Appendix A: Proof of Lemma 1

In this proof, we examine the stability of this model given a stationary path of μ_t. First, define the transformed variables $\Psi_{n,t} \equiv Y_t/(V_{n,t}N_t)$ and $\Psi_{k,t} \equiv Y_t/(V_{k,t}K_t)$. Then, differentiating $\Psi_{n,t}$ with respect to time yields

$$\frac{\dot{\Psi}_{n,t}}{\Psi_{n,t}} = \frac{\dot{Y}_t}{Y_t} - \frac{\dot{V}_{n,t}}{V_{n,t}} - \frac{\dot{N}_t}{N_t}. \tag{A.1}$$

From the final-goods resource constraint $Y_t = C_t$, the law of motion for Y_t is given by

$$\frac{\dot{Y}_t}{Y_t} = \frac{\dot{C}_t}{C_t} = R_t - \rho, \tag{A.2}$$

where the second equality stems from the Euler equation in (3). From (12), the law of motion for $V_{n,t}$ is

$$\frac{\dot{V}_{n,t}}{V_{n,t}} = R_t - \frac{\Pi_{x,t}}{V_{n,t}}. \tag{A.3}$$

where $\Pi_{x,t} = \alpha(\mu - 1)Y_t/(\mu_t N_t)$, which is obtained by applying symmetry across varieties in (4) to rewrite (5) as $\alpha Y_t/N_t = P_t(j)X_t(j)$ and substituting it into (9). Combining (A.1)-(A.3) yields

$$\frac{\dot{\Psi}_{n,t}}{\Psi_{n,t}} = \alpha\left(\frac{\mu - 1}{\mu}\right)\Psi_{n,t} - \varphi L_{r,t} - \rho, \tag{A.4}$$

where we use the fact that $\dot{N}_t/N_t = \varphi L_{r,t}$.

Using the same logic, differentiating $\Psi_{k,t}$ with respect to time yields

$$\frac{\dot{\Psi}_{k,t}}{\Psi_{k,t}} = \frac{\dot{Y}_t}{Y_t} - \frac{\dot{V}_{k,t}}{V_{k,t}} - \frac{\dot{K}_t}{K_t}. \tag{A.5}$$

From (15), the law of motion for $V_{k,t}$ is

$$\frac{\dot{V}_{k,t}}{V_{k,t}} = R_t - \frac{Q_t}{V_{k,t}}, \tag{A.6}$$

where $Q_t = \alpha\gamma Y_t/(\mu_t K_t)$, which is obtained by applying symmetry across varieties in (4) to rewrite

(5) as $\alpha Y_t/N_t = P_t(j)X_t(j)$ and substituting it into (11). Combining (A.2), (A.5), and (A.6) yields

$$\frac{\dot{\Psi}_{k,t}}{\Psi_{k,t}} = \alpha\left(\frac{\gamma}{\mu}\right)\Psi_{k,t} - \phi L_{k,t} - \rho, \tag{A.7}$$

where we use the fact that $\dot{K}_t/K_t = \phi L_{k,t}$.

Furthermore, combining (14) and (17) yields $\varphi V_{n,t}N_t = \phi V_{k,t}K_t$, which implies

$$\frac{\Psi_{n,t}}{\varphi} = \frac{\Psi_{k,t}}{\phi}, \tag{A.8}$$

and also $\dot{\Psi}_{n,t}/\Psi_{n,t} = \dot{\Psi}_{k,t}/\Psi_{k,t}$. Using this result and (A.8), we rewrite (A.7) to make $L_{k,t}$ a function of $\dot{\Psi}_{n,t}/\Psi_{n,t}$ and $\Psi_{n,t}$ such that

$$L_{k,t} = -\frac{1}{\phi}\left[\frac{\dot{\Psi}_{n,t}}{\Psi_{n,t}} - \alpha\left(\frac{\gamma}{\mu}\right)\left(\frac{\phi}{\varphi}\right)\Psi_{n,t} + \rho\right]. \tag{A.9}$$

Then, we use (10) to derive

$$L_{x,t} = \int_0^{N_t} L_{x,t}(j)dj = \frac{\left(\frac{1-\gamma}{\mu}\right)\int_0^{N_t} P_t(j)X_t(j)dj}{W_t} = \frac{\left(\frac{1-\gamma}{\mu}\right)\alpha Y_t}{W_t} = \frac{\alpha}{\varphi}\left(\frac{1-\gamma}{\mu}\right)\Psi_{n,t}, \tag{A.10}$$

where (4) and (5) are used in the third equality and (14) is used in the fourth equality.

Finally, substituting (A.9), (A.10), and the labor-market-clearing condition $L_{x,t}+L_{r,t}+L_{k,t} = 1$ into (A.4), a few steps of manipulation yield a one-dimensional differential equation in $\Psi_{n,t}$:

$$\frac{\dot{\Psi}_{n,t}}{\Psi_{n,t}} = \left(1+\frac{\varphi}{\phi}\right)^{-1}\left[\alpha\Psi_{n,t} - \varphi\left(1+\frac{\rho}{\varphi}+\frac{\rho}{\phi}\right)\right]. \tag{A.11}$$

Therefore, the dynamics of $\Psi_{n,t}$ is characterized by saddle-point stability such that $\Psi_{n,t}$ jumps immediately to its interior steady-state value given by

$$\Psi_n = \frac{\varphi}{\alpha}\left(1+\frac{\rho}{\varphi}+\frac{\rho}{\phi}\right). \tag{A.12}$$

Then, (A.4), (A.7), and (A.10) reveal that when Ψ_n and μ are stationary, L_r, L_k, and L_x must also be stationary, respectively.