VARIETY, COMPETITION, AND POPULATION IN ECONOMIC GROWTH: THEORY AND EMPIRICS

Alberto BUCCI *   Lorenzo CARBONARI †   Giovanni TROVATO ‡

ON – LINE APPENDICES A, B AND C

(*NOT INTENDED FOR PUBLICATION*)

ON – LINE APPENDIX A: EQS. (18) – (20)

From Eqs. (5) and (2) in the main text:

(A1) $$w_{It} = \left(\frac{\alpha}{m}\right) L_{Yt}^{1-\alpha} \left[\frac{1}{N_t^{\beta}} \int_0^{N_t} (x_{it})^{1/m} di\right]^{\alpha m - 1} \frac{1}{N_t^{\beta}} (x_{it})^{(1-m)/m}.$$

Using the symmetric equilibrium assumption –(Eq. 4') in the text– (A1) above can be recast as:

(A2) $$w_{It} = \left(\frac{\alpha}{m}\right)\left(\frac{L_{Yt}}{N_t}\right)^{1-\alpha}\left(\frac{L_{It}}{N_t}\right)^{\alpha-1} N_t^{\Phi}, \qquad \text{where } \Phi \equiv \alpha\left[m(1-\beta)-1\right].$$

From the aggregate production function (Eq. 1) and the hypothesis of symmetry (Eq. 4'):

(A3) $$w_{Yt} = (1-\alpha)\left(\frac{L_{Yt}}{N_t}\right)^{-\alpha}\left(\frac{L_{It}}{N_t}\right)^{\alpha} N_t^{\Phi}.$$

Using Eq. (15) in the text and equating (A2) and (A3) above yield:

(A4) $$s_Y \equiv \frac{L_{Yt}}{L_t} = m\left(\frac{1-\alpha}{\alpha}\right)s_I, \qquad \text{where } s_I \equiv \frac{L_{It}}{L_t}, \text{ and } s_Y = \frac{L_{Yt}}{L_t}.$$

Plugging (A4) into Eq. (14) in the main text delivers:

(A5) $$s_I = \left[\frac{\alpha}{m(1-\alpha)+\alpha}\right](1-s_N), \qquad \text{where } s_N \equiv \frac{L_{Nt}}{L_t}.$$

Consequently:

(A4') $$s_Y = \left[\frac{m(1-\alpha)}{m(1-\alpha)+\alpha}\right](1-s_N).$$

Using Eq. (6), Eq. (9) in the text can be re-written as:

* University of Milan (DEMM and FinGro Lab), CEIS (University of Rome 'Tor Vergata'), and RCEA (Rimini Center for Economic Analysis)

† **Corresponding Author.** University of Rome 'Tor Vergata' (DEF and CEIS), and FinGro Lab (University of Milan). E-mail address: lorenzo.carbonari@uniroma2.it

‡ University of Rome 'Tor Vergata' (DEF and CEIS), and FinGro Lab (University of Milan)

(A6) $V_{Nt}=\int_t^{\infty}\alpha\left(\frac{m-1}{m}\right)\left(\frac{L_{Y\tau}}{N_\tau}\right)^{1-\alpha}\left(\frac{L_{I\tau}}{N_\tau}\right)^{\alpha}N_\tau^{\Phi}e^{-\int_t^{\tau}r(s)ds}d\tau$.

Along a BGP, all variables depending on time grow at constant exponential rates, which implies that r is constant (see Eq. 13 in the text). Thus, V_{Nt} equals:

(A6') $V_{Nt}=\alpha\left(\frac{m-1}{m}\right)\left(\frac{L_{Yt}}{N_t}\right)^{1-\alpha}\left(\frac{L_{It}}{N_t}\right)^{\alpha}\frac{N_t^{\Phi}}{\left[r-n+(1-\Phi)\gamma_N\right]}$, $\qquad n\equiv\frac{\dot{L}_t}{L_t}$, $\qquad \gamma_N\equiv\frac{\dot{N}_t}{N_t}$.

Note that for any $\alpha\in(0;1)$, $m>1$, $L_Y>0$, $L_I>0$ and $N>0$, V_{Nt} is positive $\forall t\geq 0$ when:

$$r>n-(1-\Phi)\gamma_N .$$

Given V_{Nt}, from Eq. (8) in the main text it follows:

(A7) $w_{Nt}=\frac{1}{\chi}\frac{N_t^{\phi}}{L_{Nt}^{1-\lambda}}\alpha\left(\frac{m-1}{m}\right)\left(\frac{L_{Yt}}{N_t}\right)^{1-\alpha}\left(\frac{L_{It}}{N_t}\right)^{\alpha}\frac{N_t^{\Phi}}{\left[r-n+(1-\Phi)\gamma_N\right]}$.

We can now use again Eq. (15) in the main text and equalize (A2) and (A7), so obtaining:

(A8) $s_I=\left(\frac{\chi}{m-1}\right)\frac{L_{Nt}^{1-\lambda}}{L_tN_t^{\phi-1}}\left[r-n+(1-\Phi)\gamma_N\right]$.

From (A2), (A3) and (A7), we observe that along a BGP:

(A9) $\frac{\dot{w}_{It}}{w_{It}}=\frac{\dot{w}_{Yt}}{w_{Yt}}=\frac{\dot{w}_{Nt}}{w_{Nt}}\equiv\frac{\dot{w}_t}{w_t}=\Phi\gamma_N$.

This result is obtained by noticing (see Eq. 7 in the main text) that in a BGP equilibrium:

$$\lambda n=(1-\phi)\gamma_N .$$

In order to find out the shares of labor allocated to the final output, intermediates and research sectors ($s_Y\equiv L_{Yt}/L_t$, $s_I\equiv L_{It}/L_t$, and $s_N\equiv L_{Nt}/L_t$, respectively), first of all notice that Eq. (A8) can be recast as:

(A8') $s_I=\left(\frac{\chi}{m-1}\right)s_Ns_N^{-\lambda}\frac{L_t^{-\lambda}}{N_t^{\phi-1}}\left[r-n+(1-\Phi)\gamma_N\right]$, where $s_N^{-\lambda}\frac{L_t^{-\lambda}}{N_t^{\phi-1}}\equiv\frac{1}{\chi\gamma_N}$ by using Eq. (7) in the text.

