SSRS} = P_{SSSI} \quad (\text{B.153})$$

$$\dot{P}_{SRRS} = P_{SRIS} \quad (\text{B.154})$$

$$\dot{P}_{RSSR} = P_{ISSR} \quad (\text{B.155})$$

$$\dot{P}_{SSRR} = P_{SSIR} + P_{SSRI} \quad (\text{B.156})$$

$$\dot{P}_{SRSR} = P_{SRSI} + P_{SISR} \quad (\text{B.157})$$

$$\dot{P}_{SRRR} = P_{SRIR} + P_{SIRR} + P_{SRRI} \quad (\text{B.158})$$

$$\dot{P}_{RSRR} = P_{RSIR} + P_{ISRR} + P_{RSRI} \quad (\text{B.159})$$

$$\dot{P}_{RRRR} = P_{IRRR} + P_{RIRR} \quad (\text{B.160})$$

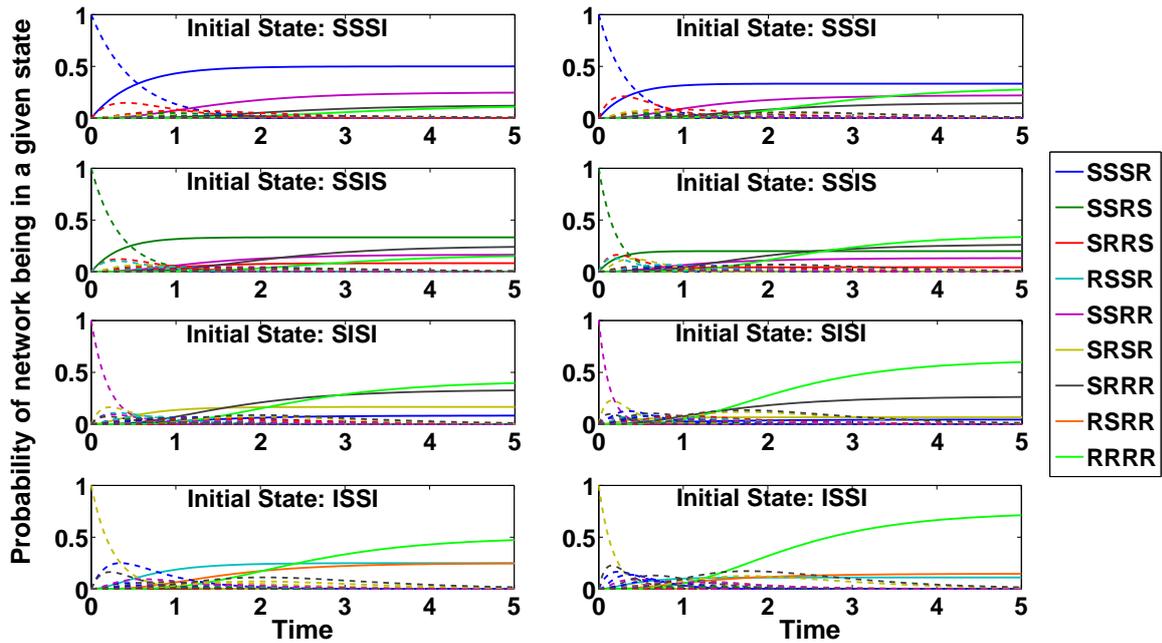


Figure B.12: Numerical results for solving the system of differential equations (B.116) - (B.160) which describes the progression of infection over time for an SIR model on a line network with $N = 4$ nodes. Left and right columns contain graphical results for $\mathcal{R} = 1$ and $\mathcal{R} = 2$ respectively for the specified initial conditions. Dashed lines represent transient states and solid lines are the absorbing states. The numerical results are in agreement with the analytical expressions for the same set of initial conditions.

B.7. Lollipop network

For the lollipop network there are three different types of nodes; node a has degree 3, nodes b and c have degree 2 and node d has degree 1. Therefore, the final size probabilities vary depending on which type of node is the initial infectious node. We denote $ISSS$ as the initial state in which node a is infectious; $SSSI$ as the initial state in which node d is infectious; $SISS$ as the initial state in which either node b or node c is infectious and $SSII$ as the initial state in which nodes b and d are infectious. The probability mass functions of the final size for the lollipop network found with four initial conditions are shown in Table 7.

