Physical Capital, Human Capital, and the Health Effects of Pollution in an OLG Model

(Supplementary Document)

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Final Version

Appendix A More Empirical Evidence Based on PM2.5

To further support our theoretical results, in this section we present the empirical evidence based on world cross-sectional data with PM2.5 serving as the proxy of pollution instead of PM10. As a preview, our theoretical results are still supported by the empirical evidence. Figure A.1 shows the negative relationship between pollution and economic growth rate from 2013 to 2015. The pattern is similar to Figure 2 in the main text.

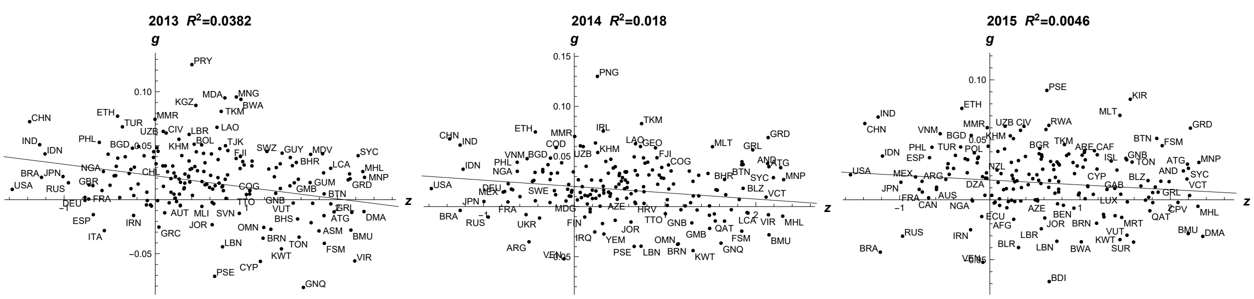

Figure A.1: The Negative Relationship between Pollution and Growth in the World (PM2.5)

To illustrate the existence of two BGPs, in Table A.1 we use K-means algorithm to conduct cluster analyses based on the growth rate of real GDP per capita ("growth rate" in the second column) and on logged values for PM2.5 weighted by population ("pollution" in the third column). The same 149 countries with complete data are endogenously divided into two groups in 1990 (Panel A of Table A.1) and in 2016 (Panel B of Table A.1). In Panel A, Group 1 consists of 75 countries and Group 2 consists of 74 countries. The average growth rate is higher and pollution is lower in Group 1 than in Groups 2. In Panel B, Group 1 consists of 73 countries and Group 2 consists of 76 countries. Again, the average growth rate is higher and pollution is lower in Group 1 than in Group 2. Thus, in 1990 and in 2016, Group 1 is on the desirable BGP whereas Group 2 is on the inferior BGP.

Table A.1: Cluster Analyses for two Groups of Countries Based on PM2.5

	Growth rate	Pollution
Panel A: Cluster analysis in 1990		
Group 1 (75 countries)	1.78% [0.03% 3.52%]	-0.01 [-0.13 0.11]
Group 2 (74 countries)	0.64% [-1.50% 2.78%]	1.40 [1.27 1.52]
Panel B: Cluster analysis in 2016		
Group 1 (73 countries)	1.82% [1.19% 2.46%]	-0.20 [-0.31 -0.09]
Group 2 (76 countries)	1.12% [0.33% 1.90%]	1.19 [1.07 1.32]

Notes. Each cell reports the group mean and 95% confidence interval in the bracket.

To illustrate the stability of the two BGPs, we carefully keep track of the transitions of each country between groups from 1990 to 2016 based on Table A.1 and construct the associated Markov transition matrix in Table A.2. We find that 97.33% of the countries that were on the desirable BGP in 1990 remain on the desirable BGP in 2016, but 2.67% of the countries that were on the desirable BGP in 1990 transition to the inferior BGP in 2016. However, no country transitions from the inferior BGP to the desirable BGP from 1990 to 2016. Therefore, the transition dynamic properties of the two BGPs are stable.

Table A.2: Markov Transition Matrix for Countries Based on PM2.5 from 1990 to 2016

	Desirable BGP	Inferior BGP
Desirable BGP	0.9733	0.0267
Inferior BGP	0	1

Notes. (1) Table A.2 is calculated based on the transitions of the same 149 countries between Groups in Table A.1.

(2) The Markov transition matrix shows the probability that a country transitions from one BGP to another. For example, the table cell indexed (1,1) says the probability that a country remains on the desirable BGP from 1990 to 2016 is 97.33%, and the table cell indexed (2,1) says the probability that a country transitions from the inferior BGP to the desirable BGP is 0.

From Figure A.1, Table A.1 and A.2, we find that if we use PM2.5 to serve as another proxy for pollution, our theoretical results are robustly consistent with the empirical evidence .

Appendix B Proof of Proposition 1 (*The PE and NPE Regimes*)

Proof. Inequality (8) is the condition under which the representative agent does not invest in private education. Substituting equations (1b), (2b), and (9b) into inequality (8) gives

$$\overline{\Phi}(k_t, z_t) < \frac{\mu\tau(1-\Delta)}{\chi\beta(1-\alpha)(1-\tau)}, \tag{B.1}$$

where $\overline{\Phi}(k_t, z_t) = \frac{\phi(k_t, z_t)}{1+\phi(k_t, z_t)}$ is the agent's propensity to save with zero private education expenditures. Because the longevity function satisfies $\phi(k_t, z_t) \in (\underline{\phi}, \overline{\phi})$, it can be shown that $\overline{\Phi}(k_t, z_t) \in \left(\frac{\underline{\phi}}{1+\underline{\phi}}, \frac{\overline{\phi}}{1+\overline{\phi}}\right)$. Three cases may arise. First, if $\frac{\mu\tau(1-\Delta)}{\chi\beta(1-\alpha)(1-\tau)} \leq \frac{\underline{\phi}}{1+\underline{\phi}}$, which says that the right-hand side of (B.1) is even smaller than the lower bound of $\overline{\Phi}(k_t, z_t)$, inequality (B.1) never holds. The representative agent always invests in private education ($e_t > 0$) for any combinations of z_t and k_t. Second, if $\frac{\mu\tau(1-\Delta)}{\chi\beta(1-\alpha)(1-\tau)} \geq \frac{\overline{\phi}}{1+\overline{\phi}}$, which says that the right-hand side of (B.1) is even larger than the upper bound of $\overline{\Phi}(k_t, z_t)$, inequality (B.1) always holds. The representative agent never invests in private education ($e_t = 0$) for any combinations of z_t and k_t. Third, if

$\frac{\underline{\phi}}{1+\underline{\phi}} < \frac{\mu\tau(1-\Delta)}{\chi\beta(1-\alpha)(1-\tau)} < \frac{\overline{\phi}}{1+\overline{\phi}}$, the representative agent may or may not invest in private education depending on the combination of z_t and k_t exogenous to her. The boundary dictating the agent's decision is defined by

$$\overline{\Phi}(k_t, z_t) = \frac{\mu\tau(1-\Delta)}{\chi\beta(1-\alpha)(1-\tau)}. \tag{B.2}$$

Because $\frac{\partial\phi(k_t,z_t)}{\partial k_t} < 0$ and $\frac{\partial\phi(k_t,z_t)}{\partial z_t} < 0$, we have $\frac{\partial\overline{\Phi}(k_t,z_t)}{\partial k_t} < 0$ and $\frac{\partial\overline{\Phi}(k_t,z_t)}{\partial z_t} < 0$. Totally differentiating (B.2) and rearranging gives

$$\frac{dk_t}{dz_t} = -\frac{\partial\overline{\Phi}(k_t,z_t)/\partial z_t}{\partial\overline{\Phi}(k_t,z_t)/\partial k_t} < 0,$$

implying that the boundary (B.2) is downward sloping in the (z_t, k_t) space. With this downward-sloping boundary, the (z_t, k_t) space is divided into the *PE* and *NPE* regimes. For the combinations of z_t and k_t lying to the bottom left of the downward-sloping boundary (B.2) and satisfying $\overline{\Phi}(k_t, z_t) > \frac{\mu\tau(1-\Delta)}{\chi\beta(1-\alpha)(1-\tau)}$, the agent invests in private education ($e_t > 0$), thus giving rise to the *PE* regime. For the combinations of z_t and k_t lying to the upper right of the downward-sloping boundary (B.2) and satisfying (B.1), the agent does not invest in private education ($e_t = 0$), thus giving rise to the *NPE* regime.

Because the third case is interesting as it shows that the agent's decision on private education expenditures are endogenously determined by the stock of pollution z_t and the ratio of physical-to-human capital k_t, we summarize the third case in Proposition 1.

Appendix C Proof of Proposition 2 (*Slope of the kk Locus*)

In this section, we prove the slopes of the *kk* loci under the *PE* and *NPE* regimes. Intuitively, the slopes of the *kk* loci reflects how the pollution stock affects the ratio of physical-to-human capital through health. We mathematically show that the slopes of the *kk* loci depend on the capital accumulation differential caused by pollution. We will first prove the slope of the *kk* locus under the *PE* regime and then under the *NPE* regime.

Proof. Under the *PE* regime, taking natural logs on both sides, totally differentiating equation (13a), and rearranging gives the slope of the kk locus in the (z_t, k_t) space:

$$\left.\frac{dk_t}{dz_t}\right|_{PE} = \frac{k_t}{z_t} \frac{E_{\Phi_{t+1},z_t} - \beta\left(E_{\Omega_{t+1},z_t} + E_{\lambda_t,z_t}\right)}{(1-\alpha+\alpha\beta) - \left(E_{\Phi_{t+1},k_t} - \beta E_{\Omega_{t+1},k_t}\right)} = \frac{k_t}{z_t} \frac{\Psi_{t,PE}}{\underbrace{1-\alpha+\alpha\beta-\Lambda_{t,PE}}_{(+)}}. \tag{C.1}$$

The denominator of equation (C.1) is positive because $1-\alpha+\alpha\beta > 0$ and $\Lambda_{t,PE} < 0$ by (17). The numerator is the capital accumulation differential of pollution under the *PE* regime in (16). As is evident in equation (C.1), the slope of the kk locus depends on the sign of the capital accumulation differential $\Psi_{t,PE}$. The kk locus slopes up if the capital accumulation differential is positive ($\Psi_{t,PE} > 0$), and slopes down if the capital accumulation differential is negative ($\Psi_{t,PE} < 0$).

Under the *NPE* regime, taking natural logs on both sides, totally differentiating equation (13b), and rearranging gives the slope of the kk locus in the (z_t, k_t) space:

$$\left.\frac{dk_t}{dz_t}\right|_{NPE} = \frac{k_t}{z_t} \frac{E_{\overline{\Phi}_{t+1},z_t} - \beta E_{\lambda_t,z_t}}{1-\alpha+\alpha\beta - E_{\overline{\Phi}_{t+1},k_t}} = \frac{k_t}{z_t} \frac{\Psi_{t,NPE}}{\underbrace{1-\alpha+\alpha\beta-\Lambda_{t,NPE}}_{(+)}}. \tag{C.2}$$

The denominator of equation (C.2) is positive because $1-\alpha+\alpha\beta > 0$ and $\Lambda_{t,NPE} < 0$ by (17). The numerator is the capital accumulation differential of pollution under the *NPE* regime in (16). Again, the slope of the kk locus depends on the sign of the capital accumulation differential $\Psi_{t,NPE}$. The kk locus slopes up if the capital accumulation differential is positive ($\Psi_{t,NPE} > 0$), and slopes down if the capital accumulation differential is negative ($\Psi_{t,NPE} < 0$).