Equalization of (A8') and (A5), then, delivers:

$$s_N=\frac{\left[\frac{\alpha}{m(1-\alpha)+\alpha}\right]}{\left[\frac{r-n+(1-\Phi)\gamma_N}{(m-1)\gamma_N}+\frac{\alpha}{m(1-\alpha)+\alpha}\right]}.$$

Along a BGP, $s_N\in(0;1)$ as long as $r>n-(1-\Phi)\gamma_N$, and $\gamma_N>0$. Given s_N, we can obtain s_I from (A5):

$$s_I=\left[\frac{\alpha}{m(1-\alpha)+\alpha}\right](1-s_N),$$

and s_Y from (A4'):

$$s_Y=\left[\frac{m(1-\alpha)}{m(1-\alpha)+\alpha}\right](1-s_N).$$

With $s_N\in(0;1)$, it is immediate to see that $s_I\in(0;1)$ and $s_Y\in(0;1)$, too. From Eq. (10) in the text, we have:

(A10) $\frac{\dot{A}_t}{A_t}=r+\frac{L_t}{A_t}w_t-\frac{C_t}{A_t}$.

Eqs. (16) in the text and (A6') above lead to:

(A11) $\gamma_A = n + \Phi\gamma_N$, $\qquad \gamma_A \equiv \frac{\dot{A}_t}{A_t}$.

We now use (A11), (A9), and the fact that along a BGP $\gamma_N = \left(\frac{\lambda}{1-\phi}\right)n$ into (A10), and get:

(A10') $\frac{C_t}{A_t} = r - n - \Phi\gamma_N + \frac{L(0)w(0)}{a(0)}$, $\qquad L(0) = 1 > 0$; $\quad a(0) = A(0) > 0$; $\qquad w(0) > 0$,

where $L(0)$, $w(0)$ and $a(0)$ are the given initial values (*i.e.*, at $t = 0$) of L_t, w_t and a_t, respectively. Eq. (A10') suggests that along a BGP, C_t and A_t grow at the same constant rate, *i.e.*:

(A10") $\frac{\dot{A}_t}{A_t} = \frac{\dot{C}_t}{C_t}$.

The aggregate production function (Eq. 1), together with the hypothesis of symmetry (Eq. 4'), delivers:

(A12) $Y_t = \left(\frac{L_{Yt}}{N_t}\right)^{1-\alpha}\left(\frac{L_{It}}{N_t}\right)^{\alpha} N_t^{1+\Phi}$.

This implies that along a BGP:

(A12') $\frac{\dot{Y}_t}{Y_t} \equiv \gamma_Y = n + \Phi\gamma_N$.

In this economy (see Eq. 10 in the main text), aggregate income ($Y_t \equiv r_t A_t + w_t L_t$) can be in part consumed (C_t) and in part used to accumulate more assets ($\dot{A}_t$). In other words:

$$Y_t = C_t + \dot{A}_t .$$

The last equation implies:

$$\frac{Y_t}{A_t} = \frac{C_t}{A_t} + \frac{\dot{A}_t}{A_t} .$$

Along a BGP, $\frac{\dot{A}_t}{A_t}$ and $\frac{C_t}{A_t}$ are constant. Therefore, it follows that (see Eqs. A10", A11 and A12' above):

(A13) $\frac{\dot{Y}_t}{Y_t} = \frac{\dot{A}_t}{A_t} = \frac{\dot{C}_t}{C_t}$.

Using Eq. (13) in the main text, we can write the growth rate of aggregate consumption as:

(A14) $\frac{\dot{C}_t}{C_t} = \frac{\dot{c}_t}{c_t} + n = \frac{1}{\theta}(r - \rho) + n$.

From (A13), (A14) and (A12') in this appendix we get:

(A15) $r = \theta\Phi\gamma_N + \rho$.

Along a BGP:

(A16) $\gamma_N = \left(\frac{\lambda}{1-\phi}\right)n$, $\qquad \frac{\dot{L}_t}{L_t} \equiv n > 0$.

From (A15) and (A16) it is immediate to obtain:

(A15') $r = \theta\Phi\left(\frac{\lambda}{1-\phi}\right)n + \rho$.

Combining (A12'), (A13) and (A16) yields:

(A17) $\frac{\dot{y}_t}{y_t} = \frac{\dot{c}_t}{c_t} = \frac{\dot{a}_t}{a_t} = \Phi\left(\frac{\lambda}{1-\phi}\right)n$.

In the end of this appendix we want also to make sure that the transversality condition:

$$\lim_{t\to+\infty} \lambda_{at} a_t = 0$$

holds. At this aim, note that from the (necessary) FOCs taken on the Hamiltonian function (J):

$$J = \left(\frac{c^{1-\theta}-1}{1-\theta}\right)e^{-(\rho-n)t} + \lambda_a\left[(r-n)a + w - c\right],$$

we have:

(A18) $\frac{\partial J}{\partial a} = -\dot{\lambda}_a \quad \Leftrightarrow \quad \frac{\dot{\lambda}_a}{\lambda_a} = -(r-n)$,

where λ_a is the co-state variable associated to the state variable, a. By using the last equation, along with (A11), the definition of $a \equiv A/L$, and $\dot{L}_t / L_t \equiv n$, the transversality condition can ultimately be recast as:

(A19) $\lim_{t\to+\infty} \lambda_{at} a_t = \lambda_a(0)a(0)\lim_{t\to+\infty} e^{-(r-n-\Phi\gamma_N)t} = 0$.