B.7.1. Catalogue of transition probabilities

Possible initial state indicator variables:

$$E_{SISS} = \begin{cases} 1, & \text{if initial state is } SISS. \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{ISSS} = \begin{cases} 1, & \text{if initial state is } ISSS. \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{SSSI} = \begin{cases} 1, & \text{if initial state is } SSSI. \\ 0, & \text{otherwise.} \end{cases}$$

$$E_{SSII} = \begin{cases} 1, & \text{if initial state is } SSII. \\ 0, & \text{otherwise.} \end{cases}$$

Probability of passing through transient states:

$$\mathcal{P}_{IISS} = \frac{2\mathcal{R}}{3\mathcal{R}+1}E_{ISSS} + \frac{\mathcal{R}}{2\mathcal{R}+1}E_{SISS}$$

$$\mathcal{P}_{ISSI} = \frac{\mathcal{R}}{3\mathcal{R}+1}E_{ISSS} + \frac{\mathcal{R}}{\mathcal{R}+1}E_{SSSI}$$

$$\mathcal{P}_{SIIS} = \frac{\mathcal{R}}{2\mathcal{R}+1}E_{SISS}$$

$$\mathcal{P}_{RSSI} = \frac{1}{2\mathcal{R}+2}\mathcal{P}_{ISSI}$$

$$\mathcal{P}_{ISSR} = \frac{1}{2\mathcal{R}+2}\mathcal{P}_{ISSI}$$

$$\mathcal{P}_{RISS} = \frac{1}{3\mathcal{R}+2}\mathcal{P}_{IISS}$$

$$\mathcal{P}_{IRSS} = \frac{1}{3\mathcal{R}+2}\mathcal{P}_{IISS}$$

$$\mathcal{P}_{IIIS} = \frac{2\mathcal{R}}{3\mathcal{R}+2}\mathcal{P}_{IISS} + \frac{2\mathcal{R}}{2\mathcal{R}+2}\mathcal{P}_{SIIS}$$

$$\mathcal{P}_{IISI} = \frac{\mathcal{R}}{3\mathcal{R}+2}\mathcal{P}_{IISS} + \frac{2\mathcal{R}}{2\mathcal{R}+2}\mathcal{P}_{ISSI} + \frac{2\mathcal{R}}{3\mathcal{R}+2}E_{SSII}$$

$$\mathcal{P}_{SIII} = \frac{\mathcal{R}}{3\mathcal{R}+2}E_{SSII}$$

$$\mathcal{P}_{SSRI} = \frac{1}{3\mathcal{R}+2}E_{SSII}$$

$$\mathcal{P}_{SSIR} = \frac{1}{3\mathcal{R}+2}E_{SSII}$$

$$\mathcal{P}_{SIRS} = \frac{2}{2\mathcal{R}+2}\mathcal{P}_{SIIS}$$

$$\mathcal{P}_{SIIR} = \frac{1}{3\mathcal{R}+3}\mathcal{P}_{SIII} + \frac{\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{SSIR}$$