To summarize, we have proved Proposition 2 that under regime i ($i = PE, NPE$), the slope of the kk locus in the (z_t, k_t) space depends on the sign of the capital accumulation differential caused by pollution, which can be positive or negative.

Appendix D Proof of Proposition 3 (*Dynamic Properties around the BGP*)

The mathematical exposition of Proposition 3 is given as follows:

Proposition. ***(Dynamic Properties around the BGP)*** *For $i = PE$ and NPE, the capital accumulation differential Ψ_i^* evaluated on the BGP is given in (16), and the own effect of the capital ratio Λ_i^* evaluated on the BGP is given in (17). The transition dynamic properties around the BGP can be described in the following three cases.*

(1) When $-\alpha(1-\beta)-(1-\theta)<\Lambda_i^*<0$, the parameters satisfy $\frac{(1-\theta)\left[\alpha(1-\beta)+\Lambda_i^*\right]-1}{\theta(1-\alpha)}<-\frac{\left[\alpha(1-\beta)+\Lambda_i^*-(1-\theta)\right]^2}{4\theta(1-\alpha)}<\frac{1-\alpha(1-\beta)-\Lambda_i^*}{1-\alpha}<\frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_i^*\right]}{\theta(1-\alpha)}$.

The BGP exhibits locally outward cycles if $\Psi_i^*<\frac{(1-\theta)\left[\alpha(1-\beta)+\Lambda_i^*\right]-1}{\theta(1-\alpha)}$*, dampened cycles if* $\frac{(1-\theta)\left[\alpha(1-\beta)+\Lambda_i^*\right]-1}{\theta(1-\alpha)}<\Psi_i^*<-\frac{\left[\alpha(1-\beta)+\Lambda_i^*-(1-\theta)\right]^2}{4\theta(1-\alpha)}$*, stability if* $-\frac{\left[\alpha(1-\beta)+\Lambda_i^*-(1-\theta)\right]^2}{4\theta(1-\alpha)}<\Psi_i^*<\frac{1-\alpha(1-\beta)-\Lambda_i^*}{1-\alpha}$*, saddle stability if* $\frac{1-\alpha(1-\beta)-\Lambda_i^*}{1-\alpha}<\Psi_i^*<\frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_i^*\right]}{\theta(1-\alpha)}$*, and instability if* $\frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_i^*\right]}{\theta(1-\alpha)}<\Psi_i^*$.

(2) When $-2-\alpha(1-\beta)-(1-\theta)<\Lambda_i^*<-\alpha(1-\beta)-(1-\theta)$, the parameters satisfy $\frac{(1-\theta)\left[\alpha(1-\beta)+\Lambda_i^*\right]-1}{\theta(1-\alpha)}<-\frac{\left[\alpha(1-\beta)+\Lambda_i^*-(1-\theta)\right]^2}{4\theta(1-\alpha)}<\frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_i^*\right]}{\theta(1-\alpha)}<\frac{1-\alpha(1-\beta)-\Lambda_i^*}{1-\alpha}$.

The BGP exhibits locally outward cycles if $\Psi_i^*<\frac{(1-\theta)\left[\alpha(1-\beta)+\Lambda_i^*\right]-1}{\theta(1-\alpha)}$*, dampened cycles if* $\frac{(1-\theta)\left[\alpha(1-\beta)+\Lambda_i^*\right]-1}{\theta(1-\alpha)}<\Psi_i^*<-\frac{\left[\alpha(1-\beta)+\Lambda_i^*-(1-\theta)\right]^2}{4\theta(1-\alpha)}$*, stability if* $-\frac{\left[\alpha(1-\beta)+\Lambda_i^*-(1-\theta)\right]^2}{4\theta(1-\alpha)}<\Psi_i^*<\frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_i^*\right]}{\theta(1-\alpha)}$*, saddle stability if* $\frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_i^*\right]}{\theta(1-\alpha)}<\Psi_i^*<\frac{1-\alpha(1-\beta)-\Lambda_i^*}{1-\alpha}$*, and instability if* $\frac{1-\alpha(1-\beta)-\Lambda_i^*}{1-\alpha}<\Psi_i^*$.

(3) When $\Lambda_i^*<-2-\alpha(1-\beta)-(1-\theta)$, the parameters satisfy $-\frac{\left[\alpha(1-\beta)+\Lambda_i^*-(1-\theta)\right]^2}{4\theta(1-\alpha)}<\frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_i^*\right]}{\theta(1-\alpha)}<\frac{1-\alpha(1-\beta)-\Lambda_i^*}{1-\alpha}$.

The BGP features outward cycles if $\Psi_i^*<-\frac{\left[\alpha(1-\beta)+\Lambda_i^*-(1-\theta)\right]^2}{4\theta(1-\alpha)}$*, saddle stability if* $\frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_i^*\right]}{\theta(1-\alpha)}<\Psi_i^*<\frac{1-\alpha(1-\beta)-\Lambda_i^*}{1-\alpha}$*, and is unstable if* $-\frac{\left[\alpha(1-\beta)+\Lambda_i^*-(1-\theta)\right]^2}{4\theta(1-\alpha)}<\Psi_i^*<\frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_i^*\right]}{\theta(1-\alpha)}$ *or* $\frac{1-\alpha(1-\beta)-\Lambda_i^*}{1-\alpha}<\Psi_i^*$.

As Proposition 3 applies to both the *PE* and *NPE* regimes, next we first prove Proposition 3 under the *PE* regime, and then under the *NPE* regime.

The Dynamic Properties under the *PE* Regime

Proof. When the zz locus intersects the kk locus under the PE regime, the local dynamics are dictated by (12a) and (14). Totally differentiating (12a) and (14) around the BGP gives

$$\begin{aligned} dk_{t+1} &= \frac{A^{1-\beta}}{B}\left[(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)\right]^{1-\beta}\left[\alpha(1-\beta)+\Lambda^*_{PE}\right]\frac{\Phi(k^*_{PE},z^*_{PE})}{[\Omega(k^*_{PE},z^*_{PE})\lambda(z^*_{PE})]^{\beta}}(k^*_{PE})^{\alpha-\alpha\beta-1}dk_t \\ &\quad +\frac{A^{1-\beta}}{B}\left[(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)\right]^{1-\beta}\frac{\Psi^*_{PE}}{z^*_{PE}}\frac{\Phi(k^*_{PE},z^*_{PE})}{[\Omega(k^*_{PE},z^*_{PE})\lambda(z^*_{PE})]^{\beta}}(k^*_{PE})^{\alpha-\alpha\beta}dz_t, \\ dz_{t+1} &= (1-\alpha)\frac{\rho}{\Delta\tau A}(k^*_{PE})^{-\alpha}dk_t+(1-\theta)dz_t, \end{aligned}$$

where the BGP values k^*_{PE} and z^*_{PE} are substituted in the partial derivatives, the capital accumulation differential is $\Psi^*_{PE}=E_{\Phi^*_{PE},z^*_{PE}}-\beta(E_{\Omega^*_{PE},z^*_{PE}}+E_{\lambda^*_{PE},z^*_{PE}})$, and the own effect of the capital ratio is $\Lambda^*_{PE}=E_{\Phi^*_{PE},k^*_{PE}}-\beta E_{\Omega^*_{PE},k^*_{PE}}$. Because k^*_{PE} and z^*_{PE} satisfy (13a) and (15), the above two equations can be simplified to

$$\begin{aligned} dk_{t+1} &= \left[\alpha(1-\beta)+\Lambda^*_{PE}\right]dk_t+\Psi^*_{PE}\frac{k^*_{PE}}{z^*_{PE}}dz_t, \\ dz_{t+1} &= \theta(1-\alpha)\frac{z^*_{PE}}{k^*_{PE}}dk_t+(1-\theta)dz_t. \end{aligned}$$

The associated Jacobian matrix is

$$J_{PE}=\begin{bmatrix} \alpha(1-\beta)+\Lambda^*_{PE} & \Psi^*_{PE}\dfrac{k^*_{PE}}{z^*_{PE}} \\ \theta(1-\alpha)\dfrac{z^*_{PE}}{k^*_{PE}} & 1-\theta \end{bmatrix}. \tag{D.1}$$

The trace and determinant of the Jacobian matrix are

$$TrJ_{PE}=\left[\alpha(1-\beta)+\Lambda^*_{PE}\right]+(1-\theta)<2, \tag{D.2}$$

$$DeJ_{PE}=(1-\theta)\left[\alpha(1-\beta)+\Lambda^*_{PE}\right]-\theta(1-\alpha)\Psi^*_{PE}. \tag{D.3}$$

Because $\alpha(1-\beta)<1$, $1-\theta<1$, and $\Lambda^*_{PE}<0$, inequality (D.2) always holds, implying the summation of the eigenvalues is smaller than 2. In (D.3), DeJ_{PE} is the product of the eigenvalues.

Define the characteristic polynomial $p(v) = v^2 - (TrJ_{PE})v + DeJ_{PE}$, where v is the eigenvalue. The sign of $(TrJ_{PE})^2 - 4DeJ_{PE}$ determines whether the eigenvalues have imaginary parts. From (D.2) and (D.3), we have

$$\begin{aligned}(TrJ_{PE})^2 - 4DeJ_{PE} &= \{[\alpha(1-\beta) + \Lambda^*_{PE}] + (1-\theta)\}^2 - 4(1-\theta)[\alpha(1-\beta) + \Lambda^*_{PE}] + 4\theta(1-\alpha)\Psi^*_{PE} \\ &= [\alpha(1-\beta) + \Lambda^*_{PE} - (1-\theta)]^2 + 4\theta(1-\alpha)\Psi^*_{PE}. \end{aligned} \tag{D.4}$$

And the following two expressions are useful to determine the eigenvalues relative to 1 and -1:

$$p(1) = 1 - TrJ_{PE} + DeJ_{PE} = \theta[1 - \alpha(1-\beta) - \Lambda^*_{PE}] - \theta(1-\alpha)\Psi^*_{PE}, \tag{D.5}$$

$$p(-1) = 1 + TrJ_{PE} + DeJ_{PE} = (2-\theta)[1 + \alpha(1-\beta) + \Lambda^*_{PE}] - \theta(1-\alpha)\Psi^*_{PE}. \tag{D.6}$$

Next, we make use of Equations (D.2)–(D.6) to characterize the transition dynamics around the BGP under the PE regime. But before we proceed, we first specify the following relationships that will turn out to be useful in the characterization of transition dynamics.