In (A19) $a(0) > 0$ and $\lambda_a(0) = \frac{1}{c(0)^\theta} > 0$ are the initial values (*i.e.*, at $t = 0$) of the state variable a and the co-state variable λ_a, respectively. Eq. (A19) reveals that the transversality condition is satisfied whenever the following inequality is met:

(TC) $r > n + \Phi\gamma_N$.

After using (A15) and (A16), it is possible to conclude that condition (TC) is equivalent to:

$$\rho > n + \Phi(1-\theta)\left(\frac{\lambda}{1-\phi}\right)n.$$

With $\gamma_c \equiv \frac{\dot{c}_t}{c_t}$ provided by (A17) above, *Assumption 1* in the main text allows the last condition (*i.e.*, the transversality condition) to be always checked.

Finally, notice that when the transversality condition (TC) is satisfied, then the condition that guarantees that V_{Nt} is positive at any time $t \geq 0$ –*i.e.*, $r > n - (1-\Phi)\gamma_N$ – is also met for any $\gamma_N > 0$, which is always true in our model's BGP equilibrium –see Eq. (18) in the main text. ■

ON – LINE APPENDIX B

In this Appendix we highlight the main differences (in the assumptions and in the results) between our model and the canonical semi-endogenous growth theory by Jones (*JPE*, 1995). In the table below, we start by comparing the two models' technological assumptions (for the sake of simplicity the time-index t is suppressed):

OUR MODEL	**JONES (*JPE*, 1995)**
$Y = L_Y^{1-\alpha}\left[\frac{1}{N^{\beta}}\int_0^N \left(x_i\right)^{1/m} di\right]^{\alpha m}$, (Eq.1) $\alpha \in (0;1)$, $m > 1$	$Y = L_Y^{\alpha}\int_0^A x_i^{1-\alpha} di$ (Eq. A1, p. 780)
$x_i = l_i$, $\forall i$ (Eq. 3)	*"…A firm that has purchased a design can… …transform each unit of **capital** into a single unit of the intermediate input…"* (p. 780)
$\dot{N} = \frac{1}{\chi} L_N^{\lambda} N^{\phi}$, (Eq. 7) $0 < \lambda \leq 1$, $\phi < 1$	$\dot{A} = \delta L_A^{\lambda} A^{\phi}$ (Eq. 6, p. 765 and the following discussion) $0 < \lambda \leq 1$, $\phi < 1$

Evidently, across the two models:

$$A = N, \qquad \text{and} \qquad L_A = L_N,$$

where $A = N$ denotes in the two frameworks the number of existing varieties (indexed by i) of intermediate inputs, and $L_A = L_N$ is the amount of the labor-input employed in the research sector. It is also apparent that in Jones (1995) labor is employed solely to produce final output and to invent new ideas (in our model, instead, labor is also an input in the production of intermediate inputs – see below).

There are two further (and, probably, more relevant) differences across the two models. The first is related to the way the optimal gross markup is determined in the monopolistically competitive intermediate sector. In Jones (1995), the elasticity of substitution (in absolute value) between two generic varieties of intermediate inputs employed in the production of final output (e) is equal to

$$e = \frac{1}{\alpha}.$$

This elasticity depends only on α, which is the labor-share in aggregate GDP (the sector that produces final output is perfectly competitive and rewards rival inputs at their marginal productivity). So, in Jones (1995), according to the usual *'markup-rule'*, the optimal markup, *m*, charged over the marginal production cost (the real interest rate, r) by each uncompetitive intermediate firm is

$$m = \frac{e}{e-1} = \frac{1}{1-\alpha} = \frac{1}{1 - Labor\ Share} > 1 \qquad \text{(see Jones, 1995, Eq. A5, p. 780).}$$

Instead, in our model the elasticity of substitution between any two generic varieties of intermediate inputs in final output production is $m/\left(m-1\right) > 1$, which is independent of the labor-share in aggregate GDP ($1-\alpha$). As a consequence, the markup (m) of price over the marginal production cost (the wage rate, w) in the intermediate sector is also independent of such share.

In brief, the first relevant difference between our model and Jones (1995) is that while our model does disentangle the gross markup of price over the marginal production cost from the shares of factor-inputs in GDP, Jones (1995) does not.

The second fundamental difference between Jones (1995) and our model is related to the production function employed by intermediate firms. In Jones (1995), it is postulated that intermediate firms produce by employing units of physical capital (more precisely, units of forgone consumption) as an input, whereas in our model they produce (one-to-one) with labor.

After briefly highlighting the main differences in the assumptions of our model with respect to Jones (1995)'s model, we are now able to show that USING THE JONES (1995)'S ASSUMPTIONS INTO OUR MODEL this model can reproduce exactly the same BGP growth rate of per capita income of Jones (1995). In this way it is possible to conclude that the different predictions of the two models about the long-run growth rate of the economy are ultimately due to the main two differences (just explained) in the basic assumptions of these models.

In a symmetric equilibrium in which:

$$x_i = x = \frac{K}{N}, \qquad \forall i \in [0; N], \qquad \text{(Jones, 1995, p. 781, Eq. A8)},$$

the production function of our model can be recast as

$$Y = L_Y^{1-\alpha}\left[\frac{1}{N^{\beta}}\int_0^N (x_i)^{1/m}\,di\right]^{\alpha m} = (1-s)^{1-\alpha}\left(L^{1-\alpha}K^{\alpha}\right)N^{\alpha[m(1-\beta)-1]}$$

where

$$L_Y = (1-s)L, \qquad L_A = sL, \qquad L_Y + L_A = L,$$

have been used (see Jones, 1995, p. 782, and Eq. 15, p. 770). Along a BGP, the decentralized economy allocates constant shares (s and $1-s$, respectively) of the available labor force/population (L) to the invention of new ideas and to the production of final output, (Jones, 1995, Eq. 10 at p. 769).
In Jones (1995, Eq. A1, p. 780), $\beta = 0$. Hence:

$$Y = L_Y^{1-\alpha}\left[\frac{1}{N^{\beta}}\int_0^N (x_i)^{1/m}\,di\right]^{\alpha m} = (1-s)^{1-\alpha}\left(L^{1-\alpha}K^{\alpha}\right)N^{\alpha(m-1)} \tag{B1}$$