$$\mathcal{P}_{SIRI} = \frac{2}{3\mathcal{R}+3}\mathcal{P}_{SIII}$$

$$\begin{aligned}
\mathcal{P}_{SIRR} &= \frac{2}{2\mathcal{R}+2}\mathcal{P}_{SIIR} + \frac{1}{2\mathcal{R}+2}\mathcal{P}_{SIRI} \\
\mathcal{P}_{RIIS} &= \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{RISS} + \frac{1}{\mathcal{R}+3}\mathcal{P}_{IIIS} \\
\mathcal{P}_{IIII} &= \frac{\mathcal{R}}{\mathcal{R}+3}\mathcal{P}_{IIIS} + \frac{2\mathcal{R}}{2\mathcal{R}+3}\mathcal{P}_{IISI} + \frac{3\mathcal{R}}{3\mathcal{R}+3}\mathcal{P}_{SIII} \\
\mathcal{P}_{RISI} &= \frac{1}{2\mathcal{R}+3}\mathcal{P}_{IISI} \\
\mathcal{P}_{IRSI} &= \frac{1}{2\mathcal{R}+3}\mathcal{P}_{IISI} + \frac{\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{IRSS} + \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{SSRI} \\
\mathcal{P}_{IISR} &= \frac{1}{2\mathcal{R}+3}\mathcal{P}_{IISI} + \frac{\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{SSIR} + \frac{2\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{ISSR} \\
\mathcal{P}_{IIRS} &= \frac{2}{\mathcal{R}+3}\mathcal{P}_{IIIS} + \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{SIRS} + \frac{\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{IRSS} \\
\mathcal{P}_{SRRI} &= \frac{1}{2\mathcal{R}+1}\mathcal{P}_{SIRI} \\
\mathcal{P}_{RRIS} &= \mathcal{P}_{RIIS} + \frac{1}{\mathcal{R}+2}\mathcal{P}_{IIRS} \\
\mathcal{P}_{IRSR} &= \frac{1}{2\mathcal{R}+2}\mathcal{P}_{IISR} + \frac{1}{\mathcal{R}+2}\mathcal{P}_{IRSI} \\
\mathcal{P}_{RISR} &= \frac{1}{2\mathcal{R}+2}\mathcal{P}_{IISR} + \frac{1}{\mathcal{R}+2}\mathcal{P}_{RISI} \\
\mathcal{P}_{RRSI} &= \frac{1}{\mathcal{R}+2}\mathcal{P}_{IRSI} + \frac{1}{\mathcal{R}+2}\mathcal{P}_{RISI} \\
\mathcal{P}_{IRRS} &= \frac{1}{\mathcal{R}+2}\mathcal{P}_{IIRS} \\
\mathcal{P}_{IIRI} &= \frac{\mathcal{R}}{\mathcal{R}+2}\mathcal{P}_{IIRS} + \frac{1}{2}\mathcal{P}_{IIII} + \frac{\mathcal{R}}{\mathcal{R}+2}\mathcal{P}_{IRSI} + \frac{2\mathcal{R}}{2\mathcal{R}+1}\mathcal{P}_{SIRI} \\
\mathcal{P}_{RIII} &= \frac{1}{4}\mathcal{P}_{IIII} + \frac{\mathcal{R}}{\mathcal{R}+2}\mathcal{P}_{RISI} \\
\mathcal{P}_{IIIR} &= \frac{1}{4}\mathcal{P}_{IIII} + \frac{2\mathcal{R}}{2\mathcal{R}+2}\mathcal{P}_{SIIR} + \frac{2\mathcal{R}}{2\mathcal{R}+2}\mathcal{P}_{IISR} \\
\mathcal{P}_{RRII} &= \frac{2}{3}\mathcal{P}_{RIII} + \frac{1}{3}\mathcal{P}_{IIRI} \\
\mathcal{P}_{RIIR} &= \frac{1}{3}(\mathcal{P}_{IIIR} + \mathcal{P}_{RIII}) + \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{RISR} \\
\mathcal{P}_{IIRR} &= \frac{\mathcal{R}}{\mathcal{R}+1}(\mathcal{P}_{IRSR} + \mathcal{P}_{SIRR}) + \frac{2}{3}\mathcal{P}_{IIIR} + \frac{1}{3}\mathcal{P}_{IIRI} \\
\mathcal{P}_{IRRI} &= \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{SRRI} + \frac{\mathcal{R}}{\mathcal{R}+1}\mathcal{P}_{IRRS} + \frac{1}{3}\mathcal{P}_{IIRI} \\
\mathcal{P}_{RRRI} &= \frac{1}{2}(\mathcal{P}_{RRII} + \mathcal{P}_{IRRI}) \\
\mathcal{P}_{RIIR} &= \mathcal{P}_{RIIR} + \frac{1}{2}(\mathcal{P}_{IIRR} + \mathcal{P}_{RRII}) \\
\mathcal{P}_{IIRR} &= \frac{1}{2}(\mathcal{P}_{IIRR} + \mathcal{P}_{IRRI})
\end{aligned}$$