(1) When $0 < TrJ_{PE} < 2 \Longleftrightarrow -\alpha(1-\beta) - (1-\theta) < \Lambda^*_{PE} < 0$, it must be true that

$$\begin{aligned}\frac{(1-\theta)[\alpha(1-\beta) + \Lambda^*_{PE}] - 1}{\theta(1-\alpha)} &< \overbrace{-\frac{[\alpha(1-\beta) + \Lambda^*_{PE} - (1-\theta)]^2}{4\theta(1-\alpha)}}^{(-)} < \overbrace{\frac{1 - \alpha(1-\beta) - \Lambda^*_{PE}}{1-\alpha}}^{(+)} \\ &< \frac{(1-\theta)[\alpha(1-\beta) + \Lambda^*_{PE}] + 1}{\theta(1-\alpha)} < \frac{(2-\theta)[1 + \alpha(1-\beta) + \Lambda^*_{PE}]}{\theta(1-\alpha)}.\end{aligned}$$

(2) When $-2 < TrJ_{PE} < 0 \Longleftrightarrow -2 - \alpha(1-\beta) - (1-\theta) < \Lambda^*_{PE} < -\alpha(1-\beta) - (1-\theta)$,

it must be true that

$$\frac{(1-\theta)[\alpha(1-\beta)+\Lambda_{PE}^{*}]-1}{\theta(1-\alpha)} < \overbrace{-\frac{[\alpha(1-\beta)+\Lambda_{PE}^{*}-(1-\theta)]^{2}}{4\theta(1-\alpha)}}^{(-)} < \frac{(2-\theta)[1+\alpha(1-\beta)+\Lambda_{PE}^{*}]}{\theta(1-\alpha)}$$
$$< \frac{(1-\theta)[\alpha(1-\beta)+\Lambda_{PE}^{*}]+1}{\theta(1-\alpha)} < \frac{\overbrace{1-\alpha(1-\beta)-\Lambda_{PE}^{*}}^{(+)}}{1-\alpha}.$$

(3) When $TrJ_{PE} < -2 \Longleftrightarrow \Lambda_{PE}^{*} < -2-\alpha(1-\beta)-(1-\theta)$, it must be true that

$$-\frac{[\alpha(1-\beta)+\Lambda_{PE}^{*}-(1-\theta)]^{2}}{4\theta(1-\alpha)} < \frac{(2-\theta)[1+\alpha(1-\beta)+\Lambda_{PE}^{*}]}{\theta(1-\alpha)} < \frac{\overbrace{(1-\theta)[\alpha(1-\beta)+\Lambda_{PE}^{*}]-1}^{(-)}}{\theta(1-\alpha)}$$
$$< \frac{(1-\theta)[\alpha(1-\beta)+\Lambda_{PE}^{*}]+1}{\theta(1-\alpha)} < \frac{\overbrace{1-\alpha(1-\beta)-\Lambda_{PE}^{*}}^{(+)}}{1-\alpha}.$$

Next, we characterize the transition dynamics in the following five sub-cases numbering from (i) to (v).

(i) The Condition for Stability. When the transition dynamics exhibit stability, the two real eigenvalues fall in the range $(-1,1)$, which requires

$$\begin{cases} (TrJ_{PE})^{2}-4DeJ_{PE} > 0 \\ 1+DeJ_{PE}+TrJ_{PE} > 0 \\ 1+DeJ_{PE}-TrJ_{PE} > 0 \\ -1 < DeJ_{PE} < 1 \end{cases}$$

Then the following four inequalities hold simultaneously:

$$\begin{cases} -\dfrac{\left[\alpha(1-\beta)+\Lambda_{PE}^*-(1-\theta)\right]^2}{4\theta(1-\alpha)} < \Psi_{PE}^* \\ \Psi_{PE}^* < \dfrac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_{PE}^*\right]}{\theta(1-\alpha)} \\ \Psi_{PE}^* < \dfrac{1-\alpha(1-\beta)-\Lambda_{PE}^*}{1-\alpha} \\ \dfrac{(1-\theta)\left[\alpha(1-\beta)+\Lambda_{PE}^*\right]-1}{\theta(1-\alpha)} < \Psi_{PE}^* < \dfrac{(1-\theta)\left[\alpha(1-\beta)+\Lambda_{PE}^*\right]+1}{\theta(1-\alpha)} \end{cases} \tag{D.7}$$

When $0 < TrJ_{PE} < 2$, the stability condition is

$$-\frac{\left[\alpha(1-\beta)+\Lambda_{PE}^*-(1-\theta)\right]^2}{4\theta(1-\alpha)} < \Psi_{PE}^* < \frac{1-\alpha(1-\beta)-\Lambda_{PE}^*}{1-\alpha}. \tag{D.8}$$

When $-2 < TrJ_{PE} < 0$, the stability condition is

$$-\frac{\left[\alpha(1-\beta)+\Lambda_{PE}^*-(1-\theta)\right]^2}{4\theta(1-\alpha)} < \Psi_{PE}^* < \frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_{PE}^*\right]}{\theta(1-\alpha)}. \tag{D.9}$$

When $TrJ_{PE} < -2$, because the four inequalities listed in (D.7) cannot hold simultaneously, it is not possible for the BGP to become stable.

(ii) The Condition for Saddle Stability. When the transition dynamics exhibit saddle stability, one real eigenvalue falls within $(-1,1)$, while the other falls outside $(-1,1)$, which requires

$$\begin{cases} (TrJ_{PE})^2-4DeJ_{PE}>0 \\ 1+DeJ_{PE}+TrJ_{PE}>0 \\ 1+DeJ_{PE}-TrJ_{PE}<0 \\ TrJ_{PE}>0 \end{cases} \quad \text{or} \quad \begin{cases} (TrJ_{PE})^2-4DeJ_{PE}>0 \\ 1+DeJ_{PE}+TrJ_{PE}<0 \\ 1+DeJ_{PE}-TrJ_{PE}>0 \\ TrJ_{PE}<0 \end{cases}.$$

Substituting (D.4), (D.5), and (D.6) into the above inequalities gives

$$\begin{cases} \Psi_{PE}^{*} > -\dfrac{\left[\alpha(1-\beta)+\Lambda_{PE}^{*}-(1-\theta)\right]^{2}}{4\theta(1-\alpha)} \\ \Psi_{PE}^{*} < \dfrac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_{PE}^{*}\right]}{\theta(1-\alpha)} \\ \Psi_{PE}^{*} > \dfrac{1-\alpha(1-\beta)-\Lambda_{PE}^{*}}{1-\alpha} \\ TrJ_{PE} > 0 \end{cases} \quad \text{or} \quad \begin{cases} \Psi_{PE}^{*} > -\dfrac{\left[\alpha(1-\beta)+\Lambda_{PE}^{*}-(1-\theta)\right]^{2}}{4\theta(1-\alpha)} \\ \Psi_{PE}^{*} > \dfrac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_{PE}^{*}\right]}{\theta(1-\alpha)} \\ \Psi_{PE}^{*} < \dfrac{1-\alpha(1-\beta)-\Lambda_{PE}^{*}}{1-\alpha} \\ TrJ_{PE} < 0 \end{cases}.$$

Reorganizing the above inequalities based on the value for TrJ_{PE} gives the following cases:

When $0 < TrJ_{PE} < 2$, the condition for saddle stability is

$$\frac{1-\alpha(1-\beta)-\Lambda_{PE}^{*}}{1-\alpha} < \Psi_{PE}^{*} < \frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_{PE}^{*}\right]}{\theta(1-\alpha)}. \tag{D.10}$$

When TrJ_{PE} is negative, which can be further divided into two ranges, $-2 < TrJ_{PE} < 0$ and $TrJ_{PE} < -2$, the condition for saddle stability is

$$\frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_{PE}^{*}\right]}{\theta(1-\alpha)} < \Psi_{PE}^{*} < \frac{1-\alpha(1-\beta)-\Lambda_{PE}^{*}}{1-\alpha}. \tag{D.11}$$

(iii) The Condition for Instability. When the transition dynamics exhibit instability, the two real eigenvalues lie outside $(-1,1)$, which requires

$$\begin{cases} (TrJ_{PE})^{2}-4DeJ_{PE} > 0 \\ 1+DeJ_{PE}+TrJ_{PE} > 0 \\ 1+DeJ_{PE}-TrJ_{PE} > 0 \\ DeJ_{PE} > 1 \end{cases} \quad \text{or} \quad \begin{cases} (TrJ_{PE})^{2}-4DeJ_{PE} > 0 \\ 1+DeJ_{PE}+TrJ_{PE} < 0 \\ 1+DeJ_{PE}-TrJ_{PE} < 0 \\ DeJ_{PE} < -1 \end{cases}.$$

Substituting (D.4), (D.5), (D.6), and (D.3) into the above inequalities gives

$$\begin{cases} \Psi^*_{PE} > -\dfrac{\left[\alpha(1-\beta)+\Lambda^*_{PE}-(1-\theta)\right]^2}{4\theta(1-\alpha)} \\ \Psi^*_{PE} < \dfrac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda^*_{PE}\right]}{\theta(1-\alpha)} \\ \Psi^*_{PE} < \dfrac{1-\alpha(1-\beta)-\Lambda^*_{PE}}{1-\alpha} \\ \Psi^*_{PE} < \dfrac{(1-\theta)\left[\alpha(1-\beta)+\Lambda^*_{PE}\right]-1}{\theta(1-\alpha)} \end{cases} \quad \text{or} \quad \begin{cases} \Psi^*_{PE} > -\dfrac{\left[\alpha(1-\beta)+\Lambda^*_{PE}-(1-\theta)\right]^2}{4\theta(1-\alpha)} \\ \Psi^*_{PE} > \dfrac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda^*_{PE}\right]}{\theta(1-\alpha)} \\ \Psi^*_{PE} > \dfrac{1-\alpha(1-\beta)-\Lambda^*_{PE}}{1-\alpha} \\ \Psi^*_{PE} > \dfrac{(1-\theta)\left[\alpha(1-\beta)+\Lambda^*_{PE}\right]+1}{\theta(1-\alpha)} \end{cases}.$$

Based on the value for TrJ_{PE}, three possible cases emerge:

When $0 < TrJ_{PE} < 2$, the condition for instability is

$$\Psi^*_{PE} > \frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda^*_{PE}\right]}{\theta(1-\alpha)}. \tag{D.12}$$

When $-2 < TrJ_{PE} < 0$, the condition for instability is

$$\Psi^*_{PE} > \frac{1-\alpha(1-\beta)-\Lambda^*_{PE}}{1-\alpha}. \tag{D.13}$$

When $TrJ_{PE} < -2$, the condition for instability is

$$-\frac{\left[\alpha(1-\beta)+\Lambda^*_{PE}-(1-\theta)\right]^2}{4\theta(1-\alpha)} < \Psi^*_{PE} < \frac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda^*_{PE}\right]}{\theta(1-\alpha)} \quad \text{or} \quad \frac{1-\alpha(1-\beta)-\Lambda^*_{PE}}{1-\alpha} < \Psi^*_{PE}.$$

(iv) The Condition for Dampened Cycles. When the BGP features dampened cycles, the absolute value for the product of the two complex eigenvalues is smaller than 1, which requires

$$\begin{cases} (TrJ_{PE})^2 - 4DeJ_{PE} < 0 \\ 1 + DeJ_{PE} + TrJ_{PE} > 0 \\ 1 + DeJ_{PE} - TrJ_{PE} > 0 \\ -1 < DeJ_{PE} < 1 \end{cases}.$$

Note that $(TrJ_{PE})^2 - 4DeJ_{PE} < 0$ implies $DeJ_{PE} > 0$, and we can make use of this fact to simplify the last inequality as $0 < DeJ_{PE} < 1$. Substituting in (D.4), (D.5), (D.6), and (D.3) gives

$$\begin{cases} \Psi^*_{PE} < -\dfrac{\left[\alpha(1-\beta)+\Lambda^*_{PE}-(1-\theta)\right]^2}{4\theta(1-\alpha)} \\ \Psi^*_{PE} < \dfrac{1-\alpha(1-\beta)-\Lambda^*_{PE}}{1-\alpha} \\ \Psi^*_{PE} < \dfrac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda^*_{PE}\right]}{\theta(1-\alpha)} \\ \dfrac{(1-\theta)\left[\alpha(1-\beta)+\Lambda^*_{PE}\right]-1}{\theta(1-\alpha)} < \Psi^*_{PE} < \dfrac{(1-\theta)\left[\alpha(1-\beta)+\Lambda^*_{PE}\right]}{\theta(1-\alpha)} \end{cases}.$$

It can be verified that the expression $-\frac{[\alpha(1-\beta)+\Lambda^*_{PE}-(1-\theta)]^2}{4\theta(1-\alpha)} < \frac{(1-\theta)[\alpha(1-\beta)+\Lambda^*_{PE}]}{\theta(1-\alpha)}$ is always true. Based on the value for TrJ_{PE}, we get the following cases:

When $0 < TrJ_{PE} < 2$ or $-2 < TrJ_{PE} < 0$, the condition for dampened cycles is

$$\frac{(1-\theta)\left[\alpha(1-\beta)+\Lambda^*_{PE}\right]-1}{\theta(1-\alpha)} < \Psi^*_{PE} < -\frac{\left[\alpha(1-\beta)+\Lambda^*_{PE}-(1-\theta)\right]^2}{4\theta(1-\alpha)}. \tag{D.14}$$

However, dampened cycles cannot emerge when $TrJ_{PE} < -2$ because the above four inequalities cannot hold simultaneously.