From (B1), the growth rate of aggregate GDP is:

$$\frac{\dot{Y}}{Y} = (1-\alpha)\frac{\dot{L}}{L} + \alpha\frac{\dot{K}}{K} + \alpha(m-1)\frac{\dot{N}}{N} = (1-\alpha)n + \alpha\frac{\dot{K}}{K} + \alpha(m-1)\frac{\dot{N}}{N}, \qquad \frac{\dot{L}}{L} \equiv n \text{ (Jones, 1995, p. 770).}$$

By defining by $y \equiv \frac{Y}{L}$ the output per worker and by $k \equiv \frac{K}{L}$ the capital/labor ratio (Jones, 1995, p. 767), from the previous equation we can obtain the growth rate of per capita income ($\dot{y}/y$) as:

$$\frac{\dot{y}}{y} = (1-\alpha)n + \alpha\left(\frac{\dot{k}}{k} + n\right) + \alpha(m-1)\frac{\dot{N}}{N} - n. \tag{B2}$$

The constancy of y/k (Jones, 1995, p. 782) allows us to re-write Eq. (B2) as:

$$(1-\alpha)\gamma = \alpha(m-1)\frac{\dot{N}}{N}, \qquad \gamma_y = \gamma_k \equiv \gamma.$$

We have already discussed above the fact that in Jones (1995):

$$m = \frac{1}{1 - Labor\ Share}.$$

In our model, the labor share in GDP is equal to $\left(1-\alpha\right)$. Hence, in our model (under the assumptions of Jones, 1995) the markup would be:

$$m=\frac{1}{\alpha}.$$

This in turn leads to:

$$\left(1-\alpha\right)\gamma=\alpha\left(m-1\right)\frac{\dot{N}}{N}=\left(1-\alpha\right)\frac{\dot{N}}{N}.$$

In other words,

$$\gamma_y=\gamma_k=\gamma_N\equiv\gamma, \qquad \frac{\dot{N}}{N}\equiv\gamma_N.$$

Finally, from our model's R&D technology we conclude that in a BGP equilibrium in which *the growth rates of all variables are constant* (Jones, 1995, p. 782):

$$\gamma_N\equiv\frac{\dot{N}}{N}=\frac{\lambda n}{\left(1-\phi\right)}=\gamma_y=\gamma_k\equiv\gamma.$$

This is the (common) growth rate of per capita variables in Jones (1995, Eq. 8, p. 767). In this case, it is evident that *"…The growth rate of the economy…depends only on the growth rate of the labor force and the parameters ϕ and λ, which determine the external returns (as well as the returns to scale) in the R&D sector…"* (Jones, 1995, p. 767).

Clearly, WITHOUT USING THE JONES' (1995) ASSUMPTIONS IN OUR MODEL, we observe that in a symmetric equilibrium in which $x_i=x=\frac{L_I}{N}$, $\forall i$ (Eq. 4' in the body-text of our paper), the aggregate production function would read as:

$$Y=L_Y^{1-\alpha}\left[\frac{1}{N^{\beta}}\int_0^N\left(x_i\right)^{1/m}di\right]^{\alpha m}=\left(s_Y^{1-\alpha}s_I^{\alpha}\right)LN^{\alpha\left[m(1-\beta)-1\right]}, \qquad s_Y\equiv\frac{L_Y}{L} \quad \text{and} \quad s_I\equiv\frac{L_I}{L}.$$

Along a BGP, the shares of labor going to each sector that employs such an input ($s_Y\equiv L_Y/L$, $s_I\equiv L_I/L$, and $s_N\equiv L_N/L$) are constant – see *On-line Appendix A*. Hence, the growth rate of aggregate GDP is:

$$\frac{\dot{Y}}{Y}=\frac{\dot{L}}{L}+\alpha\left[m\left(1-\beta\right)-1\right]\gamma_N=n+\alpha\left[m\left(1-\beta\right)-1\right]\gamma_N, \qquad \frac{\dot{L}}{L}\equiv n, \qquad \frac{\dot{N}}{N}\equiv\gamma_N.$$

By defining by $y\equiv\frac{Y}{L}$ output per capita, $\dot{y}/y$ will be equal to:

$$\frac{\dot{y}}{y}=\underbrace{\alpha\left[m\left(1-\beta\right)-1\right]}_{\equiv\Phi}\gamma_N=\Phi\gamma_N.$$

Notice that in our model the markup (m) is independent of the factor-input shares in GDP. From our R&D technology, we see that in the BGP equilibrium

$$\gamma_N\equiv\frac{\dot{N}}{N}=\underbrace{\frac{\lambda}{\left(1-\phi\right)}}_{\equiv\Psi}n.$$

Therefore,

$$\frac{\dot{y}}{y} = \alpha\left[m(1-\beta)-1\right]\gamma_N = \Phi\Psi n\,.$$

It is apparent that in our model (Eq. 19 in the text) the growth rate of the economy depends not only on the growth rate of the labor force (n) and the parameters of the R&D technology ϕ and λ (as in Jones, 1995), but also on α (the share of GDP that goes to intermediate inputs), the markup (m), and, more importantly, the parameter β, which represents the main novelty of our work.

