**Supplementary Materials**

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# General Review on the Modified Strain Gradient Theory

In this section, some basic relations of the modified strain gradient elasticity theory which are essential for the dynamic analysis of the microbeams are briefly presented. In the proposed theory, there are only three independent higher-order material length scale parameters in addition to the two classical material constants for isotropic linear elastic materials. According to this theory, the strain energy *U* in a deformed isotropic linear elastic material occupying region is derived as ([Lam *et al.*, 2003](#_ENREF_3); [Kong *et al.*, 2009](#_ENREF_2))

|  |  |
| --- | --- |
|   | (1) |

where

|  |  |
| --- | --- |
|   | (2) |
|  | (3) |
|  | (4) |
|  | (5) |

in the above equations, , , and  represent the components of the strain tensor, dilatation gradient vector, deviatoric stretch gradient tensor and symmetric rotation gradient tensor, respectively. Also, is the differential operator,  is the displacement vector and,  and denote the Kronecker delta and the alternate tensor, respectively. Moreover, the stresses correspond to the kinematic parameters , , and are symbolized by , , , and , respectively, in whichrepresents the classical stress tensor and , , and stand for higher-order stress tensors and are given by ([Lam *et al.*, 2003](#_ENREF_3))

|  |  |
| --- | --- |
|  | (6) |
|  | (7) |
|  | (8) |
|  | (9) |

where  is deviatoric strainand ,  and  are the additional independent material length scale parameters which appear in the constitutive equations of higher order stresses. Also, and represent the bulk and shear modulus, respectively, and are given in terms of the Young’s modulus, *E* and the Poisson’s ratio, as  and.

# Potential energy of Bernoulli–Euler beams based on strain gradient energy

Consider a straight beam with the length, *L*, which is subjected to a transverse loading *q(x,t)* distributed along the longitudinal axis *x* of the beam, as shown in Fig. 1. According to the Euler–Bernoulli beam hypothesis, the displacement field of a beam in bending is assumed as

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  | (10) |

where ,  and  represent the displacement along *x*, *y* and *z* axes, respectively. Substituting Eq. (10) into Eq. (2), one obtains the non-zero strain component as

|  |  |
| --- | --- |
|  | (11) |

and also the non- zero deviatoric strain components are

|  |  |
| --- | --- |
|   | (12) |

Moreover, inserting Eq. (11) into Eqs. (3)-(5), the non-zero components of dilatation gradient vector, deviatoric stretch gradient tensor and symmetric rotation gradient tensor are respectively determined as



**Figure 1:** Schematic diagram of a microbeam subjected to a transverse loading

|  |  |
| --- | --- |
|   | (13) |
|  ,  , ,  | (14) |
|  | (15) |

Accordingly, employing the components of , , ,  and  obtained before and considering the classical and higher order stress tensors introduced in Eqs. (6)-(9), the non-zero components of these stress tensors are given as

|  |  |
| --- | --- |
|   | (16) |
|   | (17) |
| ,  , ,  | (18) |
|  | (19) |

It should be noted that for a slender beam with a large aspect ratio, the Poisson effect is minor (secondary) and may be neglected to facilitate the governing equations ([Kong *et al.*, 2009](#_ENREF_2); [Akgoz & Civalek., 2011](#_ENREF_1)). Therefore, for writing Eq. (16), the Poisson’s effect has been neglected.

Substituting Eqs. (11), (13)-(19) into Eq. (1) and integrating over the volume of the cantilever, the elastic potential energy of a simple Euler–Bernoulli microbeam can be written as

|  |  |
| --- | --- |
|   | (20) |

where *I* is the usual second moment of cross-sectional area and *A* is the cross-sectional area of the beam. Moreover, *R* and *K* are defined as

|  |  |  |
| --- | --- | --- |
|  |  | (21) |

**References**

Akgoz, B. & Civalek., O. (2011) Strain gradient elasticity and modified couple stress models for buckling analysis of axially loaded micro-scaled beams. *International Journal of Engineering Science*, **49**, 1268-1280.

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