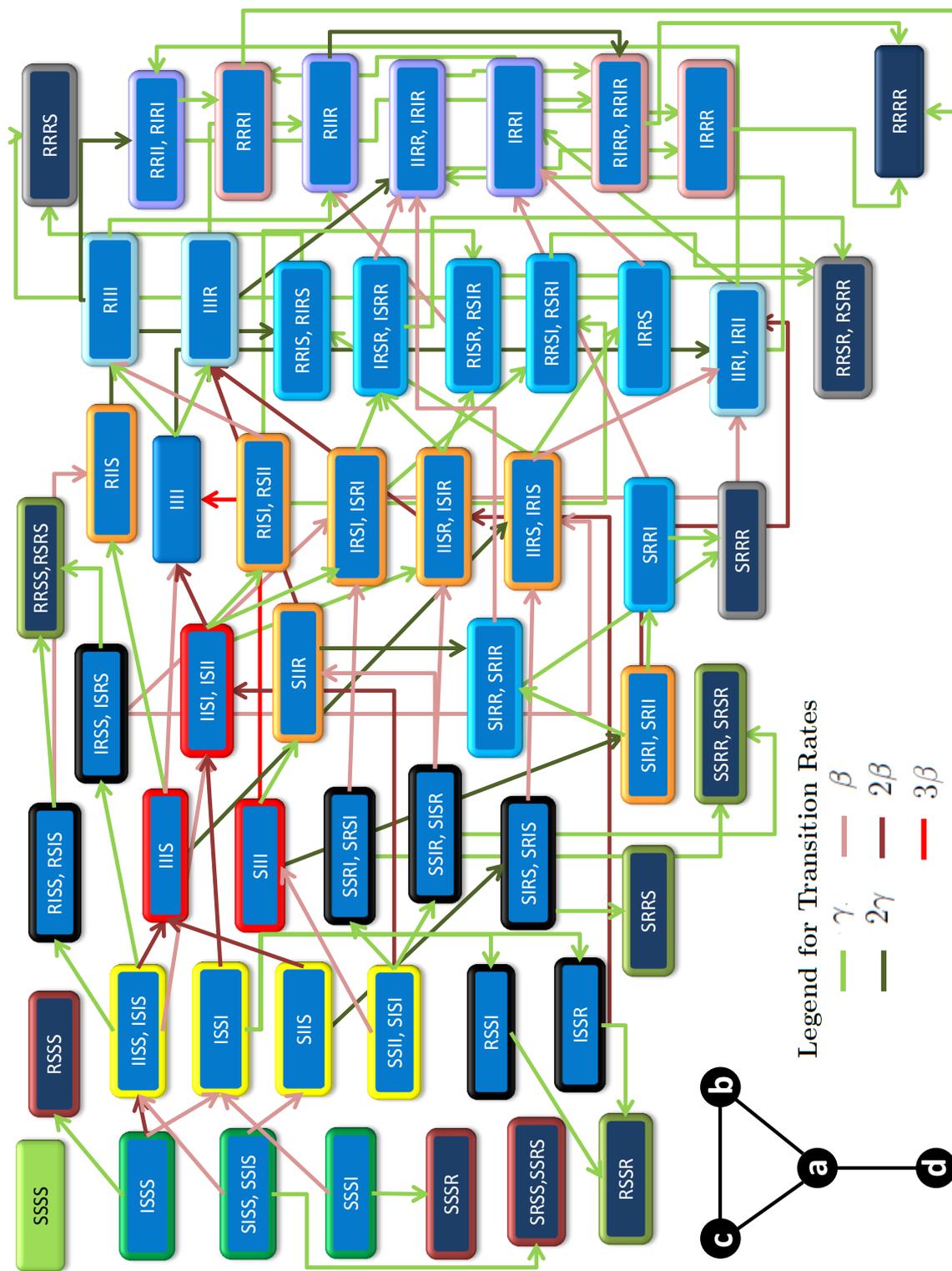


Figure B.13: Lollipop network with $N = 4$ nodes.