All in all, from above it can be inferred that the difference between our economy's growth rate (Eq. 19 in the text),

$$\frac{\dot{y}}{y} = \alpha\left[m(1-\beta)-1\right]\left(\frac{\lambda}{1-\phi}\right)n\,,$$

and that obtained by Jones (1995, Eq. 8, p. 767),

$$\frac{\dot{y}}{y} = \frac{\lambda}{(1-\phi)}n\,,$$

can be entirely explained by the presence of two different assumptions across the two approaches, namely by the fact that in our model:

(1) Intermediate firms produce with labor (rather than physical capital), and

(2) The gross markup of price over the marginal cost of production has been disentangled from the factor-input shares in GDP. ■

ON – LINE APPENDIX C: VERSION OF THE MODEL WITH HOUSEHOLD'S INVESTMENT IN PHYSICAL CAPITAL (THE ONLY INPUT IN THE PRODUCTION OF INTERMEDIATE INPUTS)

In this Appendix we formally show that our model's basic results do not qualitatively change if physical capital (as opposed to labor), accumulated through households' savings, is assumed to be the only input in the production of intermediate inputs. At this aim, we start from Eq. (1) in the text:

$$Y_t = L_{Yt}^{1-\alpha}\left[\frac{1}{N_t^{\beta}}\int_0^{N_t}\left(x_{it}\right)^{1/m} di\right]^{\alpha m}, \qquad 0<\alpha<1, \qquad m>1. \tag{1}$$

Using Eq. (1), it is possible to compute the inverse demand function for the *i-th* intermediate:

$$p_{it} = \alpha L_{Yt}^{1-\alpha}\left[\frac{1}{N_t^{\beta}}\int_0^{N_t}\left(x_{it}\right)^{1/m} di\right]^{\alpha m-1}\frac{1}{N_t^{\beta}}\left(x_{it}\right)^{(1-m)/m}. \tag{2}$$

Following Romer (1990), we now postulate that monopolistically competitive firms have access to the same technology employing solely physical capital (namely, forgone output), k, as an input:

$$x_{it} = k_{it}, \qquad \forall i\in\left[0; N_t\right], \qquad N_t\in[0;\infty), \tag{3}$$

Thus, the marginal cost of production is now the real interest rate, r. For given N_t, Eq. (3) implies that the total amount of capital employed in the intermediate sector at time *t* (K_t) is:

$$\int_0^{N_t} x_{it}di = \int_0^{N_t} k_{it}di \equiv K_t\ . \tag{4}$$

Under the assumption that there exists no strategic interaction across intermediate firms,[1] maximization of the generic *i-th* firm's instantaneous profit with respect to x_{it} leads to the canonical *mark-up rule*:

$$p_{it} = mr_{it}, \qquad \forall i\in\left[0;N_t\right]. \tag{5}$$

We follow the literature (see, among others, Bucci and Raurich, 2017, p. 188) in focusing on a *symmetric equilibrium* where:

$$r_{it} = r_t; \qquad k_{it} = k_t, \qquad \forall i\in\left[0; N_t\right],$$

with k_t denoting the average amount of physical capital input employed by existing intermediate firms. Hence,

$$p_{it} = p_t \qquad \text{and} \qquad x_{it} = x_t, \qquad \forall i\in\left[0; N_t\right].$$

The hypothesis of symmetry leads to:

$$x_{it} = x_t = \frac{K_t}{N_t}, \qquad \forall i\in\left[0;N_t\right] \tag{4'}$$

$$\pi_{it} = \alpha\left(\frac{m-1}{m}\right)\left(\frac{L_{Yt}}{N_t}\right)^{1-\alpha}\left(\frac{K_t}{N_t}\right)^{\alpha} N_t^{\alpha\left[m(1-\beta)-1\right]} = \pi_t, \qquad \forall i\in\left[0;N_t\right]. \tag{6}$$

The aggregate R&D technology is still:

$$\dot{N}_t = \frac{1}{\chi}L_{Nt}^{\lambda}N_t^{\phi}, \qquad N(0)>0, \qquad \chi>0, \qquad \lambda\in(0;1], \qquad \phi<1 \tag{7}$$

Because the R&D sector is competitive, there is free entry into this market:

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[1] More precisely, we assume that each of these intermediate firms is so small that it takes $\left[\frac{1}{N_t^{\beta}}\int_0^{N_t}\left(x_{it}\right)^{1/m} di\right]^{\alpha m-1}$ as given.

$$w_{Nt} = \frac{1}{\chi}\frac{N_t^{\phi}}{L_{Nt}^{1-\lambda}} V_{Nt}, \tag{8}$$

where V_{Nt} is given by

$$V_{Nt} = \int_t^{\infty} \pi_\tau e^{-\int_t^{\tau} r(s)ds} d\tau, \qquad \tau > t, \tag{9}$$

and satisfies the *no-arbitrage condition*: $\dot{V}_{Nt} = r_t V_{Nt} - \pi_t$.

HOUSEHOLDS

The representative household uses savings to accumulate physical capital, an input into the production of intermediate inputs:

$$\dot{K}_t = \underbrace{\underbrace{\left(r_t K_t + w_t L_t\right)}_{\equiv Y_t} - C_t}_{\text{Household's Savings}}, \qquad K(0) > 0, \tag{10}$$

In Eq. (10) K, C and L denote, respectively, aggregate physical capital, aggregate consumption and the aggregate labor-input,[2] and r is the real rate of return on K. According to this equation, household's investment in physical capital (the left hand side) equals household's savings (the right hand side). In turn, household's savings are equal to the difference between household's income (the sum of interest income, rK, and labor income, wL) and household's consumption (C). For the sake of simplicity, physical capital does not depreciate (see Romer, 1990, p. S82, Eq. 2). Given the above expression, the law of motion of per-capita capital is:

$$\dot{k}_t = \left(r_t - n\right)k_t + w_t - c_t, \qquad k(0) > 0, \qquad \frac{\dot{L}}{L} \equiv n > 0, \tag{11}$$

with $k \equiv K/L$ and $c \equiv C/L$ representing per-capita physical capital and per-capita consumption, respectively. With a *constant inter-temporal elasticity of substitution* (CIES) instantaneous utility function, the objective of the household is to maximize, under constraint, the discounted utility of per capita consumption of all its members:

$$\underset{\{c_t, k_t\}_{t=0}^{t=+\infty}}{\text{Max}}\ U \equiv \int_0^{+\infty} \left(\frac{c_t^{1-\theta} - 1}{1-\theta}\right) e^{-(\rho - n)t} dt, \qquad \left(\rho - n\right) > \left(1-\theta\right)\gamma_c, \qquad \theta > 0, \tag{12}$$