(v) The Condition for Outward Cycles. When the BGP features outward cycles, the absolute value for the product of the two complex eigenvalues is greater than 1, which requires

$$\begin{cases} (TrJ_{PE})^2 - 4DeJ_{PE} < 0 \\ 1 + DeJ_{PE} + TrJ_{PE} > 0 \\ 1 + DeJ_{PE} - TrJ_{PE} > 0 \\ DeJ_{PE} > 1 \quad \text{or} \quad DeJ_{PE} < -1 \end{cases}.$$

Again, $(TrJ_{PE})^2 - 4DeJ_{PE} < 0$ implies that it is only possible that $DeJ_{PE} > 1$, and thus the possibility of $DeJ_{PE} < -1$ can be ruled out. Substituting in (D.4), (D.5), (D.6), and (D.3) gives

$$\begin{cases} \Psi_{PE}^{*} < -\dfrac{\left[\alpha(1-\beta)+\Lambda_{PE}^{*}-(1-\theta)\right]^{2}}{4\theta(1-\alpha)} \\ \Psi_{PE}^{*} < \dfrac{1-\alpha(1-\beta)-\Lambda_{PE}^{*}}{1-\alpha} \\ \Psi_{PE}^{*} < \dfrac{(2-\theta)\left[1+\alpha(1-\beta)+\Lambda_{PE}^{*}\right]}{\theta(1-\alpha)} \\ \Psi_{PE}^{*} < \dfrac{(1-\theta)\left[\alpha(1-\beta)+\Lambda_{PE}^{*}\right]-1}{\theta(1-\alpha)} \end{cases}.$$

The following cases can emerge based on the value for TrJ_{PE}:

When $0 < TrJ_{PE} < 2$ and $-2 < TrJ_{PE} < 0$, the condition for outward cycles is

$$\Psi_{PE}^{*} < \frac{(1-\theta)\left[\alpha(1-\beta)+\Lambda_{PE}^{*}\right]-1}{\theta(1-\alpha)}. \tag{D.15}$$

When $TrJ_{PE} < -2$, the condition for outward cycles is

$$\Psi_{PE}^{*} < -\frac{\left[\alpha(1-\beta)+\Lambda_{PE}^{*}-(1-\theta)\right]^{2}}{4\theta(1-\alpha)}. \tag{D.16}$$

Summary of the Transition Dynamic Properties under the *PE* Regime

Based on the above five sub-cases (i)-(v) establishing the conditions for stability, saddle stability, instability, dampened cycles, and outward cycles, the transition dynamic properties around the BGP under the *PE* regime can be summarized as follows. This summary constitutes the structure of the mathematical exposition for Proposition 3.

(1) When $-\alpha(1-\beta)-(1-\theta) < \Lambda_{PE}^{*} < 2-\alpha(1-\beta)-(1-\theta)$, the parameters satisfy $\frac{(1-\theta)[\alpha(1-\beta)+\Lambda_{PE}^{*}]-1}{\theta(1-\alpha)} < -\frac{[\alpha(1-\beta)+\Lambda_{PE}^{*}-(1-\theta)]^{2}}{4\theta(1-\alpha)} < \frac{1-\alpha(1-\beta)-\Lambda_{PE}^{*}}{1-\alpha} < \frac{(2-\theta)[1+\alpha(1-\beta)+\Lambda_{PE}^{*}]}{\theta(1-\alpha)}$.

The BGP exhibits locally outward cycles if $\Psi_{PE}^{*} < \frac{(1-\theta)[\alpha(1-\beta)+\Lambda_{PE}^{*}]-1}{\theta(1-\alpha)}$, dampened cycles if $\frac{(1-\theta)[\alpha(1-\beta)+\Lambda_{PE}^{*}]-1}{\theta(1-\alpha)} < \Psi_{PE}^{*} < -\frac{[\alpha(1-\beta)+\Lambda_{PE}^{*}-(1-\theta)]^{2}}{4\theta(1-\alpha)}$, stability if $-\frac{[\alpha(1-\beta)+\Lambda_{PE}^{*}-(1-\theta)]^{2}}{4\theta(1-\alpha)} < \Psi_{PE}^{*} < \frac{1-\alpha(1-\beta)-\Lambda_{PE}^{*}}{1-\alpha}$, saddle stability if $\frac{1-\alpha(1-\beta)-\Lambda_{PE}^{*}}{1-\alpha} < \Psi_{PE}^{*} < \frac{(2-\theta)[1+\alpha(1-\beta)+\Lambda_{PE}^{*}]}{\theta(1-\alpha)}$, and instability if $\frac{(2-\theta)[1+\alpha(1-\beta)+\Lambda_{PE}^{*}]}{\theta(1-\alpha)} < \Psi_{PE}^{*}$.

(2) When $-2-\alpha(1-\beta)-(1-\theta)<\Lambda^*_{PE}<-\alpha(1-\beta)-(1-\theta)$, the parameters satisfy $\frac{(1-\theta)[\alpha(1-\beta)+\Lambda^*_{PE}]-1}{\theta(1-\alpha)}<-\frac{[\alpha(1-\beta)+\Lambda^*_{PE}-(1-\theta)]^2}{4\theta(1-\alpha)}<\frac{(2-\theta)[1+\alpha(1-\beta)+\Lambda^*_{PE}]}{\theta(1-\alpha)}<\frac{1-\alpha(1-\beta)-\Lambda^*_{PE}}{1-\alpha}$.

The BGP exhibits locally outward cycles if $\Psi^*_{PE}<\frac{(1-\theta)[\alpha(1-\beta)+\Lambda^*_{PE}]-1}{\theta(1-\alpha)}$, dampened cycles if $\frac{(1-\theta)[\alpha(1-\beta)+\Lambda^*_{PE}]-1}{\theta(1-\alpha)}<\Psi^*_{PE}<-\frac{[\alpha(1-\beta)+\Lambda^*_{PE}-(1-\theta)]^2}{4\theta(1-\alpha)}$, stability if $-\frac{[\alpha(1-\beta)+\Lambda^*_{PE}-(1-\theta)]^2}{4\theta(1-\alpha)}<\Psi^*_{PE}<\frac{(2-\theta)[1+\alpha(1-\beta)+\Lambda^*_{PE}]}{\theta(1-\alpha)}$, saddle stability if $\frac{(2-\theta)[1+\alpha(1-\beta)+\Lambda^*_{PE}]}{\theta(1-\alpha)}<\Psi^*_{PE}<\frac{1-\alpha(1-\beta)-\Lambda^*_{PE}}{1-\alpha}$, and instability if $\frac{1-\alpha(1-\beta)-\Lambda^*_{PE}}{1-\alpha}<\Psi^*_{PE}$.

(3) When $\Lambda^*_{PE}<-2-\alpha(1-\beta)-(1-\theta)$, the parameters satisfy $-\frac{[\alpha(1-\beta)+\Lambda^*_{PE}-(1-\theta)]^2}{4\theta(1-\alpha)}<\frac{(2-\theta)[1+\alpha(1-\beta)+\Lambda^*_{PE}]}{\theta(1-\alpha)}<\frac{1-\alpha(1-\beta)-\Lambda^*_{PE}}{1-\alpha}$.

The BGP features outward cycles if $\Psi^*_{PE}<-\frac{[\alpha(1-\beta)+\Lambda^*_{PE}-(1-\theta)]^2}{4\theta(1-\alpha)}$, saddle stability if $\frac{(2-\theta)[1+\alpha(1-\beta)+\Lambda^*_{PE}]}{\theta(1-\alpha)}<\Psi^*_{PE}<\frac{1-\alpha(1-\beta)-\Lambda^*_{PE}}{1-\alpha}$, and is unstable if $-\frac{[\alpha(1-\beta)+\Lambda^*_{PE}-(1-\theta)]^2}{4\theta(1-\alpha)}<\Psi^*_{PE}<\frac{(2-\theta)[1+\alpha(1-\beta)+\Lambda^*_{PE}]}{\theta(1-\alpha)}$ or $\frac{1-\alpha(1-\beta)-\Lambda^*_{PE}}{1-\alpha}<\Psi^*_{PE}$.

The Dynamic Properties under the *NPE* Regime

Proposition 3 carries over to the *NPE* regime after the capital accumulation differential evaluated on the BGP Ψ^*_{PE} is replaced by $\Psi^*_{NPE}=E_{\overline{\Phi}^*_{NPE},z^*_{NPE}}-\beta E_{\lambda^*_{NPE},z^*_{NPE}}$ and the own effect of the capital ratio evaluated on the BGP is updated to $\Lambda^*_{NPE}=E_{\overline{\Phi}^*_{NPE},k^*_{NPE}}$. When the zz locus intersects with the kk locus under the *NPE* regime, the local dynamics are dictated by (12b) and (14). Totally differentiating (12b) and (14), substituting in (13b) and (15) evaluated on the BGP, and simplifying gives

$$dk_{t+1}=[\alpha(1-\beta)+\Lambda^*_{NPE}]dk_t+\Psi^*_{NPE}\frac{k^*_{NPE}}{z^*_{NPE}}dz_t,$$
$$dz_{t+1}=\theta(1-\alpha)\frac{z^*_{NPE}}{k^*_{NPE}}dk_t+(1-\theta)dz_t.$$

The associated Jacobian matrix is

$$J_{NPE}=\begin{bmatrix}\alpha(1-\beta)+\Lambda^*_{NPE} & \Psi^*_{NPE}\frac{k^*_{NPE}}{z^*_{NPE}}\\ \theta(1-\alpha)\frac{z^*_{NPE}}{k^*_{NPE}} & 1-\theta\end{bmatrix}. \tag{D.17}$$

The trace and determinant of the Jacobian matrix are

$$TrJ_{NPE} = [\alpha(1-\beta) + \Lambda^*_{NPE}] + (1-\theta) < 2, \tag{D.18}$$

$$DeJ_{NPE} = (1-\theta)\,[\alpha(1-\beta) + \Lambda^*_{NPE}] - \theta(1-\alpha)\Psi^*_{NPE}. \tag{D.19}$$

Define the characteristic polynomial $p(v) = v^2 - (TrJ_{NPE})v + DeJ_{NPE}$, where v represents the eigenvalue. Whether the eigenvalues have imaginary parts depends on

$$(TrJ_{NPE})^2 - 4DeJ_{NPE} = [\alpha(1-\beta) + \Lambda^*_{NPE} - (1-\theta)]^2 + 4\theta(1-\alpha)\Psi^*_{NPE}. \tag{D.20}$$

And the following two terms determine the eigenvalues relative to 1 and -1:

$$p(1) = 1 - TrJ_{NPE} + DeJ_{NPE} = \theta\,[1 - \alpha(1-\beta) - \Lambda^*_{NPE}] - \theta(1-\alpha)\Psi^*_{NPE}, \tag{D.21}$$

$$p(-1) = 1 + TrJ_{NPE} + DeJ_{NPE} = (2-\theta)\,[1 + \alpha(1-\beta) + \Lambda^*_{NPE}] - \theta(1-\alpha)\Psi^*_{NPE}. \tag{D.22}$$

Thus far, we can use Equations (D.18)−(D.22) to characterize the transition dynamics around the BGP under the *NPE* regime. A careful comparison of Equations (D.2)−(D.6) and (D.18)−(D.22) reveals that Ψ^*_i and Λ^*_i ($i = PE, NPE$) apply to both the *PE* and *NPE* regimes. Therefore, we have proved Proposition 3.