Probability of terminating in absorbing states:

$$\begin{aligned}
\mathcal{P}_{RSSS} &= \frac{1}{3\mathcal{R} + 1} \mathbb{E}_{ISSS} \\
\mathcal{P}_{SSSR} &= \frac{1}{\mathcal{R} + 1} \mathbb{E}_{SSSI} \\
\mathcal{P}_{SRSS} &= \frac{1}{2\mathcal{R} + 1} \mathbb{E}_{SISS} \\
\mathcal{P}_{RRSS} &= \frac{1}{2\mathcal{R} + 1} \mathcal{P}_{ISSR} + \mathcal{P}_{RSSI} \\
\mathcal{P}_{RRSS} &= \frac{1}{\mathcal{R} + 1} \mathcal{P}_{RISS} + \frac{1}{2\mathcal{R} + 1} \mathcal{P}_{IRSS} \\
\mathcal{P}_{SRRS} &= \frac{1}{\mathcal{R} + 1} \mathcal{P}_{SIRS} \\
\mathcal{P}_{SSRR} &= \frac{1}{2\mathcal{R} + 1} \mathcal{P}_{SSIR} + \frac{1}{\mathcal{R} + 1} \mathcal{P}_{SSRI} \\
\mathcal{P}_{SRRR} &= \frac{1}{\mathcal{R} + 1} (\mathcal{P}_{SRRI} + \mathcal{P}_{SIRR}) \\
\mathcal{P}_{RRSR} &= \frac{1}{\mathcal{R} + 1} (\mathcal{P}_{RISR} + \mathcal{P}_{IRSR}) + \mathcal{P}_{RRSI} \\
\mathcal{P}_{RRRS} &= \frac{1}{\mathcal{R} + 1} \mathcal{P}_{IRRS} + \mathcal{P}_{RRIS} \\
\mathcal{P}_{RRRR} &= \mathcal{P}_{IRRR} + \mathcal{P}_{RIRR} + \mathcal{P}_{RRRI}
\end{aligned}$$

To find the equations for the final size probabilities we evaluated the following:

$$\begin{aligned}
\mathbb{P}(\text{Final Size} = 1) &= \mathcal{P}_{RSSS} + \mathcal{P}_{SSSR} + \mathcal{P}_{SRSS} \\
\mathbb{P}(\text{Final Size} = 2) &= \mathcal{P}_{RRSS} + \mathcal{P}_{RRSS} + \mathcal{P}_{SRRS} + \mathcal{P}_{SSRR} \\
\mathbb{P}(\text{Final Size} = 3) &= \mathcal{P}_{SRRR} + \mathcal{P}_{RRSR} + \mathcal{P}_{RRRS} \\
\mathbb{P}(\text{Final Size} = 4) &= \mathcal{P}_{RRRR}
\end{aligned}$$

Simplifying the above we obtained the final size equations for the lollipop network with four nodes as shown in Table 7.

Table 7: Lollipop network final size PMFs

Initial State	$SSSI$	$SISS$	$ISSS$	$SSII$
$\mathbb{P}(\text{Final Size}=1)$	$\frac{1}{\mathcal{R}+1}$	$\frac{1}{2\mathcal{R}+1}$	$\frac{1}{3\mathcal{R}+1}$	0
$\mathbb{P}(\text{Final Size}=2)$	$\frac{\mathcal{R}}{(\mathcal{R}+1)(2\mathcal{R}+1)}$	$\frac{\mathcal{R}(3\mathcal{R}+2)}{(\mathcal{R}+1)^2(2\mathcal{R}+1)^2}$	$\frac{\mathcal{R}(\mathcal{R}+3)}{(\mathcal{R}+1)(2\mathcal{R}+1)(3\mathcal{R}+1)}$	$\frac{1}{(\mathcal{R}+1)(2\mathcal{R}+1)}$
$\mathbb{P}(\text{Final Size}=3)$	$\frac{2\mathcal{R}^2}{(\mathcal{R}+1)^3(2\mathcal{R}+1)}$	$\frac{\mathcal{R}^2(4\mathcal{R}^2+10\mathcal{R}+5)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2}$	$\frac{4\mathcal{R}^2(\mathcal{R}+2)}{(\mathcal{R}+1)^2(2\mathcal{R}+1)(3\mathcal{R}+1)}$	$\frac{\mathcal{R}(2\mathcal{R}+3)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)}$
$\mathbb{P}(\text{Final Size}=4)$	$\frac{2\mathcal{R}^3(\mathcal{R}+2)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)}$	$\frac{\mathcal{R}^3(4\mathcal{R}^2+10\mathcal{R}+5)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)^2}$	$\frac{2\mathcal{R}^3(3\mathcal{R}+5)}{(\mathcal{R}+1)^2(2\mathcal{R}+1)(3\mathcal{R}+1)}$	$\frac{\mathcal{R}^2(2\mathcal{R}+3)(\mathcal{R}+2)}{(\mathcal{R}+1)^3(2\mathcal{R}+1)}$
Expected FS	$4 - \frac{8\mathcal{R}^3+21\mathcal{R}^2+14\mathcal{R}+3}{(\mathcal{R}+1)^3(2\mathcal{R}+1)}$	$4 - \frac{5\mathcal{R}^3+16\mathcal{R}^2+13\mathcal{R}+3}{(\mathcal{R}+1)^3(2\mathcal{R}+1)}$	$4 - \frac{4\mathcal{R}^2+9\mathcal{R}+3}{(\mathcal{R}+1)^2(2\mathcal{R}+1)}$	$4 - \frac{4\mathcal{R}^2+7\mathcal{R}+2}{(\mathcal{R}+1)^3(2\mathcal{R}+1)}$