s.t.: $\quad \dot{k}_t = \left(r_t - n\right)k_t + w_t - c_t, \qquad k(0) > 0.$

In Eq. (12) we have normalized population at time 0 to one, $L(0) \equiv 1$. The condition $\left(\rho - n\right) > \left(1-\theta\right)\gamma_c$ (where γ_c is the long-run constant growth rate of per-capita consumption) ensures that the attainable utility, U, is bounded and that the transversality condition holds (see below). The representative dynastic family chooses the optimal path of per-capita consumption $\{c\}_{t=0}^{t=+\infty}$, taking the real interest rate r_t and the wage rate w_t as given. The solution to this problem gives the usual *Ramsey-Keynes rule*:

$$\gamma_c \equiv \frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left(r_t - \rho\right). \tag{13}$$

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[2] In this setting, the labor-force coincides with population, hence per-capita and per-worker variables are the same. Moreover, all labor is employed and at equilibrium obtains the same wage, w.

THE LABOR MARKET AND THE BGP EQUILIBRIUM

Since labor is fully employed and distributed across production of consumption goods and invention of new ideas, at equilibrium the following equalities must hold:

$$L_t = L_{Yt} + L_{Nt}\,, \qquad \forall t \geq 0 \tag{14}$$

$$w_{Yt} = w_{Nt}\,. \tag{15}$$

DEFINITION: *BGP EQUILIBRIUM*

A BGP Equilibrium in this economy is an equilibrium-path along which:

(i) *All variables depending on time grow at constant exponential rates;*

(ii) *The sectoral shares of labor employment (* $s_j \equiv L_{jt} / L_t$, with $j \equiv Y, N$ *) are constant.*

In order to characterize the BGP equilibrium of the model, we proceed as follows. From Eq. (1), under the hypothesis of symmetry (Eq. 4'):

$$w_{Yt} = (1-\alpha)\left(\frac{L_{Yt}}{N_t}\right)^{-\alpha}\left(\frac{K_t}{N_t}\right)^{\alpha} N_t^{\Phi}\,, \qquad \Phi \equiv \alpha\left[m(1-\beta)-1\right]. \tag{16}$$

Using Eq. (6), Eq. (9) can be re-written as:

$$V_{Nt} = \int_t^{\infty} \alpha\left(\frac{m-1}{m}\right)\left(\frac{L_{Y\tau}}{N_\tau}\right)^{1-\alpha}\left(\frac{K_\tau}{N_\tau}\right)^{\alpha} N_\tau^{\Phi} e^{-\int_t^{\tau} r(s)ds} d\tau\,. \tag{17}$$

Along the BGP, due to the fact that all variables depending on time grow at constant exponential rates, r is also constant (see Eq. 13). Thus, after some algebra, V_{Nt} can ultimately be recast as:

$$V_{Nt} = \alpha\left(\frac{m-1}{m}\right)\left(\frac{L_{Yt}}{N_t}\right)^{1-\alpha}\left(\frac{K_t}{N_t}\right)^{\alpha}\frac{N_t^{\Phi}}{\left[r-(1-\alpha)n-\alpha\gamma_K+(1-\Phi)\gamma_N\right]}, \tag{17'}$$

$$n \equiv \frac{\dot{L}_t}{L_t}, \qquad \gamma_N \equiv \frac{\dot{N}_t}{N_t}, \qquad \gamma_K \equiv \frac{\dot{K}_t}{K_t}.$$

Note that for any $\alpha \in (0;1)$, $m > 1$, $L_Y > 0$, $K > 0$, and $N > 0$, V_{Nt} is positive at any time $t \geq 0$ when:

$$r > (1-\alpha)n + \alpha\gamma_K - (1-\Phi)\gamma_N\,. \tag{18}$$

Given V_{Nt}, from Eq. (8) it follows:

$$w_{Nt} = \frac{1}{\chi}\frac{N_t^{\phi}}{L_{Nt}^{1-\lambda}}\alpha\left(\frac{m-1}{m}\right)\left(\frac{L_{Yt}}{N_t}\right)^{1-\alpha}\left(\frac{K_t}{N_t}\right)^{\alpha}\frac{N_t^{\Phi}}{\left[r-(1-\alpha)n-\alpha\gamma_K+(1-\Phi)\gamma_N\right]}. \tag{19}$$

We can now use Eq. (15) and equalize (16) and (19), so obtaining:

$$s_Y \equiv \frac{L_{Yt}}{L_t} = \chi\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{m}{m-1}\right)\frac{L_{Nt}^{1-\lambda}}{L_t N_t^{\phi-1}}\left[r-(1-\alpha)n-\alpha\gamma_K+(1-\Phi)\gamma_N\right]. \tag{20}$$

From (16) and (19) we also observe that along a BGP:

$$\frac{\dot{w}_{Yt}}{w_{Yt}} = \frac{\dot{w}_{Nt}}{w_{Nt}} \equiv \frac{\dot{w}_t}{w_t} = \alpha\gamma_K - \alpha n + \Phi\gamma_N\,. \tag{21}$$

This result has been obtained by noticing (see Eq. 7) that in the BGP equilibrium:

$$\gamma_N = \left(\frac{\lambda}{1-\phi}\right)n\,.$$

Under the hypothesis of symmetry, and using Eqs. (5), (2) and (4'), we can express r as:

$$r = \frac{\alpha L_{Yt}^{1-\alpha} K_t^{\alpha-1} N_t^{\Phi}}{m}. \tag{22}$$

Because r and the sectoral labor-shares are constant along a BGP, Eq. (22) implies that:

$$\Phi\gamma_N = (1-\alpha)(\gamma_K - n). \tag{23}$$

Using (23) into (18) and into (21) delivers, respectively:

$$r > \gamma_K - \gamma_N. \tag{18'}$$

$$\frac{\dot{w}_{Yt}}{w_{Yt}} = \frac{\dot{w}_{Nt}}{w_{Nt}} \equiv \frac{\dot{w}_t}{w_t} = \gamma_K - n. \tag{21'}$$

After combining (20), (23), (7) and (14), in the end we can determine the share of labor allocated to the production of final goods as:

$$s_Y \equiv \frac{L_{Yt}}{L_t} = \frac{m(1-\alpha)(r+\gamma_N-\gamma_K)}{m(1-\alpha)(r+\gamma_N-\gamma_K)+\alpha(m-1)\gamma_N}. \tag{24}$$

From (14):

$$s_N = 1 - s_Y,$$

where s_Y is given by (24).