Appendix E Proof of Proposition 4 (*Pairwise Relationships Among the BGP Variables*)

Proof. For simplicity of the expressions, define

$$\Gamma^*_i = \begin{cases} (\alpha + E_{\Omega^*_{PE},k^*_{PE}})E_{\Phi^*_{PE},z^*_{PE}} + (1-\alpha-E_{\Phi^*_{PE},k^*_{PE}})(E_{\Omega^*_{PE},z^*_{PE}} + E_{\lambda^*_{PE},z^*_{PE}}), & i = PE \\ \alpha E_{\overline{\Phi}^*_{NPE},z^*_{NPE}} + (1-\alpha-E_{\overline{\Phi}^*_{NPE},k^*_{NPE}})E_{\lambda^*_{NPE},z^*_{NPE}}. & i = NPE \end{cases}$$

The newly-defined term $\Gamma^*_i < 0$ for regime $i = PE, NPE$ because $E_{\Phi^*_{PE},k^*_{PE}} = E_{\Omega^*_{PE},k^*_{PE}} < 0$, $E_{\Phi^*_{PE},z^*_{PE}} = E_{\Omega^*_{PE},z^*_{PE}} < 0$, $E_{\overline{\Phi}^*_{NPE},z^*_{NPE}} < 0$, and $E_{\lambda^*_{NPE},z^*_{NPE}} < 0$.

From equations (19a) and (19b) under the *PE* regime, and from equations (20a) and (20b) under the *NPE* regime, we derive the following partials that apply to regime i ($i = PE, NPE$):

$$\frac{\partial k_i^*}{\partial g_i^*}\frac{g_i^*}{k_i^*} = \frac{g_i^*}{\beta}\frac{\Psi_i^*}{\Gamma_i^*} \quad \text{and} \quad \frac{\partial z_i^*}{\partial g_i^*}\frac{g_i^*}{z_i^*} = \frac{g_i^*}{\beta}\frac{1-\alpha+\alpha\beta-\Lambda_i^*}{\Gamma_i^*} < 0, \tag{E.1}$$

where Ψ_i^* is the capital accumulation differential evaluated on the BGP given in (16), and $\Lambda_i^* < 0$ is the own effect of the capital ratio evaluated on the BGP given in (17). By (E.1), the relationship between the economic growth rate g_i^* and the pollution stock z_i^* is negative, while the relationship between g_i^* and k_i^* depends on the sign of Ψ_i^*. Therefore, we have proved Proposition 4.

Appendix F Proof of Proposition 5 (*The Policy Effects on the BGP Variables*)

Proof. The expression of Θ_i ($i = PE, NPE$) is given in (21). Using equations (19a)-(19c) under the *PE* regime, and using equations (20a)-(20c) under the *NPE* regime, we derive the policy effects of τ on k_i^*, z_i^*, and g_i^* in the form of elasticities under regime i ($i = PE, NPE$):

$$\frac{dk_i^*}{d\tau}\frac{\tau}{k_i^*} = \frac{\Psi_i^* + \beta + \frac{1}{1-\tau}(\tau - \Theta_i)}{(1-\alpha)\Psi_i^* - \left(1-\alpha+\alpha\beta-\Lambda_i^*\right)}, \tag{F.1a}$$

$$\frac{dz_i^*}{d\tau}\frac{\tau}{z_i^*} = \frac{\beta + \frac{1-\alpha}{1-\tau}(1-\Theta_i) - \Lambda_i^*}{(1-\alpha)\Psi_i^* - \left(1-\alpha+\alpha\beta-\Lambda_i^*\right)} < 0, \tag{F.1b}$$

$$\frac{dg_i^*}{d\tau}\frac{\tau}{g_i^*} = \frac{(1-\alpha)\left(\frac{dk_i^*}{d\tau}\frac{\tau}{k_i^*}\right) - \left(\frac{dz_i^*}{d\tau}\frac{\tau}{z_i^*}\right) - 1}{\left(\frac{\partial z_i^*}{\partial g_i^*}\frac{g_i^*}{z_i^*}\right) - (1-\alpha)\left(\frac{\partial k_i^*}{\partial g_i^*}\frac{g_i^*}{k_i^*}\right)}. \tag{F.1c}$$

Because we focus on the BGP that exhibits locally dampened cycles or local stability, the denominators of (F.1a) and (F.1b) are negative by the mathematical exposition of Proposition 3 in Appendix D, and thus $(1-\alpha)\Psi_i^* - (1-\alpha+\alpha\beta-\Lambda_i^*) < 0$. In (F.1a), the sign of $\frac{dk_i^*}{d\tau}\frac{\tau}{k_i^*}$ depends on the numerator $\Psi_i^* + \beta + \frac{1}{1-\tau}(\tau - \Theta_i)$. In (F.1b), it is always true that $\frac{dz_i^*}{d\tau}\frac{\tau}{z_i^*} < 0$ because $\beta + \frac{1-\alpha}{1-\tau}(1-\Theta_i) - \Lambda_i^* > 0$. In (F.1c), we can see that the sign of $\frac{dg_i^*}{d\tau}\frac{\tau}{g_i^*}$ depends on the relationship between z_i^* and g_i^* and the relationship between k_i^* and g_i^* given in (E.1), and on the effects of τ on k_i^* and z_i^* given in (F.1a) and (F.1b).

Similarly, the policy effects of Δ on k_i^*, z_i^*, and g_i^* are

$$\frac{dk_i^*}{d\Delta}\frac{\Delta}{k_i^*} = \frac{\Psi_i^* + \frac{\Delta}{1-\Delta}(\Theta_i - \beta)}{(1-\alpha)\Psi_i^* - \left(1-\alpha+\alpha\beta-\Lambda_i^*\right)}, \tag{F.2a}$$

$$\frac{dz_i^*}{d\Delta}\frac{\Delta}{z_i^*} = \frac{(1-\alpha+\alpha\beta-\Lambda_i^*) + (1-\alpha)\frac{\Delta}{1-\Delta}(\Theta_i-\beta)}{(1-\alpha)\Psi_i^* - \left(1-\alpha+\alpha\beta-\Lambda_i^*\right)}, \tag{F.2b}$$

$$\frac{dg_i^*}{d\Delta}\frac{\Delta}{g_i^*} = \frac{(1-\alpha)\left(\frac{dk_i^*}{d\Delta}\frac{\Delta}{k_i^*}\right) - \left(\frac{dz_i^*}{d\Delta}\frac{\Delta}{z_i^*}\right) - 1}{\left(\frac{\partial z_i^*}{\partial g_i^*}\frac{g_i^*}{z_i^*}\right) - (1-\alpha)\left(\frac{\partial k_i^*}{\partial g_i^*}\frac{g_i^*}{k_i^*}\right)}. \tag{F.2c}$$

Again, $(1-\alpha)\Psi_i^* - (1-\alpha+\alpha\beta-\Lambda_i^*) < 0$. In (F.2a), the sign of $\frac{dk_i^*}{d\Delta}\frac{\Delta}{k_i^*}$ depends on the numerator $\Psi_i^* + \frac{\Delta}{1-\Delta}(\Theta_i - \beta)$. In (F.2b), the sign of $\frac{dz_i^*}{d\Delta}\frac{\Delta}{z_i^*}$ depends on the regime. When $i = PE$, $\Theta_{PE} - \beta > 0$, the numerator $(1-\alpha+\alpha\beta-\Lambda_{PE}^*) + (1-\alpha)\frac{\Delta}{1-\Delta}(\Theta_{PE}-\beta) > 0$, and thus $\frac{dz_{PE}^*}{d\Delta}\frac{\Delta}{z_{PE}^*} < 0$. In contrast, when $i = NPE$, the numerator becomes $(1-\alpha+\alpha\beta-\Lambda_{NPE}^*) - (1-\alpha)\frac{\Delta}{1-\Delta}\beta$, and thus $\frac{dz_{NPE}^*}{d\Delta}\frac{\Delta}{z_{NPE}^*} \lesseqgtr 0$ if $\Delta \lesseqgtr \frac{1-\alpha+\alpha\beta-\Lambda_{NPE}^*}{1-\alpha+\beta-\Lambda_{NPE}^*}$. In (F.2c), the sign of $\frac{dg_i^*}{d\Delta}\frac{\Delta}{g_i^*}$ depends on $\frac{\partial z_i^*}{\partial g_i^*}\frac{g_i^*}{z_i^*}$ and $\frac{\partial k_i^*}{\partial g_i^*}\frac{g_i^*}{k_i^*}$ given in (E.1), and on $\frac{dk_i^*}{d\Delta}\frac{\Delta}{k_i^*}$ and $\frac{dz_i^*}{d\Delta}\frac{\Delta}{z_i^*}$ given in (F.2a) and (F.2b).

Appendix G Proof of Proposition 6 (*A Necessary Condition for the Emergence of Two Stable BGPs*)

Proof. To prove the necessary condition for two stable BGPs, we first establish two mathematical facts. First, evaluated on a stable BGP, the slope of the kk locus must be smaller than that of the zz locus. This fact applies to both regimes. For regime i ($i = PE, NPE$), by (C.1) and (C.2), the slope of the kk locus evaluated on the BGP is

$$\left.\frac{dk_t}{dz_t}\right|_{i,BGP} = \frac{k_i^*}{z_i^*}\frac{\Psi_i^*}{1-\alpha+\alpha\beta-\Lambda_i^*}.$$

By (15), the slope of the zz locus evaluated on the BGP is

$$\left.\frac{dk_t}{dz_t}\right|_{i,BGP} = \frac{k_i^*}{z_i^*}\frac{1}{1-\alpha} > 0.$$

By the mathematical exposition of Proposition 3 shown in Appendix D, a locally stable BGP implies that the following must be true:

$$\Psi_i^* < \frac{1-\alpha+\alpha\beta-\Lambda_i^*}{1-\alpha},$$

which after rearrangement leads to

$$\frac{k_i^*}{z_i^*}\frac{\Psi_i^*}{1-\alpha+\alpha\beta-\Lambda_i^*} < \frac{k_i^*}{z_i^*}\frac{1}{1-\alpha}.$$

We can see that the left-hand side is the slope of the kk locus, the right-hand side is the slope of the zz locus, and both are evaluated on the BGP. Thus, we have proved the first mathematical fact.