B.7.2. Progression of infection over time

Equations describing the probability that the network is in a given state at time t for an SIR model on the lollipop network are:

Initial states:

$$\dot{P}_{SSSS} = 0 \quad (\text{B.161})$$

$$\dot{P}_{ISSS} = -(3\mathcal{R}+1)P_{ISSS} \quad (\text{B.162})$$

$$\dot{P}_{SISS} = -(2\mathcal{R}+1)P_{SISS} \quad (\text{B.163})$$

$$\dot{P}_{SSSI} = -(\mathcal{R}+1)P_{SSSI} \quad (\text{B.164})$$

$$\dot{P}_{SSII} = -(3\mathcal{R}+2)P_{SSII} \quad (\text{B.165})$$

Transient states:

$$\dot{P}_{IISS} = 2\mathcal{R}P_{ISSS} + \mathcal{R}P_{SISS} - (3\mathcal{R}+2)P_{IISS} \quad (\text{B.166})$$

$$\dot{P}_{IISS} = \mathcal{R}(P_{ISSS} + P_{SSSI}) - 2(\mathcal{R}+1)P_{IISS} \quad (\text{B.167})$$

$$\dot{P}_{SIIS} = \mathcal{R}P_{SISS} - 2(\mathcal{R}+1)P_{SIIS} \quad (\text{B.168})$$

$$\dot{P}_{RSSI} = P_{IISS} - P_{RSSI} \quad (\text{B.169})$$

$$\dot{P}_{ISSR} = P_{IISS} - P_{ISSR} \quad (\text{B.170})$$

$$\dot{P}_{RISS} = P_{IISS} - (\mathcal{R}+1)P_{RISS} \quad (\text{B.171})$$

$$\dot{P}_{IIIS} = 2\mathcal{R}(P_{IISS} + P_{SIIS}) - (\mathcal{R}+3)P_{IIIS} \quad (\text{B.172})$$

$$\dot{P}_{SIII} = \mathcal{R}P_{SSII} - 3(\mathcal{R}+1)P_{SIII} \quad (\text{B.173})$$

$$\dot{P}_{SSRI} = P_{SSII} - (\mathcal{R}+1)P_{SSRI} \quad (\text{B.174})$$

$$\dot{P}_{SSIR} = P_{SSII} - (2\mathcal{R}+1)P_{SSIR} \quad (\text{B.175})$$

$$\dot{P}_{SIRS} = 2P_{SIIS} - (\mathcal{R}+1)P_{SIRS} \quad (\text{B.176})$$

$$\dot{P}_{IRSS} = P_{IISS} - (2\mathcal{R}+1)P_{IRSS} \quad (\text{B.177})$$

$$\dot{P}_{IISI} = 2\mathcal{R}(P_{IISS} + P_{SSII}) + \mathcal{R}P_{IISS} - (2\mathcal{R}+3)P_{IISI} \quad (\text{B.178})$$

$$\dot{P}_{SIIR} = \mathcal{R}P_{SSIR} + P_{SIII} - 2(\mathcal{R} + 1)P_{SIIR} \quad (\text{B.179})$$