With $\alpha \in (0;1)$, $m > 1$, $\gamma_N > 0$ and $r > \gamma_K - \gamma_N$ (Eq. 18'), it is immediate to see that $s_Y \in (0;1)$ and $s_N \in (0;1)$, too. Using Eq. (10):

$$\frac{C_t}{K_t} = r + \frac{w_t L_t}{K_t} - \gamma_K. \tag{25}$$

We now use (21), (23), and $n \equiv \frac{\dot{L}_t}{L_t}$ into (25) and obtain:

$$\frac{C_t}{K_t} = r + \frac{w(0)L(0)}{K(0)} - \gamma_K, \qquad L(0) \equiv 1; \qquad K(0) > 0; \tag{25'}$$

$$w(0) > 0,$$

where $L(0)$, $w(0)$, and $K(0)$ are the given initial values (*i.e.*, at $t=0$) of L_t, w_t, and K_t, respectively. Eq. (25') is important because it suggests that along a BGP, C_t and K_t grow at the same constant rate:

$$\frac{\dot{C}_t}{C_t} \equiv \gamma_C = \frac{\dot{K}_t}{K_t} \equiv \gamma_K. \tag{25''}$$

The aggregate production function (Eq. 1), together with the hypothesis of symmetry (Eq. 4'), delivers:

$$Y_t = \left(\frac{L_{Yt}}{N_t}\right)^{1-\alpha} \left(\frac{K_t}{N_t}\right)^{\alpha} N_t^{1+\Phi}. \tag{26}$$

After taking logs and deriving with respect to time, and using (23) into it, Eq. (26) allows us to conclude that along a BGP:

$$\frac{\dot{Y}_t}{Y_t} \equiv \gamma_Y = \gamma_K. \tag{26'}$$

Therefore, by combining (26') and (25''), we see that:

$$\frac{\dot{C}_t}{C_t} \equiv \gamma_C = \frac{\dot{K}_t}{K_t} \equiv \gamma_K = \frac{\dot{Y}_t}{Y_t} \equiv \gamma_Y. \tag{27}$$

Using Eq. (13) and the definition of $c_t \equiv C_t / L_t$, we can write the growth rate of aggregate consumption as:

$$\frac{\dot{C}_t}{C_t} = \frac{\dot{c}_t}{c_t} + n = \frac{1}{\theta}(r - \rho) + n . \tag{28}$$

From (27), (28) and (23) we get:

$$r = \frac{\theta\Phi}{(1-\alpha)}\gamma_N + \rho . \tag{29}$$

Along a BGP:

$$\gamma_N = \left(\frac{\lambda}{1-\phi}\right)n . \tag{7'}$$

Hence:

$$r = \frac{\theta\Phi}{(1-\alpha)}\left(\frac{\lambda}{1-\phi}\right)n + \rho . \tag{29'}$$

Combining (27), (13) and (29') yields:

$$\frac{\dot{c}_t}{c_t} \equiv \gamma_c = \frac{\dot{k}_t}{k_t} \equiv \gamma_k = \frac{\dot{y}_t}{y_t} \equiv \gamma_y \equiv \gamma = \frac{\Phi}{(1-\alpha)}\left(\frac{\lambda}{1-\phi}\right)n , \tag{30}$$

where $y \equiv Y / L$ is per-capita GDP.

By comparing Eqs. (7'), (30) and (29') in this appendix with the corresponding Eqs. (18), (19) and (20) in the main text, it is immediate to see that, qualitatively, results do not change at all if one considers explicitly the hypothesis that the representative household saves and invests savings in the accumulation of physical capital, an input in the production of intermediate goods [in Romer, 1990, intermediate firms are postulated to employ physical capital (*i.e.*, forgone consumption), as opposed to labor, as the only input].

In the end of this appendix we want also to make sure that the transversality condition:

$$\lim_{t\to+\infty} \lambda_{kt} k_t = 0$$

does hold in the BGP equilibrium. From the (necessary) FOCs taken on the Hamiltonian function (J):

$$J = \left(\frac{c^{1-\theta}-1}{1-\theta}\right)e^{-(\rho-n)t} + \lambda_k\left[(r-n)k + w - c\right],$$

we have:

$$\frac{\partial J}{\partial k} = -\dot{\lambda}_k \qquad \Leftrightarrow \qquad \frac{\dot{\lambda}_k}{\lambda_{ak}} = -(r-n), \tag{31}$$

where λ_k is the co-state variable associated to the state variable, k. By using the last equation, the transversality condition can be recast as:

$$\lim_{t\to+\infty} \lambda_{kt} k_t = \lambda_k(0)k(0)\lim_{t\to+\infty} e^{-(r-n-\gamma_k)t} = 0 . \tag{32}$$

In (32) $k(0) > 0$ and $\lambda_k(0) = \frac{1}{c(0)^{\theta}} > 0$ are the initial values (*i.e.*, at $t = 0$) of the state variable, *k*, and the co-state variable, λ_k. Eq. (32) reveals that the transversality condition is satisfied when the following inequality is met:

$$r > n + \gamma_k ,$$

that is to say when: $r > \gamma_K$, as $\gamma_k = \gamma_K - n$. (TC)