Second, the two kk loci are continuous when the regime switches from PE to NPE, which implies that the kk loci are everywhere continuous. Denote the switching point as (z^o, k^o), which lies on the boundary separating the PE and NPE regimes defined in Proposition 1. Suppose the kk locus under the PE regime intersects the boundary at point (z^o, k^o). If the kk locus under the NPE regime also intersects the boundary at point (z^o, k^o), we can conclude that there is no discontinuity of the kk loci when the regime switches. Thus, the basic idea is that we know (z^o, k^o) satisfies (13a) and (B.2), and we show that solving (13b) and (B.2) still yields (z^o, k^o). To achieve this goal, we show that (13a) after manipulation is identical to (13b) in form. Recall that under the PE regime, the agent's propensity to save is $\Phi(k_t, z_t) = \frac{1}{1+\chi\beta+[1/\phi(k_t,z_t)]}$ and the propensity to invest in private education is $\Omega(k_t, z_t) = \frac{\chi\beta}{1+\chi\beta+[1/\phi(k_t,z_t)]}$. Under the NPE regime, the propensity to save is $\overline{\Phi}(k_t, z_t) = \frac{1}{1+[1/\phi(k_t,z_t)]}$. When evaluated at (z^o, k^o), both $\Phi(k^o, z^o)$ and

$\Omega(k^o,z^o)$ can be expressed in terms of $\overline{\Phi}(k^o,z^o)$:

$$\Phi(k^o,z^o) = \frac{1}{\chi\beta + [1/\overline{\Phi}(k^o,z^o)]} \quad \text{and} \quad \Omega(k^o,z^o) = \frac{\chi\beta}{\chi\beta + [1/\overline{\Phi}(k^o,z^o)]}. \tag{G.1}$$

As (z^o,k^o) lies on the boundary, substituting (B.2) into (G.1) to eliminate $\overline{\Phi}(k^o,z^o)$ gives

$$\Phi(k^o,z^o) = \frac{1}{\chi\beta}\frac{\mu\tau(1-\Delta)}{(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)} \quad \text{and} \quad \Omega(k^o,z^o) = \frac{\mu\tau(1-\Delta)}{(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)}. \tag{G.2}$$

Substituting (z^o,k^o) and (G.2) into (13a) gives the kk locus under the PE regime evaluated on the switching point:

$$(k^o)^{1-\alpha+\alpha\beta} = \frac{A^{1-\beta}}{B}\frac{[\mu\tau(1-\Delta)]^{1-\beta}}{\chi\beta}\frac{1}{\lambda(z^o)^{\beta}}. \tag{G.3}$$

Using (B.2), (G.3) becomes:

$$(k^o)^{1-\alpha+\alpha\beta} = \frac{A^{1-\beta}}{B}\frac{(1-\alpha)(1-\tau)}{[\mu\tau(1-\Delta)]^{\beta}}\frac{\overline{\Phi}(k^o,z^o)}{\lambda(z^o)^{\beta}}, \tag{G.4}$$

which is identical to (13b) in form. So solving the kk locus under the NPE regime (13b) and the boundary (B.2) for z_t and k_t still yields (z^o,k^o). We conclude that (z^o,k^o) also satisfies the kk locus under the NPE regime (13b) and the boundary (B.2), and there is no discontinuity on the kk loci when the regime switches at (z^o,k^o). Therefore, we have proved the second mathematical fact.

The combination of the above two established mathematical facts implies that for two stable BGPs to arise, the kk locus must intersect the zz locus from below. Thus, on the intersection of the kk and zz loci, i.e., the BGP, the slope of the kk locus is larger than that of the zz locus. We have

$$\frac{k_i^*}{z_i^*}\frac{\Psi_i^*}{1-\alpha+\alpha\beta-\Lambda_i^*} > \frac{k_i^*}{z_i^*}\frac{1}{1-\alpha},$$

which after rearrangement leads to $\frac{1-\alpha+\alpha\beta-\Lambda_i^*}{1-\alpha} < \Psi_i^*$, implying that the BGP can be locally saddle or unstable.

Appendix H Proof of Proposition 7 (*Ranking of BGPs*)

Proof. To prove that one BGP is preferred over the other when multiple BGPs emerge, we need to compare the economic growth rate and intergenerational welfare improvement associated with each BGP. Substituting the BGP capital ratio k^*_{PE} and pollution stock z^*_{PE} into either (10a) or (11a) and taking natural logs gives the economic growth rate on the BGP under the *PE* regime:

$$g^*_{PE} = \ln\left\{[(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)]\frac{\rho}{\Delta\tau\theta}\frac{\Phi(k^*_{PE},z^*_{PE})}{z^*_{PE}}\right\}. \tag{H.1}$$

Similarly, the economic growth rate on the BGP under the *NPE* regime is

$$g^*_{NPE} = \ln\left[(1-\tau)(1-\alpha)\frac{\rho}{\Delta\tau\theta}\frac{\overline{\Phi}(k^*_{NPE},z^*_{NPE})}{z^*_{NPE}}\right]. \tag{H.2}$$

The first part of Proposition 7 says that under the same regime, policymakers prefer the BGP with a lower stock of pollution because the BGP also features a higher economic growth rate and higher intergenerational welfare improvement. To prove this fact, we check the *PE* regime and the idea also carries over to the *NPE* regime. Because the zz locus slopes up, a lower stock of pollution must be associated with a lower ratio of physical-to-human capital. Consider two BGPs under the *PE* regime, $(z^*_{PE,low},k^*_{PE,low})$ and $(z^*_{PE,high},k^*_{PE,high})$, where the subscripts *low* and *high* denote lower and higher values for the pollution stock and the capital ratio. It must be true that $0<z^*_{PE,low}<z^*_{PE,high}$ and $0<k^*_{PE,low}<k^*_{PE,high}$. By assumption, $\frac{\partial\Phi(k_t,z_t)}{\partial z_t}<0$ and $\frac{\partial\Phi(k_t,z_t)}{\partial k_t}<0$. Therefore, $\frac{\Phi(k^*_{PE,low},z^*_{PE,low})}{z^*_{PE,low}}>\frac{\Phi(k^*_{PE,high},z^*_{PE,high})}{z^*_{PE,high}}$, and by equation (H.1), we get $g^*_{PE,low}>g^*_{PE,high}$. Also by assumption, $\frac{\partial\phi(k_t,z_t)}{\partial z_t}<0$ and $\frac{\partial\phi(k_t,z_t)}{\partial k_t}<0$. Thus, $\phi(k^*_{PE,low},z^*_{PE,low})>\phi(k^*_{PE,high},z^*_{PE,high})$. By (22), we have

$$\left[1+(1+\chi)\phi(k^*_{PE,low},z^*_{PE,low})\right]g^*_{PE,low}>\left[1+(1+\chi)\phi(k^*_{PE,high},z^*_{PE,high})\right]g^*_{PE,high},$$

$$W^*_{PE,low}>W^*_{PE,high},$$

implying that under the *PE* regime, a lower stock of pollution is associated with a higher BGP economic growth rate and higher intergenerational welfare improvement, whereas a higher stock of pollution is associated with a lower BGP economic growth rate and lower intergenerational welfare improvement. This conclusion also applies to the *NPE* regime.

The second part of Proposition of 7 says that policymakers prefer the BGP under the *PE* regime over the BGP under the *NPE* regime. The reason is that the BGP under the *PE* regime features a lower stock of pollution, a higher economic growth rate, and higher intergenerational welfare improvement. From equations (H.1) and (H.2), it is difficult to directly compare the economic growth rates g^*_{PE} and g^*_{NPE}. But we know $z^*_{PE} < z^o < z^*_{NPE}$ and $k^*_{PE} < k^o < k^*_{NPE}$ because both the pollution stock and the capital ratio lie on the monotonically upward-sloping zz locus. Besides, (z^*_{PE}, k^*_{PE}) lies to the lower left of the boundary, (z^*_{NPE}, k^*_{NPE}) lies to the upper right of the boundary, and (z^o, k^o) lies on the boundary separating the *PE* and *NPE* regimes. Therefore, we can rely on (z^o, k^o) on the boundary as a baseline to indirectly compare the two economic growth rates. Recall equation (G.2) gives the propensity to save under the *PE* regime evaluated at the switching point, $\Phi(k^o, z^o) = \frac{1}{\chi\beta}\frac{\mu\tau(1-\Delta)}{(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)}$. Because $\frac{\partial\Phi(k_t,z_t)}{\partial z_t} < 0$ and $\frac{\partial\Phi(k_t,z_t)}{\partial k_t} < 0$, $k^*_{PE} < k^o$ and $z^*_{PE} < z^o$, we have $\frac{\Phi(k^*_{PE},z^*_{PE})}{z^*_{PE}} > \frac{\Phi(k^o,z^o)}{z^o}$. By equations (H.1) and (G.2), the economic growth rate on the BGP under the *PE* regime satisfies

$$\begin{aligned} g^*_{PE} &= \ln\left\{[(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)]\frac{\rho}{\Delta\tau\theta}\frac{\Phi(k^*_{PE},z^*_{PE})}{z^*_{PE}}\right\} \\ &> \ln\left\{[(1-\tau)(1-\alpha)+\mu\tau(1-\Delta)]\frac{\rho}{\Delta\tau\theta}\frac{\Phi(k^o,z^o)}{z^o}\right\} = \ln\left[\frac{\rho\mu(1-\Delta)}{\chi\beta\Delta\theta}\frac{1}{z^o}\right]. \end{aligned} \tag{H.3}$$

Similarly, recall equation (B.2) gives the propensity to save under the *NPE* regime evaluated at the switching point, $\overline{\Phi}(k^o, z^o) = \frac{\mu\tau(1-\Delta)}{\chi\beta(1-\alpha)(1-\tau)}$. Because $\frac{\partial\overline{\Phi}(k_t,z_t)}{\partial z_t} < 0$ and $\frac{\partial\overline{\Phi}(k_t,z_t)}{\partial k_t} < 0$, $z^o < z^*_{NPE}$ and $k^o < k^*_{NPE}$, we have $\frac{\overline{\Phi}(k^*_{NPE},z^*_{NPE})}{z^*_{NPE}} < \frac{\overline{\Phi}(k^o,z^o)}{z^o}$. By equations (H.2) and (B.2), the economic growth rate on the BGP under the *NPE* regime satisfies

$$g^*_{NPE} = \ln\left[(1-\tau)(1-\alpha)\frac{\rho}{\Delta\tau\theta}\frac{\overline{\Phi}(k^*_{NPE}, z^*_{NPE})}{z^*_{NPE}}\right]$$
$$< \ln\left[(1-\tau)(1-\alpha)\frac{\rho}{\Delta\tau\theta}\frac{\overline{\Phi}(k^o, z^o)}{z^o}\right] = \ln\left[\frac{\rho\mu(1-\Delta)}{\chi\beta\Delta\theta}\frac{1}{z^o}\right]. \tag{H.4}$$

Comparing equations (H.3) and (H.4) yields $g^*_{PE} > \ln\left[\frac{\rho\mu(1-\Delta)}{\chi\beta\Delta\theta}\frac{1}{z^o}\right] > g^*_{NPE}$. Thus, the BGP economic growth rate is higher under the *PE* regime than under the *NPE* regime. At last, it can be easily verified that

$$[1+(1+\chi)\phi(k^*_{PE}, z^*_{PE})]\, g^*_{PE} > [1+(1+\chi)\phi(k^*_{NPE}, z^*_{NPE})]\, g^*_{NPE},$$
$$W^*_{PE} > W^*_{NPE}.$$

Therefore, the BGP intergenerational welfare improvement is also higher under the *PE* regime than under the *NPE* regime.

Appendix I The Difference in the Slopes of the *kk* Loci When the Regime Switches

Because we have proved in Appendix G that the *kk* loci are continuous at the switching point (z^o, k^o), we compare the slope of the *kk* locus under the *PE* regime (C.1) and the slope of the *kk* locus under the *NPE* regime (C.2), both of which are evaluated at (z^o, k^o).