$$\dot{P}_{SIRR} = 2P_{SIIR} + P_{SIRI} - (\mathcal{R} + 1)P_{SIRR} \quad (\text{B.180})$$

$$\dot{P}_{SIRI} = 2P_{SIII} - 2(\mathcal{R} + 1)P_{SIRI} \quad (\text{B.181})$$

$$\dot{P}_{RIIS} = P_{IIIS} + \mathcal{R}P_{RISS} - 2P_{RISS} \quad (\text{B.182})$$

$$\dot{P}_{IIII} = \mathcal{R}(P_{IIIS} + 2P_{IISI} + 3P_{SIII}) - 4P_{IIII} \quad (\text{B.183})$$

$$\dot{P}_{RISI} = P_{IISI} - (\mathcal{R} + 2)P_{RISI} \quad (\text{B.184})$$

$$\dot{P}_{IRSI} = \mathcal{R}(P_{IRSS} + P_{SSRI}) + P_{IISI} - (\mathcal{R} + 2)P_{IRSI} \quad (\text{B.185})$$

$$\dot{P}_{IISR} = \mathcal{R}P_{SSIR} + P_{IISI} - 2(\mathcal{R} + 1)P_{IISR} \quad (\text{B.186})$$

$$\dot{P}_{IIRS} = 2P_{IIIS} + \mathcal{R}(P_{SIRS} + P_{IRSS}) - (\mathcal{R} + 2)P_{IIRS} \quad (\text{B.187})$$

$$\dot{P}_{SRRI} = P_{SIRI} - (\mathcal{R} + 1)P_{SRRI} \quad (\text{B.188})$$

$$\dot{P}_{RIII} = P_{IIII} + \mathcal{R}P_{RISI} - 3P_{RIII} \quad (\text{B.189})$$

$$\dot{P}_{IIIR} = P_{IIII} + 2\mathcal{R}(P_{SIIR} + P_{IISR}) - 3P_{IIIR} \quad (\text{B.190})$$

$$\dot{P}_{RRIS} = 2P_{RIIS} + P_{IIRS} - P_{RRIS} \quad (\text{B.191})$$

$$\dot{P}_{IRSR} = P_{IRSI} + P_{IISR} - (\mathcal{R} + 1)P_{IRSR} \quad (\text{B.192})$$

$$\dot{P}_{RISR} = P_{IISR} + P_{RSIR} - (\mathcal{R} + 1)P_{RISR} \quad (\text{B.193})$$

$$\dot{P}_{RRSI} = P_{IRSI} + P_{RISI} - P_{RRSI} \quad (\text{B.194})$$

$$\dot{P}_{IRRS} = P_{IIRS} - (\mathcal{R} + 1)P_{IRRS} \quad (\text{B.195})$$

$$\dot{P}_{IIRI} = 2\mathcal{R}P_{SIRI} + \mathcal{R}(P_{IIRS} + P_{IRSI}) + 2P_{IIII} - 3P_{IIRI} \quad (\text{B.196})$$

$$\dot{P}_{RRII} = 2P_{RIII} + P_{IRII} - 2P_{RRII} \quad (\text{B.197})$$

$$\dot{P}_{RRRI} = P_{RRII} + P_{IRRI} - P_{RRRI} \quad (\text{B.198})$$

$$\dot{P}_{RIIR} = P_{RIII} + P_{IIIR} + \mathcal{R}P_{RISR} - 2P_{RIIR} \quad (\text{B.199})$$

$$\dot{P}_{IIRR} = 2P_{IIIR} + \mathcal{R}(P_{IRSR} + P_{SIRR}) - 2P_{IIRR} + P_{IIRI} \quad (\text{B.200})$$

$$\dot{P}_{IRRI} = \mathcal{R}(P_{SRRI} + P_{IRRS}) + P_{IIRI} - 2P_{IRRI} \quad (\text{B.201})$$

$$\dot{P}_{RIRR} = 2P_{RIIR} + P_{RRII} + P_{IIRR} - P_{RIRR} \quad (\text{B.202})$$

$$\dot{P}_{IRRR} = P_{IRRI} + P_{IIRR} - P_{IRRR} \quad (\text{B.203})$$

Absorbing states:

$$\dot{P}_{RSSS} = P_{ISSS} \quad (\text{B.204})$$

$$\dot{P}_{SSSR} = P_{SSSI} \quad (\text{B.205})$$

$$\dot{P}_{SRSS} = P_{SISS} \quad (\text{B.206})$$

$$\dot{P}_{RSSR} = P_{ISSR} + P_{RSSI} \quad (\text{B.207})$$

$$\dot{P}_{RRSS} = P_{RISS} + P_{IRSS} \quad (\text{B.208})$$

$$\dot{P}_{SRRS} = P_{SIRS} \quad (\text{B.209})$$

$$\dot{P}_{SSRR} = P_{SSRI} + P_{SSIR} \quad (\text{B.210})$$

$$\dot{P}_{SRRR} = P_{SIRR} + P_{SRRR} \quad (\text{B.211})$$

$$\dot{P}_{RRSR} = P_{RISR} + P_{RRSI} + P_{IRSR} \quad (\text{B.212})$$

$$\dot{P}_{RRRS} = P_{RRIS} + P_{IRRS} \quad (\text{B.213})$$

$$\dot{P}_{RRRR} = P_{IRRR} + P_{RRRI} + P_{RIRR} \quad (\text{B.214})$$

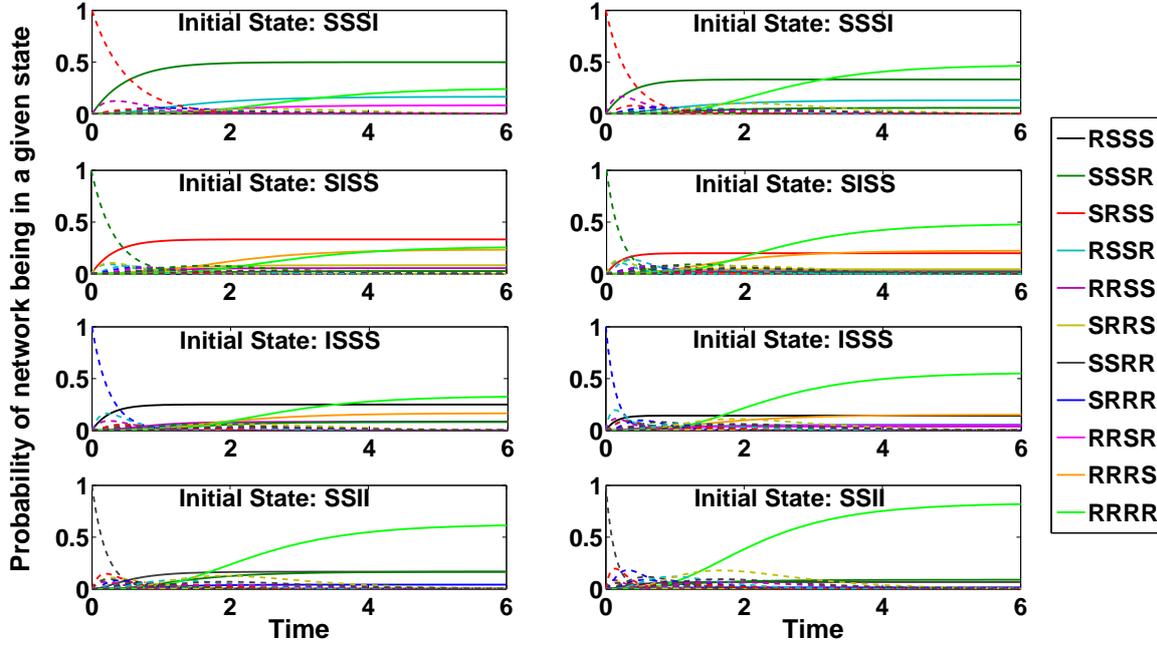


Figure B.14: Numerical results for solving the system of differential equations (B.161) - (B.214) which describes the progression of infection over time for an *SIR* model on a line network with $N = 4$ nodes. Left and right columns contain graphical results for $\mathcal{R} = 1$ and $\mathcal{R} = 2$ respectively for the specified initial conditions. Dashed lines represent transient states and solid lines are the absorbing states. The numerical results are in agreement with the analytical expressions for the same set of initial conditions.

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