Notice that when the transversality condition (TC) is satisfied, then the condition that guarantees that V_{Nt} is positive at any time $t \geq 0$ –*i.e.*, $r > \gamma_K - \gamma_N$ (see Eqs. 18 and 23)– is simultaneously met for any $\gamma_N > 0$, which is always true in the BGP equilibrium of our model.
Making use of (29') and (30), the transversality condition (TC) leads to the following inequality:

$$\underbrace{\frac{\theta\Phi}{(1-\alpha)}\left(\frac{\lambda}{1-\phi}\right)n+\rho}_{\equiv\,\theta\gamma_c+\rho} > \underbrace{\frac{\Phi}{(1-\alpha)}\left(\frac{\lambda}{1-\phi}\right)n+n}_{\equiv\,\gamma_c+n}.$$

This inequality is always checked when $(\rho - n) > (1-\theta)\gamma_c$, see Eq. 12 above. ■

ON – LINE APPENDIX D: BALASSA DENSITY INDEX, LAFAY DENSITY INDEX AND NETWORK TRADE INDEX

Following Ferrarini and Scaramozzino [20], we provide now the definitions of the complexity indexes employed in our regressions.

Let $X(i,j)$ denote country *i*'s exports of product *j* and $M(i,j)$ denote country *c*'s imports of product *i*.

The *Balassa index* is essentially a normalized export share, i.e.

$$BI(i,j) = \frac{X(i,j)/\sum_j X(i,j)}{\sum_{n \neq i} \frac{X(i,j)}{\sum_j X(i,j)}}$$

where the numerator is the share of industry *j* in country *i*'s export and the denominator is the share of industry *j* in the total exports of the countries in the sample. If $BI(i,j) > 1$, indutry *j* is more important for country *i*'s exports than for the exports of the other countries in the sample. Hence, country *i* has a *revealed comparative advantage* in the production of good *j*.

The normalized trade balance $Z(i,j)$can be defined as:

$$X(i,j) = \frac{X(i,j) - M(i,j)}{X(i,j) + M(i,j)}$$

The trade specialization index, $TS(i,j)$ for each sector *j* is computed as the difference between a country's $Z(i,j)$ and its total trade balance, across sectors:

$$TS(i,j) = Z(i,j) - \sum_j Z(i,j)$$

The *Lafay index*, $LI(i,j)$, is obtained by weighing the $TS(i,j)$ by the sector *j* contribution to trade:

$$LI(i,j) = TS(i,j) \frac{X(i,j) + M(i,j)}{\sum_j X(i,j) + \sum_j M(i,j)}$$

If country *i* specializes in sector *j*, then $LI(i,j) > 0$.

We can use the $BI(i,j)$ or the $LI(i,j)$to build the following *indicator of trade specialization* $q(i,j)$:

$$q(i,j) = \boldsymbol{I}_{\Lambda(\mathrm{i,j})>o}$$

where $\Lambda(i,j)$ can be either the $BI(i,j)$or the $LI(i,j)$ and $\boldsymbol{I}$ is an indicator function giving 1 if $\Lambda(i,j) > 0$ and 0 otherwise.

The degree of closeness of any two production sectors *j* and *k* in the global product space is measured by the index of proximity $\theta(j,k)$:

$$\theta(j,k) = min\{Prob[\mathrm{q}(i,j) = 1|\mathrm{q}(i,k) = 1], Prob[\mathrm{q}(i,k) = 1|\mathrm{q}(i,j) = 1]\}$$

The *density index* for sector *j* is the weighted average of the trade specialization indicators, in which the weights are the proximities of sector *j* with all the other sectors:

$$\omega(i,j) = \frac{\sum_k \theta(j,k)\mathrm{q}(i,k)}{\sum_k \theta(j,k)}$$

The average density of country *i* is obtained as an average of the density indexes across all sectors:

$$\widetilde{\omega} = \sum\nolimits_j \omega(i,j)$$

The average density refers to the number of paths out of all possible paths within the product space that lead to the products that are already part of a country's export basket.

As Ferrarini and Scaramozzino point out, this concept of product space and the related density measure account for intra-industry trade within coarse-grained sector balances. To account also for vertical trade in the empirical investigation of a complexity-growth model, they suggest to employ the *Network Trade Index* (NTI) as a measure of the intensity of trade among countries participating in the international production network (Ferrarini [16]). The $\mathrm{NTI}(i,z)$ of country *i* with respect to its trade partner *z* is defined as follows:

$$\mathrm{NTI}(i,z) \sum\nolimits_j \sum\nolimits_{i \neq z} \frac{M_{i,j}^z}{\sum_z M_{i,j}^z} \frac{X_j^i}{\sum_j X_j^i}$$

where: $M_{i,j}^z$ is the value of imports to country *i* of components of industry *j* from country *z*, $M_{i,j}^z / \sum_z M_{i,j}^z$ is the share of country z's components of industry *j* on total imports of *j* in *i* and $X_j^i / \sum_j X_j^i$ is the share of sector *j* on total exports from country *i*. The higher NTI, the greater the importance of country *z* in the network of productive relations of country *i*. In its aggregate form, the index is derived as a geometric average across sectors.

Throughout the paper, NTI, $\widetilde{\omega}_{BI}$ and $\widetilde{\omega}_{LI}$ have been normalized in the interval [0,1].

ON – LINE APPENDIX E

Table 7: Estimated intermediate sector's markup

Country	$\tilde{m}$
Australia	1.665
Austria	1.510
Belgium	1.650
Canada	1.755
Cyprus	1.796
Czech Republic	1.420
Denmark	1.572
Estonia	1.793
Finland	1.668
France	1.569
Germany	1.665
Greece	1.751
Hungary	1.620
Ireland	1.518
Italy	1.599
Japan	1.540
Latvia	1.368
Lithuania	1.678
Malta	1.510
Netherlands	1.688
Poland	1.596
Portugal	1.683
Slovak Republic	1.717
Slovenia	1.906
Spain	1.608
Sweden	1.538
United Kingdom	1.686
United States	1.713
mean	1.635

Note: all parameters are significant at 1%.