Under the *PE* regime, the propensity to save is $\Phi(k_t, z_t) = \frac{\phi(k_t, z_t)}{(1+\chi\beta)\phi(k_t, z_t)+1}$. The elasticities of the propensity to save with respect to the pollution stock and with respect to the capital ratio can be rewritten as expressions consisting of longevity and the elasticities of longevity with respect to the pollution stock and the capital ratio:

$$E_{\Phi(k_t, z_t), z_t} = \frac{\partial\Phi(k_t, z_t)}{\partial z_t}\frac{z_t}{\Phi(k_t, z_t)} = \frac{\frac{\partial\phi(k_t, z_t)}{\partial z_t}\frac{z_t}{\phi(k_t, z_t)}}{(1+\chi\beta)\phi(k_t, z_t)+1} = \frac{E_{\phi(k_t, z_t), z_t}}{(1+\chi\beta)\phi(k_t, z_t)+1} < 0, \tag{I.1a}$$
$$E_{\Phi(k_t, z_t), k_t} = \frac{\partial\Phi(k_t, z_t)}{\partial k_t}\frac{k_t}{\Phi(k_t, z_t)} = \frac{\frac{\partial\phi(k_t, z_t)}{\partial k_t}\frac{k_t}{\phi(k_t, z_t)}}{(1+\chi\beta)\phi(k_t, z_t)+1} = \frac{E_{\phi(k_t, z_t), k_t}}{(1+\chi\beta)\phi(k_t, z_t)+1} < 0. \tag{I.1b}$$

Under the *NPE* regime, in contrast, the propensity to save is $\overline{\Phi}(k_t, z_t) = \frac{\phi(k_t, z_t)}{\phi(k_t, z_t)+1}$. The elasticities of the propensity to save with respect to the pollution stock and with respect to the capital ratio are

$$E_{\overline{\Phi}(k_t,z_t),z_t} = \frac{\partial \overline{\Phi}(k_t,z_t)}{\partial z_t}\frac{z_t}{\overline{\Phi}(k_t,z_t)} = \frac{\frac{\partial \phi(k_t,z_t)}{\partial z_t}\frac{z_t}{\phi(k_t,z_t)}}{\phi(k_t,z_t)+1} = \frac{E_{\phi(k_t,z_t),z_t}}{\phi(k_t,z_t)+1} < 0, \tag{I.2a}$$

$$E_{\overline{\Phi}(k_t,z_t),k_t} = \frac{\partial \overline{\Phi}(k_t,z_t)}{\partial k_t}\frac{k_t}{\overline{\Phi}(k_t,z_t)} = \frac{\frac{\partial \phi(k_t,z_t)}{\partial k_t}\frac{k_t}{\phi(k_t,z_t)}}{\phi(k_t,z_t)+1} = \frac{E_{\phi(k_t,z_t),k_t}}{\phi(k_t,z_t)+1} < 0. \tag{I.2b}$$

Comparing (I.1a) and (I.2a) evaluated on (z^o, k^o) yields $0 > \frac{E_{\phi(k^o,z^o),z^o}}{(1+\chi\beta)\phi(k^o,z^o)+1} > \frac{E_{\phi(k^o,z^o),z^o}}{\phi(k^o,z^o)+1}$, and comparing (I.1b) and (I.2b) evaluated on (z^o, k^o) yields $0 > \frac{E_{\phi(k^o,z^o),k^o}}{(1+\chi\beta)\phi(k^o,z^o)+1} > \frac{E_{\phi(k^o,z^o),k^o}}{\phi(k^o,z^o)+1}$. Thus, we have

$$0 > E_{\Phi(k^o,z^o),z^o} > E_{\overline{\Phi}(k^o,z^o),z^o}, \tag{I.3a}$$

$$0 > E_{\Phi(k^o,z^o),k^o} > E_{\overline{\Phi}(k^o,z^o),k^o}. \tag{I.3b}$$

The capital accumulation differentials caused by pollution (16) and the own effects of the capital ratio (17) under the *PE* and *NPE* regimes when evaluated on (z^o, k^o) are as follows:

$$\Psi^o_{PE} = E_{\Phi(k^o,z^o),z^o} - \beta\left(E_{\Omega(k^o,z^o),z^o} + E_{\lambda(k^o,z^o),z^o}\right),$$

$$\Lambda^o_{PE} = E_{\Phi(k^o,z^o),k^o} - \beta E_{\Omega(k^o,z^o),k^o},$$

$$\Psi^o_{NPE} = E_{\overline{\Phi}(k^o,z^o),z^o} - \beta E_{\lambda(k^o,z^o),z^o},$$

$$\Lambda^o_{NPE} = E_{\overline{\Phi}(k^o,z^o),k^o}.$$

By (I.3a) and $E_{\Omega(k^o,z^o),z^o} < 0$,
$\Psi^o_{PE} - \Psi^o_{NPE} = \left(E_{\Phi(k^o,z^o),z^o} - E_{\overline{\Phi}(k^o,z^o),z^o}\right) - \beta E_{\Omega(k^o,z^o),z^o} > 0$. Then evaluated on (z^o, k^o), there are three possible cases for the relative slopes of the *kk* loci when the regime switches: (1) $\Psi^o_{PE} > \Psi^o_{NPE} > 0$; (2) $\Psi^o_{PE} > 0 > \Psi^o_{NPE}$; (3) $0 > \Psi^o_{PE} > \Psi^o_{NPE}$.

By (I.3b) and $E_{\Omega(k^o,z^o),k^o} < 0$,

$\Lambda^o_{PE} - \Lambda^o_{NPE} = \left(E_{\Phi(k^o,z^o),k^o} - E_{\overline{\Phi}(k^o,z^o),k^o}\right) - \beta E_{\Omega(k^o,z^o),k^o} > 0$. It implies

$$
\begin{aligned}
(1-\alpha+\alpha\beta) - \Lambda^o_{NPE} > (1-\alpha+\alpha\beta) - \Lambda^o_{PE} > 0, \\
\frac{1}{(1-\alpha+\alpha\beta) - \Lambda^o_{PE}} > \frac{1}{(1-\alpha+\alpha\beta) - \Lambda^o_{NPE}} > 0.
\end{aligned}
\tag{I.4}
$$

Subtracting equation (C.2) from (C.1) yields the difference in the slopes of the *kk* loci when the regime switches at (z^o, k^o):

$$
\left.\frac{dk_t}{dz_t}\right|_{PE} - \left.\frac{dk_t}{dz_t}\right|_{NPE} = \frac{k^o}{z^o}\frac{\Psi^o_{PE}}{(1-\alpha+\alpha\beta) - \Lambda^o_{PE}} - \frac{k^o}{z^o}\frac{\Psi^o_{NPE}}{(1-\alpha+\alpha\beta) - \Lambda^o_{NPE}}. \tag{I.5}
$$

By (I.5) and (I.4), when (1) $\Psi^o_{PE} > \Psi^o_{NPE} > 0$ and (2) $\Psi^o_{PE} > 0 > \Psi^o_{NPE}$, i.e., the *kk* locus slopes up under the *PE* regime by (C.1), the slope of the *kk* locus under the *PE* regime is larger than that of the *kk* locus under the *NPE* regime at the switching point (z^o, k^o). But when (3) $0 > \Psi^o_{PE} > \Psi^o_{NPE}$, i.e., the *kk* locus slopes down under the *PE* regime by (C.1), the relative slopes of the *kk* loci when the regime switches cannot be determined.

Appendix J An Alternative Model: Robustness Check

In this section, we establish an alternative model that isolates mechanisms that could possibly blur our results, thus allowing for robustness check of the results derived from the basic model. The alternative model simplifies the basic model in two major ways. First, the alternative model does not involve parents' utility derived from their children's human capital. Thus, the agent does not invest in private education and the accumulation of human capital is supported by public education expenditures only. Second, both physical and human capital fully depreciate within one period. The assumption of different types of capital depreciating at the same rate is not essential to the results and is widely employed by the literature for simplicity (see, for example, Mankiw et al., 1992; Goenka and Liu, 2020). Full depreciation of physical and human capital is a special case where both types of capital depreciate at the same rate, which makes the analysis as simple as possible. More importantly, full depreciation of physical and human capital implies no capital can be directly left from one period to the next, thus serving as another advantage of the alternative model to confirm the robustness of our results. Because parents cannot leave heritage to their children in the form of either physical or human capital, the alternative model stays away from the heritage issue. Based on the above two points, the prominent feature of the alternative model is to *completely* shut down altruism, a mechanism that turns out to be important in affecting the capital ratio (Chakraborty and Das, 2019). As is explained in the Section "Related Literature", altruistic parents favor heritage left for their children in the form of physical capital rather than in the form of human capital in the event of uncertain mortality, thus prompting parents to invest more in physical capital than in human capital. Therefore, the alternative model isolates the effect of altruism on the capital ratio from the health effects of pollution on the capital ratio, and focuses on the capital accumulation differential caused by pollution through health. Thus, our model deviates from (Chakraborty and Das, 2019). We show that our primary results survive the alternative model.

Because altruism is assumed away, the altruism parameter in equation (5) is $\chi = 0$ and the lifetime utility of the representative agent born at the beginning of period $t-1$ becomes

$$U_{t-1} = \ln c_t + \phi_{t+1} \ln d_{t+1}. \tag{J.1}$$

As the representative agent does not derive utility from her child's human capital, her private education expenditures are zero $e_t = 0$. The adulthood budget constraint becomes

$$w_t = c_t + s_t. \tag{J.2}$$

The elderhood budget constraint remains the same. The representative agent maximizes (J.1) subject to (J.2) and (6b) by choosing adulthood consumption c_t, elderhood consumption d_{t+1}, and savings s_t. Solving the agent's utility maximization problem yields the savings function:

$$s_t = \overline{\Phi}_{t+1} w_t, \tag{J.3}$$

where $\overline{\Phi}_{t+1} = \frac{\phi_{t+1}}{\phi_{t+1}+1}$ is the propensity to save when $e_t = 0$.

Because the absence of altruism leads the representative agent not to invest in private education and human capital fully depreciates within one period, private education expenditures $e_t = 0$ and the representative agent's human capital H_t does not directly come into the formation of her child's human capital H_{t+1}. As a result, the evolution of human capital (4) becomes

$$H_{t+1} = B(\lambda_t \mu m_t). \tag{J.4}$$

A comparison of the human capital evolution in the main text and the above reveals that (J.4) is a special case of (4) when $e_t = 0$ and $\beta = 1$.

In the equilibrium, $K_{t+1} = s_t$ because physical capital also fully depreciates within one period. Substituting (1b) into (J.3), (2b) into (J.4), and applying $k_t = {}^{K_t}/_{H_t}$ yields the non-linear difference equations describing the evolution of physical and human capital:

$$\frac{K_{t+1}}{K_t} = \overline{\Phi}(k_t, z_t)(1-\alpha)(1-\tau)Ak_t^{\alpha-1}, \tag{J.5}$$

$$\frac{H_{t+1}}{H_t} = B\lambda(z_t)\mu(1-\Delta)\tau Ak_t^{\alpha}, \tag{J.6}$$

where the propensity to save $\overline{\Phi}_{t+1} = \overline{\Phi}(k_t, z_t)$ is written as a function of k_t and z_t, because $\phi_{t+1} = \phi(k_t, z_t)$.

From equations (J.5) and (J.6), the ratio of physical-to-human capital evolves according to

$$k_{t+1} = \frac{(1-\alpha)(1-\tau)}{B\mu\tau(1-\Delta)} \frac{\overline{\Phi}(k_t, z_t)}{\lambda(z_t)}. \tag{J.7}$$

The evolution of the pollution stock remains the same according to (14) in the main text. To proceed, we define the capital accumulation differential caused by pollution in the alternative model:

$$\overline{\Psi}_t = E_{\overline{\Phi}_{t+1},z_t} - E_{\lambda_t,z_t},$$

where $E_{\overline{\Phi}_{t+1},z_t} = \frac{\partial \overline{\Phi}(k_t,z_t)}{\partial z_t} \frac{z_t}{\overline{\Phi}(k_t,z_t)} < 0$ is the elasticity of the propensity to save with respect to the pollution stock, which captures the physical capital effect of pollution, and $E_{\lambda_t,z_t} = \frac{\lambda'(z_t)}{\lambda(z_t)} z_t < 0$ is the elasticity of education expenditures effectiveness with respect to the pollution stock, which captures the human capital effect of pollution. Evaluated on the BGP, the capital accumulation differential is $\overline{\Psi}^* = E_{\overline{\Phi}^*,z^*} - E_{\lambda^*,z^*}$.

We also define the own effect of the capital ratio in the alternative model:

$$\overline{\Lambda}_t = E_{\overline{\Phi}_{t+1},k_t},$$

where $E_{\overline{\Phi}_{t+1},k_t} = \frac{\partial \overline{\Phi}(k_t,z_t)}{\partial k_t} \frac{k_t}{\overline{\Phi}(k_t,z_t)} < 0$ is the elasticity of the propensity to save with respect to the capital ratio, which captures how capital ratio affects itself. Evaluated on the BGP, the own effect of the capital ratio is $\overline{\Lambda}^* = E_{\overline{\Phi}^*,k^*}$.

From (J.7) and (14), we write the Jacobian matrix as

$$J = \begin{bmatrix} \overline{\Lambda}^* & \overline{\Psi}^* \dfrac{k^*}{z^*} \\ \theta(1-\alpha)\dfrac{z^*}{k^*} & 1-\theta \end{bmatrix}. \tag{J.8}$$

From (J.8), the trace and determinant of the Jacobian matrix are

$$TrJ = \overline{\Lambda}^* + (1-\theta) < 1,$$
$$DeJ = (1-\theta)\overline{\Lambda}^* - \theta(1-\alpha)\overline{\Psi}^*.$$

To characterize the transition dynamics, we define the characteristic polynomial $p(v) = v^2 - (TrJ)v + DeJ$, where v is the eigenvalue. The following equation determines whether the eigenvalues have imaginary parts:

$$(TrJ)^2 - 4DeJ = \left[\overline{\Lambda}^* - (1-\theta)\right]^2 + 4\theta(1-\alpha)\overline{\Psi}^*.$$

And the following two equations determine the eigenvalues relative to 1 and -1:

$$p(1) = 1 + DeJ - TrJ = \theta(1-\overline{\Lambda}^*) - \theta(1-\alpha)\overline{\Psi}^*,$$
$$p(-1) = 1 + DeJ + TrJ = (2-\theta)(1+\overline{\Lambda}^*) - \theta(1-\alpha)\overline{\Psi}^*.$$

Based on the above equations, we apply the method in Appendix D and get the results regarding the transition dynamics in three possible cases.

First, when $0 < TrJ < 1 \iff -(1-\theta) < \overline{\Lambda}^* < 0$, the following relationships must hold:

$$\frac{(1-\theta)\overline{\Lambda}^* - 1}{\theta(1-\alpha)} < -\frac{\left[\overline{\Lambda}^* - (1-\theta)\right]^2}{4\theta(1-\alpha)} < \frac{1-\overline{\Lambda}^*}{1-\alpha} < \frac{(2-\theta)\left(\overline{\Lambda}^*+1\right)}{\theta(1-\alpha)}.$$

The transition dynamics around the BGP exhibit outward cycles if $\overline{\Psi}^* < \frac{(1-\theta)\overline{\Lambda}^*-1}{\theta(1-\alpha)}$, dampened cycles if $\frac{(1-\theta)\overline{\Lambda}^*-1}{\theta(1-\alpha)} < \overline{\Psi}^* < -\frac{\left[\overline{\Lambda}^*-(1-\theta)\right]^2}{4\theta(1-\alpha)}$, stability if $-\frac{\left[\overline{\Lambda}^*-(1-\theta)\right]^2}{4\theta(1-\alpha)} < \overline{\Psi}^* < \frac{1-\overline{\Lambda}^*}{1-\alpha}$, saddle stability if $\frac{1-\overline{\Lambda}^*}{1-\alpha} < \overline{\Psi}^* < \frac{(2-\theta)\left(\overline{\Lambda}^*+1\right)}{\theta(1-\alpha)}$, and instability if $\frac{(2-\theta)\left(\overline{\Lambda}^*+1\right)}{\theta(1-\alpha)} < \overline{\Psi}^*$.

Second, when $-2 < TrJ < 0 \iff -2-(1-\theta) < \overline{\Lambda}^* < -(1-\theta)$, the following relationships must hold:

$$\frac{(1-\theta)\overline{\Lambda}^* - 1}{\theta(1-\alpha)} < -\frac{\left[\overline{\Lambda}^* - (1-\theta)\right]^2}{4\theta(1-\alpha)} < \frac{(2-\theta)\left(\overline{\Lambda}^*+1\right)}{\theta(1-\alpha)} < \frac{1-\overline{\Lambda}^*}{1-\alpha}.$$

The transition dynamics around the BGP exhibit outward cycles if $\overline{\Psi}^* < \frac{(1-\theta)\overline{\Lambda}^*-1}{\theta(1-\alpha)}$, dampened cycles if $\frac{(1-\theta)\overline{\Lambda}^*-1}{\theta(1-\alpha)} < \overline{\Psi}^* < -\frac{\left[\overline{\Lambda}^*-(1-\theta)\right]^2}{4\theta(1-\alpha)}$, stability if $-\frac{\left[\overline{\Lambda}^*-(1-\theta)\right]^2}{4\theta(1-\alpha)} < \overline{\Psi}^* < \frac{(2-\theta)\left(\overline{\Lambda}^*+1\right)}{\theta(1-\alpha)}$, saddle stability if $\frac{(2-\theta)\left(\overline{\Lambda}^*+1\right)}{\theta(1-\alpha)} < \overline{\Psi}^* < \frac{1-\overline{\Lambda}^*}{1-\alpha}$, and instability if $\frac{1-\overline{\Lambda}^*}{1-\alpha} < \overline{\Psi}^*$.

Third, when $TrJ < -2 \Longleftrightarrow \overline{\Lambda}^* < -2-(1-\theta)$, the following relationships must hold:

$$-\frac{\left[\overline{\Lambda}^* - (1-\theta)\right]^2}{4\theta(1-\alpha)} < \frac{(2-\theta)\left(\overline{\Lambda}^*+1\right)}{\theta(1-\alpha)} < \frac{1-\overline{\Lambda}^*}{1-\alpha}.$$

The transition dynamics around the BGP exhibit outward cycles if $\overline{\Psi}^* < -\frac{\left[\overline{\Lambda}^*-(1-\theta)\right]^2}{4\theta(1-\alpha)}$, saddle stability if $\frac{(2-\theta)\left(\overline{\Lambda}^*+1\right)}{\theta(1-\alpha)} < \overline{\Psi}^* < \frac{1-\overline{\Lambda}^*}{1-\alpha}$, and instability if $-\frac{\left[\overline{\Lambda}^*-(1-\theta)\right]^2}{4\theta(1-\alpha)} < \overline{\Psi}^* < \frac{(2-\theta)\left(\overline{\Lambda}^*+1\right)}{\theta(1-\alpha)}$ or $\frac{1-\overline{\Lambda}^*}{1-\alpha} < \overline{\Psi}^*$.

From the above results, we see that in the alternative model, the capital accumulation differential caused by pollution $\overline{\Psi}^*$ still drives the transition dynamics around the BGP. But due to the absence of altruism, the agent never invests in private education and only one regime exists. As a result, two stable BGPs lying under two regimes cannot emerge simultaneously as depicted in Figure 7. Besides, as a simplified version of the basic model, the above results also can be verified by setting $\beta = 1$ and $\chi = 0$ under the *NPE* regime in the basic model.

Evaluated on the BGP, the following partials reflecting the relationships among the BGP variables are derived, which are similar to those in the basic model:

$$\frac{\partial k^*}{\partial g^*}\frac{g^*}{k^*} = g^*\frac{\overline{\Psi}^*}{\overline{\Gamma}^*},$$
$$\frac{\partial z^*}{\partial g^*}\frac{g^*}{z^*} = g^*\frac{1-\overline{\Lambda}^*}{\overline{\Gamma}^*} < 0,$$

where $\overline{\Gamma}^* = \alpha E_{\overline{\Phi}^*,z^*} + (1-\alpha-E_{\overline{\Phi}^*,k^*})E_{\lambda^*,z^*} < 0$.

The effects of τ on k^*, z^*, and g^* are

$$\frac{dk^*}{d\tau}\frac{\tau}{k^*} = \frac{\overline{\Psi}^* + \frac{1}{1-\tau}}{(1-\alpha)\overline{\Psi}^* - \left(1-\overline{\Lambda}^*\right)},$$

$$\frac{dz^*}{d\tau}\frac{\tau}{z^*} = \frac{1 + \frac{1-\alpha}{1-\tau} - E_{\overline{\Phi}^*,k^*}}{(1-\alpha)\overline{\Psi}^* - \left(1-\overline{\Lambda}^*\right)},$$

$$\frac{dg^*}{d\tau}\frac{\tau}{g^*} = \frac{(1-\alpha)\left(\frac{dk^*}{d\tau}\frac{\tau}{k^*}\right) - \left(\frac{dz^*}{d\tau}\frac{\tau}{z^*}\right) - 1}{\left(\frac{\partial z^*}{\partial g^*}\frac{g^*}{z^*}\right) - (1-\alpha)\left(\frac{\partial k^*}{\partial g^*}\frac{g^*}{k^*}\right)}.$$

The effects of Δ on k^*, z^*, and g^* are

$$\frac{dk^*}{d\Delta}\frac{\Delta}{k^*} = \frac{\overline{\Psi}^* - \frac{\Delta}{1-\Delta}}{(1-\alpha)\overline{\Psi}^* - \left(1-\overline{\Lambda}^*\right)},$$

$$\frac{dz^*}{d\Delta}\frac{\Delta}{z^*} = \frac{\left(1 - E_{\overline{\Phi}^*,k^*}\right) - (1-\alpha)\frac{\Delta}{1-\Delta}}{(1-\alpha)\overline{\Psi}^* - \left(1-\overline{\Lambda}^*\right)},$$

$$\frac{dg^*}{d\Delta}\frac{\Delta}{g^*} = \frac{(1-\alpha)\left(\frac{dk^*}{d\Delta}\frac{\Delta}{k^*}\right) - \left(\frac{dz^*}{d\Delta}\frac{\Delta}{z^*}\right) - 1}{\left(\frac{\partial z^*}{\partial g^*}\frac{g^*}{z^*}\right) - (1-\alpha)\left(\frac{\partial k^*}{\partial g^*}\frac{g^*}{k^*}\right)}.$$

To conclude, the alternative model completely shuts down altruism by dropping the agent's utility derived from her child's human capital and by assuming full depreciation of both physical and human capital. The alternative model simplifies the basic model under the *NPE* regime by setting $\chi = 0$ and $\beta = 1$. The results derived from the alternative model show that the capital accumulation differential caused by pollution through health still is the key driving force.

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